Journal of Sound and Vibration (2002) **253**(3), 724–732
doi:10.1006/jsvi.2001.4054, available online at http://www.idealibrary.com on IDEAL®

A NOTE ON THE VIBRATION OF TRANSVERSELY ISOTROPIC SOLID SPHERES

G. R. Buchanan and G. R. Ramirez

Department of Civil and Environmental Engineering, Tennessee Technological University, Cookeville, TN 38501, *U.S.A. E-mail*: *gbuchanan@tntech.edu*

(*Received* 10 *October* 2001)

1. INTRODUCTION

The vibration of solid isotropic spheres has been a topic of study for well over a century with the first published results being credited to Lamb [1]. Since then Sato and Usami [2] published their classic work and that was expanded by the more complete treatise of Lapwood and Usami [3]. Noteworthy additions to the published literature have apeared and among them Heyliger and Jilani [4] verify the results of Sato and Usami [2] for isotropic spheres and give some results for inhomogeneous spheres as well as an excellent list of references. Chau [5] has given toroidal frequencies and mode shapes for transversely isotropic solid spheres. Recently, Chen and Ding [6] outlined their work on layered transversely isotropic hollow spheres. In this letter, the toroidal frequencies and mode shapes given by Chau [5] are verified and augmented. New results are tabulated for spheroidal frequencies of transversely isotropic spheres. Illustrative mode shapes for both spheroidal and toroidal frequencies are presented graphically as a plane slice removed from the cross-section of the sphere.

2. ANALYSIS

The finite element method is used to model the sphere in (r, θ, ϕ) co-ordinates. The derivation and accuracy of the finite element used in this study has been established previously by Buchanan and Rich [7] for thick sphereical shells and need not be repeated in this letter. It should be noted that the formulation is three-dimensional but the finite element was derived in two dimensions while maintaining three degrees of freedom by assuming the following solution that satisfies the circumferential displacement and also defines a circular frequency:

$$u_r(r, \theta, \phi, t) = U(r, \theta) \cos n\theta \cos \omega t, \; u_\theta(r, \theta, \phi, t) = W(r, \theta) \cos n\theta \cos \omega t,)$$

$$u_\phi(r, \theta, \phi, t = V(r, \theta) \sin n\theta \cos \omega t, \tag{1}$$

where n is the circumferential wave number, ω is the circular frequency and u_r, u_θ and u_z as well as U, W and V are the displacements in the r, θ and ϕ co-ordinate directions respectively. The analysis for solid spheres with completely free boundary conditions corresponds to the solution for circumferential wave number $n = 0$, where spheroidal motion defined by U and W is uncoupled from toroidal motion defined by V.

0022-460X/02/$35.00 © 2002 Elsevier Science Ltd. All rights reserved.

A nine-node Lagrangian finite element was used for the analysis. The cross-section of the sphere was modelled using 50 elements, 231 nodes or 693 degrees of freedom.

3. FREQUENCIES AND MODE SHAPES

The reliability of the finite element analysis was verified by comparing results for frequency with the frequencies reported by Sato and Usami [2]. The first two columns of Table 1 correspond to a solid isotropic sphere with 0·25 for the Poisson ratio. The agreement between the finite element solution and the exact solution for the first 20 frequencies is acceptable (1·1% difference for the 20th frequency). It follows that seven of the first 20 frequencies are toroidal when $\nu = 0{\cdot}25$. Frequencies are given in terms of a non-dimensional frequency Ω defined as

$$\Omega = \omega a\sqrt{\rho/C_{44}}, \tag{2}$$

where a is the radius of the solid sphere, ρ is the density and C_{44} is the material constant that relates the shear stresses $\sigma_{r\theta}$ and $\sigma_{r\phi}$ to their respective strains. Material constants for stress–strain relations in spherical co-ordinates are defined in most of the referenced papers. Chau [5] defines material constants for spherical isotropy and this work will follow his description. Chau [5] defines a material modulus β that is essentially the material constant C_{66} for a hexagonal material non-dimensionalized with respect to C_{44}:

$$\beta = (C_{11} - C_{12})/2C_{44}. \tag{3}$$

The non-dimensional β allows a comparison of the degree of spherical isotropy for hexagonal crystals and minerals, where $\beta = 1{\cdot}0$ for isotropic materials. Material constants are non-dimensional with respect to C_{44} in this report.

