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FURTHER SIMPLEST-EXPRESSION INTEGRALS INVOLVING BEAM EIGENFUNCTIONS AND DERIVATIVES

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1. INTRODUCTION

Dynamic analysis, including free vibration and stability, of distributed structures such as beams, plates, shells, frames and other multibody structures composed of the former ones, can be carried out through the classical Rayleigh–Ritz method and its generalization for composite systems, the substructure synthesis method [1, 2]. In the analyses, admissible functions such as Euler–Bernoulli beam eigenfunctions have the advantage, compared to other functions, of being dynamically related to the foregoing structures, which implies excellent convergence characteristics of the methodologies [3, 2], and of being orthogonal. In fact, they were applied successfully in analyzing those structures throughout the last century [4–9]. Nevertheless, the application of beam eigenfunctions has not been wide because they have the drawback of being troublesome to work with, as they are complicated and involve hyperbolic functions. In avoiding beam functions, some researchers utilized trigonometric functions [10], non-orthogonal polynomials [11] and orthogonal polynomials [12], while others developed the finite element method which, as far as dynamics is concerned, represents a Rayleigh–Ritz method [13]. The finite element method is not the most indicated one for beams, plates, shells and multibody structures with simple geometry; indeed, the superior convergence characteristics of basic Rayleigh–Ritz methodologies based on beam characteristic functions have been shown in these cases [3, 2, 9].

Therefore, if convergence, more importantly, if speed of convergence is critical in analyzing these structures, the Rayleigh–Ritz method and the substructure synthesis method should be preferred; as a consequence, integrals involving beam eigenfunctions and derivatives must be carried out. Moreover, because any of the Rayleigh–Ritz approaches is a numerical method itself, there is the strongest interest in reducing the number of computer operations in the algorithms, needless to say, in avoiding secondary numerical routines as numerical integration. Reducing the number of computer operations is especially important because badly behaved hyperbolic functions are involved. Hence, the solution of the integrals must be simple.

In 1950, in a sometimes forgotten and not well-disseminated work, R. P. Felgar presented a table with that type of integrals. The results were reprinted by Blevins [14] and are neat and amazing; in other words, the simplicity and order of their structure is so marvellous that they represent an example of the close relationship among mathematics, physics and beauty. Incidentally, it seems that Sharma [15] was not aware of this preceding work while developing a paper on this subject. In contrast, in another work Leung [16] extended the

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original by Felgar; indeed, some erroneous results were corrected. However, integrals in their simplest form were not attained as in Felgar's work; actually, the objective of Leung's investigation was to generalize Felgar's results and to develop a computational algorithm for obtaining the definite integrals.

We present additional integrals in their simplest form. Most of them involve pinned–clamped and clamped–pinned beam eigenfunctions. In addition, some integrals that appeared in reference [14] that can be written in an even simpler form are also presented. All the integrals arise in vibrational and buckling analysis of frames [9]. It is emphasized that simpler results improve the accuracy of Rayleigh–Ritz-based approximate methods for dynamic analysis of continuous systems.

2. DEFINITIONS AND RESULTS

The vibrational eigenfunctions satisfy the ordinary differential equation

$$Y'''' = \beta^4 Y, \tag{1}$$

wherein the independent variable is the spatial variable x and beta is given by

$$\beta^4 = \frac{m\omega^2}{EI}, \tag{2}$$

where m is the mass per unit length, ω is the natural frequency and EI is the flexural rigidity. The characteristic functions for clamped–free, clamped–clamped, clamped–pinned and pinned–clamped beams are, respectively, given by

$$Y_{cf} = \cosh\beta_1 x - \cos\beta_1 x - \sigma_1(\sinh\beta_1 x - \sin\beta_1 x), \tag{3a}$$

$$Y_{cc} = \cosh\beta_2 x - \cos\beta_2 x - \sigma_2(\sinh\beta_2 x - \sin\beta_2 x), \tag{3b}$$

$$Y_{cp} = \cosh\beta_3 x - \cos\beta_3 x - \sigma_3(\sinh\beta_3 x - \sin\beta_3 x), \tag{3c}$$

$$Y_{pc} = \sigma_4 \sinh\beta_3 x - \sigma_5 \sin\beta_3 x, \tag{3d}$$

where the characteristic betas are defined by

$$\cos\beta_1 L \cosh\beta_1 L = -1, \qquad \cos\beta_2 L \cosh\beta_2 L = 1, \qquad \mathrm{tg}\,\beta_3 L = \mathrm{tgh}\,\beta_3 L \tag{4a, b, c}$$

and the characteristic sigmas by

$$\sigma_1 = \frac{\sinh\beta_1 L - \sin\beta_1 L}{\cosh\beta_1 L + \cos\beta_1 L} = \frac{\cosh\beta_1 L + \cos\beta_1 L}{\sinh\beta_1 L + \sin\beta_1 L}, \tag{5a}$$

$$\sigma_2 = \frac{\sinh\beta_2 L + \sin\beta_2 L}{\cosh\beta_2 L - \cos\beta_2 L} = \frac{\cosh\beta_2 L - \cos\beta_2 L}{\sinh\beta_2 L - \sin\beta_2 L}, \tag{5b}$$

$$\sigma_3 = \operatorname{ctg}\beta_3 L = \operatorname{ctgh}\beta_3 L = \frac{\cosh\beta_3 L + \cos\beta_3 L}{\sinh\beta_3 L + \sin\beta_3 L} = \frac{\cosh\beta_3 L - \cos\beta_3 L}{\sinh\beta_3 L - \sin\beta_3 L}, \tag{5c}$$

$$\sigma_4 = \operatorname{csch}\beta_3 L, \qquad \sigma_5 = \csc\beta_3 L, \tag{5d, e}$$

where L is the beam length. Five points for noting are the simple form for Y_{pc} that has been obtained, that a normalized spatial variable has not been considered as in reference [16], the several forms for σ_3, the introduction of σ_4 and σ_5 and the fact that the usual normalizing constant has not been considered in equations (3) for simplicity. Finally, in the table of integrals that follows, the indices r and s indicate the mode and the primes indicate derivatives.

2.1. INTEGRALS

1. $\displaystyle\int_0^L Y_{pc}^2\,\mathrm{d}x = L,$

2. $\displaystyle\int_0^L Y_{pc}'^2\,\mathrm{d}x = \beta_3\sigma_3(\beta_3 L\sigma_3 - 1),$

3. $\displaystyle\int_0^L Y_{pc}''^2\,\mathrm{d}x = \beta_3^4 L,$

4. $\displaystyle\int_0^L Y_{cp}\,\mathrm{d}x = \frac{1}{\beta_3}(2\sigma_3 - \sigma_4 - \sigma_5),$

5. $\displaystyle\int_0^L Y_{pc}\,\mathrm{d}x = \frac{1}{\beta_3}(2\sigma_3 - \sigma_4 - \sigma_5),$

6. $\displaystyle\int_0^L xY_{cp}\,\mathrm{d}x = \frac{1}{\beta_3}\left(\frac{2}{\beta_3} - L(\sigma_4 + \sigma_5)\right),$

7. $\displaystyle\int_0^L xY_{pc}\,\mathrm{d}x = \frac{2}{\beta_3}\left(L\sigma_3 - \frac{1}{\beta_3}\right),$

8. $\displaystyle\int_0^L Y_{cp_r}Y_{pc_s}\,\mathrm{d}x = \frac{2\beta_{3r}\beta_{3s}}{\beta_{3r}^4 - \beta_{3s}^4}(\beta_{3r}(\sigma_{4s} - \sigma_{5s}) - \beta_{3s}(\sigma_{4r} - \sigma_{5r})) \quad \text{for } r \neq s,$

9. $\displaystyle\int_0^L Y_{cp_r}Y_{pc_r}\,\mathrm{d}x = \frac{1}{2\beta_{3r}}(\sigma_{4r} - \sigma_{5r})(\beta_{3r}L\sigma_{3r} + 1),$

10. $\displaystyle\int_0^L Y_{cf_r}Y_{cp_s}\,\mathrm{d}x = \frac{(-1)^r 2\beta_{3s}^3}{\beta_{3s}^4 - \beta_{1r}^4}(\sigma_{4s} + \sigma_{5s}),$

11. $$\int_0^L Y_{cc_r} Y_{cp_s}\,\mathrm{d}x = \frac{(-1)^r 2\beta_{2r}^2\beta_{3s}}{\beta_{3s}^4 - \beta_{2r}^4}(\sigma_{4s} - \sigma_{5s}),$$