Material constants for the hexagonal materials that were studied by Chau [5] were tabulated by Payton [8] and were used to compute the frequencies of Table 1. Comparison with the toroidal frequencies given by Chau [5] indicates the accuracy that can be expected and it is acceptable. There are several toroidal frequencies given in Table 1 that are not included in Chau's work because they correspond to a higher harmonic than he reported. The nature of the formulation of the finite element solution gives the frequencies in a numerically ordered manner.

The toroidal results for isotropic materials are independent of the Poisson ratio; however, the first few columns of Table 1 show that varying the Poisson ratio does effect the spheroidal frequencies. The magnitudes of the toroidal frequencies remain unchanged but their numerical ordering does change. The first 20 frequencies are given in Table 1 and as β increases the spheroidal motion becomes more pronounced. Mode shapes are shown in Figures 1 and 2 and were plotted using the results for thallium spheres. The numbering of the mode shapes corresponds to the order of the frequencies in Table 1. The first 12 spheroidal mode shapes for thallium are shown in Figure 1 where the deformed shape is superposed on an undeformed plane sliced from the sphere. Spheroidal mode shapes for Ω_2, Ω_9, Ω_{11}, Ω_{18}, Ω_{20} and Ω_{26} are antisymmetric with respect to a horizontal plane ($\theta = 90°$). Mode shapes for Ω_7, Ω_{19} and Ω_{22} show a radially symmetric type of motion. The remaining mode shapes show a translatory motion along the vertical axis but somewhat symmetrical with respect to a horizontal plane. The frequencies of Figure 2 show torsional motion of a plane section of the sphere as a three-dimensional plot. The same motion is shown (using a smaller scale) as a contour plot to aid in visualizing the mode shape. The non-dimensional toroidal frequency $\Omega = 5{\cdot}765$ is common to all materials that are reported in Table 1 and according to Chau [5] is the first root corresponding to a first-degree spherical harmonic.

TABLE 1

Frequency $\Omega = \omega a(\rho/C_{44})^{1/2}$ *for solid spheres with transversely isotropic material properties corresponding to various minerals and crystals,* $n = 0$, $\beta = (C_{11} - C_{12})/2C_{44}$; t *indicates toroidal frequency*

	Isotropic				Thallium		Titanium		Beryllium	
	$\beta = 1{\cdot}0000$				$\beta = 0{\cdot}3719$		$\beta = 0{\cdot}7495$		$\beta = 0{\cdot}8188$	
Mode	Reference [2]	$\nu = 0{\cdot}25$	$\nu = 0{\cdot}30$	$\nu = 0{\cdot}40$	Reference [5]		Reference [5]		Reference [5]	
1	2·501*t*	2.501*t*	2·501*t*	2·501*t*	1·531*t*	1·551*t*	2·178*t*	2·178*t*	2·273*t*	2.273*t*
2	2·640	2·641	2·647	2·658		2·182		2·484		2·465
3	3·424	3·477	3·588	3·792	2·415*t*	2·416*t*	3·373*t*	3·374*t*		2·823
4	3·865*t*	3·867*t*	3·867*t*	3·867*t*	3·197*t*	3·201*t*		3·815		2·980
5	3·916	3·920	3·942	3·979		3·585		3·820	3·517*t*	3·519*t*
6	4·440	4·440	4·996	5·102*t*		3·952*t*	4·450*t*	4·456*t*		3·661
7	4·865	4·866	5·009	5·130		4·245		4·986		3·990
8	5·009	5·024	5·061	5·265		4·689*t*		5·171	4·639*t*	4·646*t*
9	5·095*t*	5·102*t*	5·102*t*	5·765*t*		4·832		5·189		4·679
10	5·763*t*	5·765*t*	5·765*t*	6·226*t*		5·425*t*		5·493*t*		5·405
11	6·033	6·070	6·124	6·287*t*		5·539	5·764*t*	5·765*t*		5·726*t*
12	6·266*t*	6·287*t*	6·287*t*	6·891	5·764*t*	5·765*t*		6·088		5·644
13	6·454	6·496	6·615	7·095		6·013		6·512*t*		5·755
14	6·771	6·918	7·175	7·141*t*		6·172*t*		6·777	5·764*t*	5·765*t*
15	7·023	7·105	7·141*t*	7·316	6·368*t*	6·372*t*	6·855*t*	6·859*t*		6·600
16	7·136*t*	7·141*t*	7·285	7·453*t*		6·969*t*		7·172		6·788
17	7·404*t*	7·453*t*	7·453*t*	7·566	7·060*t*	7·065*t*		7·531		6·874
18	7·744	7·810	8·095	8·429		7·175		7·622	6·936*t*	6·940*t*
19	7·995	8·078	8·234	8·455*t*		7·185	7·955*t*	7·964*t*		7·238
20	8·062	8·153	8·241	8·504		7·305		8·251		7·531

Table 1

Continued

	Hafnium		Cobalt		Yttrium		Magnesium		Rhenium	
	$\beta = 0{\cdot}9336$		$\beta = 0{\cdot}9404$		$\beta = 1{\cdot}0016$		$\beta = 1{\cdot}0213$		$\beta = 1{\cdot}0556$	
Mode	Reference [5]		Reference [5]		Reference [5]		Reference [5]		Reference [5]	
1	2·420*t*	2.420*t*	2·429*t*	2·429*t*	2·503*t*	2·503*t*	2·527*t*	2·527*t*	2·567*t*	2.567*t*
2		2·620		2·691		2·676		2·697		2·748
3		3·671	3·754*t*	3·756*t*		3·746		3·814	3·965*t*	3·966*t*
4	3·742*t*	3·743*t*		4·096	3·868*t*	3·869*t*	3·903*t*	3·905*t*		3·975
5		3·938		4·222		3·996		4·031		4·116
6		4·787	4·950*t*	4·957*t*		4·583		5·046		5·169
7	4·933*t*	4·940*t*		5·318	5·098*t*	5·106*t*	5·145*t*	5·153*t*	5·226*t*	5·233*t*
8		5·079		5·515		5·136		5·186		5·300
9		5·080	5·764*t*	5·765*t*		5·213		5·311		5·536
10	5·764*t*	5·765*t*		5·814	5·764*t*	5·765*t*	5·764*t*	5·765*t*	5·764*t*	5·765*t*
11		6·088*t*		6·104*t*		6·215		6·280		6·419
12		6·159		6·472		6·291*t*		6·349*t*		6·448*t*
13		6·708	7·072*t*	7·076*t*		6·899		7·012	7·195*t*	7·200
14	7·069*t*	7·069*t*		7·241*t*		7·038	7·159*t*	7·164*t*		7·310
15		7·217*t*		7·604	7·138*t*	7·143*t*		7·358		7·520
16		7·224		7·641		7·278		7·527*t*		7·644*t*
17		7·368		8·245		7·458*t*		7·553	8·546*t*	8·556*t*
18		8·006	8·334*t*	8·344*t*		8·051		8·169		7·877
19		8·300		8·375*t*		8·352		8·448		8·288
20	8·321*t*	8·331*t*		8·534	8·448*t*	8·458*t*	8·484*t*	8·494*t*		8·631