12. $$\int_0^L Y'_{cf_r} Y'_{cc_s}\,\mathrm{d}x = \frac{4\beta_{1r}^2\beta_{2s}^2}{\beta_{2s}^4 - \beta_{1r}^4}(\beta_{1r}\sigma_{1r} - \beta_{2s}\sigma_{2s}),$$

13. $$\int_0^L Y'_{cf_r} Y'_{cp_s}\,\mathrm{d}x = \frac{2\beta_{1r}\beta_{3s}}{\beta_{3s}^4 - \beta_{1r}^4}((-1)^r\beta_{1r}^3(\sigma_{5s} - \sigma_{4s}) - 2\beta_{1r}\beta_{3s}(\beta_{3s}\sigma_{3s} - \beta_{1r}\sigma_{1r})),$$

14. $$\int_0^L Y'_{cc_r} Y'_{cp_s}\,\mathrm{d}x = \frac{2\beta_{2r}^2\beta_{3s}^2}{\beta_{2r}^4 - \beta_{3s}^4}((-1)^r\beta_{3s}(\sigma_{5s} + \sigma_{4s}) + 2(\beta_{3s}\sigma_{3s} - \beta_{2r}\sigma_{2r})),$$

15. $$\int_0^L Y''_{cf_r} Y''_{cc_s}\,\mathrm{d}x = \frac{(-1)^{r+s}4\beta_{1r}^4\beta_{2s}^2}{\beta_{1r}^4 - \beta_{2s}^4}(\beta_{1r}\sigma_{1r} - \beta_{2s}\sigma_{2s}),$$

16. $$\int_0^L Y''_{cp_r} Y''_{pc_s}\,\mathrm{d}x = \frac{2\beta_{3s}^2\beta_{3r}^2}{\beta_{3r}^4 - \beta_{3s}^4}(\beta_{3s}^3(\sigma_{4s} - \sigma_{5s}) - \beta_{3r}^3(\sigma_{4r} - \sigma_{5r})) \quad \text{for } r \neq s,$$

17. $$\int_0^L Y''_{cp_r} Y''_{pc_r}\,\mathrm{d}x = \frac{\beta_{3r}^3}{2}(\sigma_{4r} - \sigma_{5r})(\beta_{3r}L\sigma_{3r} - 3),$$

18. $$\int_0^L Y''_{cf_r} Y''_{cp_s}\,\mathrm{d}x = \frac{(-1)^{r+1}2\beta_{1r}^4\beta_{3s}^3}{\beta_{1r}^4 - \beta_{3s}^4}(\sigma_{4s} + \sigma_{5s}),$$

19. $$\int_0^L Y''_{cc_r} Y''_{cp_s}\,\mathrm{d}x = \frac{(-1)^{r+1}2\beta_{2r}^2\beta_{3s}^5}{\beta_{2r}^4 - \beta_{3s}^4}(\sigma_{4s} - \sigma_{5s}).$$

3. REMARKS ON THESE AND PREVIOUS RESULTS

Apart from assuring neatness in mechanics, the practical implication of this work and the original work by Felgar is that simple and computer-friendly expressions for the characteristic integrals have been attained. The results are important because complex approximate methods such as the ones based on the Rayleigh–Ritz theory, including the finite element method, ask for easy system matrices construction and for curtailing the number of both computer operations and subordinate numerical methods in order to protect accuracy. In other words, the interest is in that the convergence of the approximate technique be controlled only by its inherent numerical characteristics and not by round-off errors associated with operations involving hyperbolic functions, for example, or secondary approximate methods such as numerical integration. Leung's important contribution [16], although broader in scope, does not ensure that the Rayleigh–Ritz procedures based on the algorithms presented therein are numerically optimal in the sense explained before.

4. CONCLUSIONS

Further simplest-expression integrals that contain eigenfunctions of Euler–Bernoulli beam boundary-value problems have been obtained; the expressions are written in terms of beam length and a few characteristic constants (L, β_i and σ_i). This type of integrals appears in optimal structural applications of Rayleigh–Ritz methodologies. These results supplement the work by Felgar, which proves powerful for simplifying those methodologies and improving their convergence characteristics.

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