Table 1

Continued

	Ice (275 K)		Apatite		Beryl		Zinc		Cadmium	
	$\beta = 1{\cdot}1041$		$\beta = 1{\cdot}1606$		$\beta = 1{\cdot}3309$		$\beta = 1{\cdot}6919$		$\beta = 1{\cdot}8897$	
Mode	Reference [5]		Reference [5]		Reference [5]		Reference [5]		Reference [5]	
1	2·622*t*	2.623*t*		2·612	2·867*t*	2·867*t*		2·926*t*		3.192
2		2·786	2·686*t*	2·686*t*		2·921		3·152	3·386*t*	3·387*t*
3	4·049*t*	4·051*t*		2·796		4·071	3·213*t*	3·214*t*		3·668
4		4·084		3·756		4·273		4·065		4·383
5		4·164		3·811	4·421*t*	4·424*t*		4·766	5·212*t*	5·215*t*
6	5·337*t*	5·344*t*		4·111		5·120		4·810		5·442
7		5·356	4·146*t*	4·148*t*		5·434	4·949*t*	4·951*t*		5·565
8		5·579		4·735	5·764*t*	5·765*t*		5·034	5·764*t*	5·765*t*
9		5·706	5·463*t*	5·472*t*	5·825*t*	5·833*t*		5·599		6·116
10	5·764*t*	5·765*t*		5·547		5·792	5·764*t*	5·765*t*		6·457
11		6·484		5·676		6·535		5·971		6·616
12		6·585*t*	5·764*t*	5·765*t*		7·187*t*		6·340	6·863*t*	6·874*t*
13	7·245*t*	7·250*t*		5·945	7·471*t*	7·477*t*	6·517*t*	6·527*t*		7·256
14		7·522		6·619		7·489		6·917		7·476
15		7·596		6·742*t*		7·623		7·191	7·969*t*	7·976*t*
16		7·806*t*		6·980		7·683		7·657		8·340
17		8·082		7·222		8·345		7·759		8·469*t*
18		8·570	7·303*t*	7·309*t*		8·520*t*	7·801*t*	7·807*t*		8·524
19	8·632*t*	8·643*t*		7·319		8·724		7·892		8·860
20		8·719		7·584	9·013*t*	9·025*t*		8·041*t*		8·896

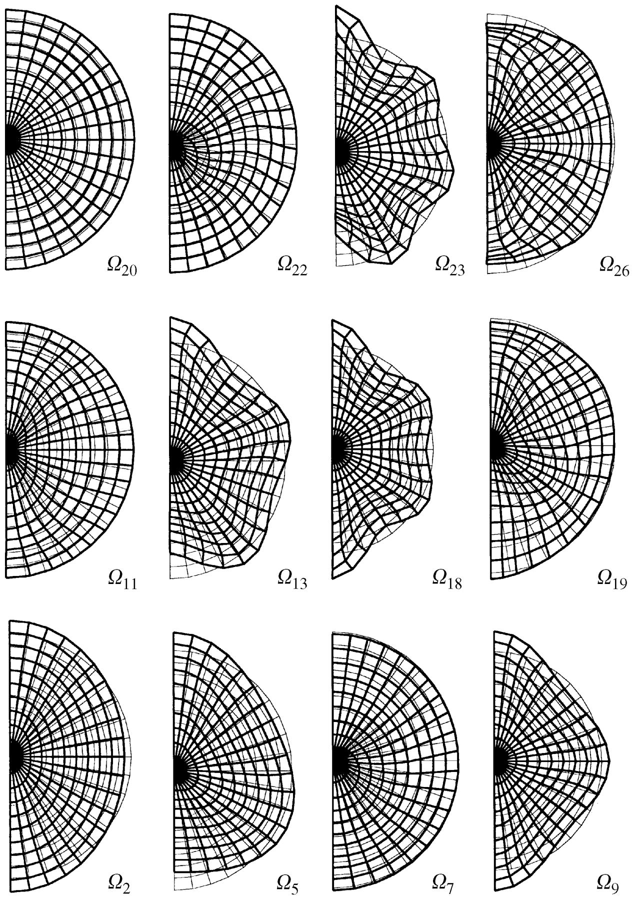

Figure 1. Spheroidal mode shapes for solid spheres with free boundary conditions corresponding to material properties for thallium (see Table 1).

For thallium ($\beta = 0{\cdot}3719$), $\Omega = 5{\cdot}765$ is the seventh toroidal root, but for an isotropic material ($\beta = 1{\cdot}0$) it is the fourth toroidal root, while for cadmium ($\beta = 1{\cdot}8897$) $\Omega = 5{\cdot}765$ is the third toroidal root. The effect on frequency and mode shape caused by changing material properties is easily identified. Chau [5] and Hosseini-Hashemi and Anderson [9] gave torsional mode shapes as deformed meshes on a spherical surface. The mode shape corresponding to $\Omega = 5{\cdot}765$ has been given by Hosseini-Hashemi and Anderson [9, Figure 3(b)] and when compared with Figure 2 it is Ω_{12}. The corresponding motion inside the sphere is illustrated as points inside the sphere that are moving opposite to points on the surface of the sphere. Similarly, Ω_1 and Ω_{15}, as well as Ω_3 and Ω_{17} of Figure 2 would appear to have similar mode shapes when viewed as motion on the surface of the sphere, but Figure 2 shows the motion of the interior of the solid sphere. It follows that Figure 2 gives an improved visualization of the mode shapes.

Frequencies for the spheroidal motion of a solid isotropic sphere with completely fixed boundary conditions were reported by Schafbuch *et al.* [10]. Table 2 gives the finite element results and comparison for $\nu = 0{\cdot}25$ since that was the Poisson ratio used by Sato and Usami [2]. The first five torsional frequencies are also listed in Table 2. Chen *et al.* [11]

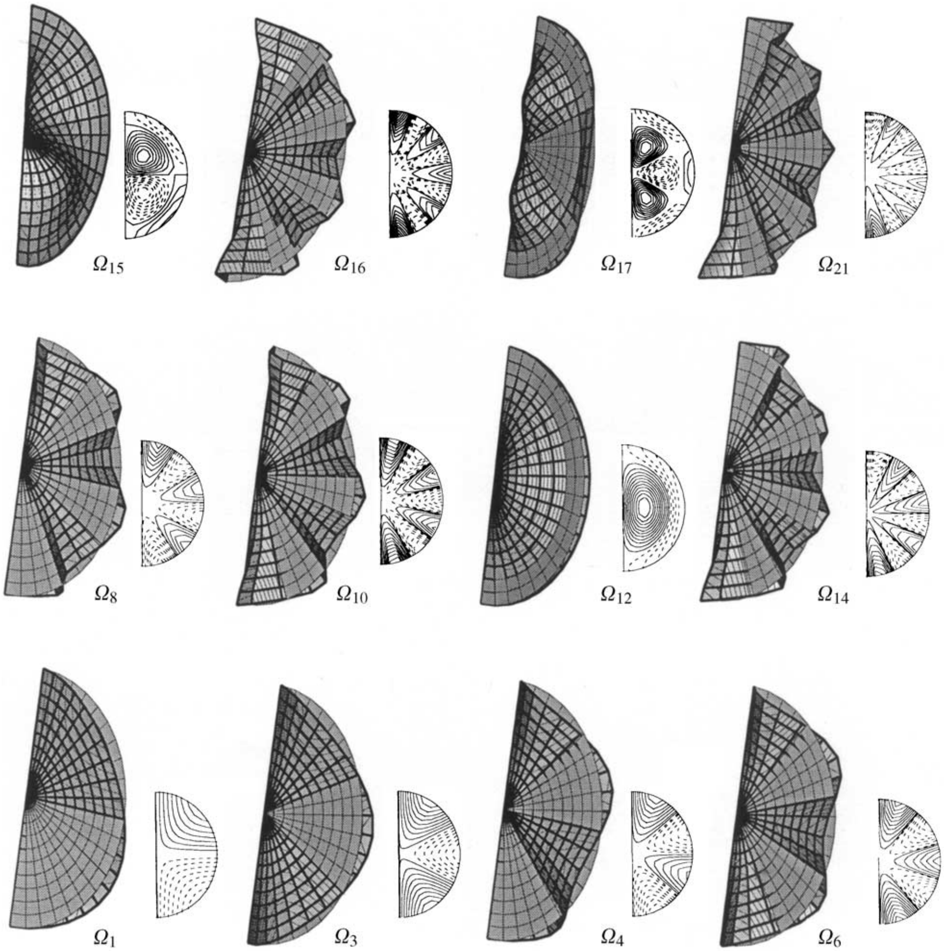

Figure 2. Toroidal mode shapes for solid spheres with free boundary conditions corresponding to material properties for thallium (see Table 1).

computed frequencies for fixed spheres with transversely isotropic properties using material properties that were postulated in an early paper by Cohen *et al.* [12] and those results are also compared with the finite element results in Table 2.

Frequencies are given in Table 3 for three materials with distinctly different values of β so that effects of material properties on frequency can be illustrated for spheres with completely fixed boundaries. The solid spheres of Table 3 have fixed displacements at the boundary of the sphere. The first torsional frequency is the same for all solid spheres with these boundary conditions. The results show a trend that as β decreases the first spheroidal frequency increases, but the second toroidal frequency tends to decrease.

4. CONCLUDING REMARKS

Frequencies and mode shapes for solid spheres with transversely isotropic material properties have been computed using a finite element that was formulated in spherical

TABLE 2

Frequency $\Omega = \omega a(\rho/C_{44})^{1/2}$ for solid spheres with completely fixed boundary conditions and $n = 0$

Mode	Isotropic $\nu = 0{\cdot}25$			Transversely isotropic†		
	Reference [9]	Sph.	Tor.	Reference [10]	Sph.	Tor.
1	3·990	4·072	4·494	3·489	3·570	4·494
2	5·775	5·777	5·765	6·565	6·569	7·739
3	6·203	6·296	6·992	6·881	7·043	7·917
4	7·293	7·301	7·739	8·068	8·084	10·526
5	7·736	7·747	8·194	9·319	9·359	10·992

† $C_{11} = 20{\cdot}0$, $C_{12} = 12{\cdot}0$, $C_{33} = 2{\cdot}0$, $C_{13} = 2{\cdot}0$, $C_{44} = 1{\cdot}0$, $C_{66} = 4{\cdot}0$.

TABLE 3

Frequency $\Omega = \omega a(\rho/C_{44})^{1/2}$ for solid transversely isotropic spheres with completely fixed boundary conditions and $n = 0$

Mode	Ω					
	Cadmium, $\beta = 1{\cdot}8897$		Isotropic, $\nu = 0{\cdot}3$		Thallium, $\beta = 0{\cdot}3719$	
	Sph.	Tor.	Sph.	Tor.	Sph.	Tor.
1	3·251	4·494	4·244	4·494	5·492	4·494
2	5·315	6·544	5·979	5·765	6·862	5·052
3	5·908	7·739	6·334	6·992	6·953	5·695
4	6·653	8·307	7·482	7·739	8·283	6·375
5	7·801	9·968	7·877	8·194	8·874	7·074

co-ordinates. Results in the literature have been extended to include a variety of hexagonal material properties. Selected mode shapes have been presented graphically and give an improved illustration for motion of the cross-section of the sphere during free vibration.

REFERENCES

1. H. LAMB 1882 *Proceedings London Mathematical Society* **13**, 189–212. On the vibrations of an elastic sphere.
2. Y. SATO and T. USAMI 1962 *Geophysical Magazine* **31**, 15–24. Basic study on the oscillation of a homogeneous elastic sphere I: frequency of free oscillations.
3. E. R. LAPWOOD and T. USAMI 1961 *Free Oscillations of the Earth*. Cambridge: Cambridge University Press.
4. P. R. HEYLIGER and A. JILANI 1992 *International Journal of Solids and Structures* **29**, 2689–2708. The free vibrations of inhomogeneous elastic cylinders and spheres.
5. K. T. CHAU 1998 *Journal of Applied Mechanics* **65**, 59–65. Toroidal vibrations of anisotropic spheres with spherical isotropy.
6. W. Q. CHEN and H. J. DING 2001 *International Journal of Mechanical Sciences* **43**, 667–680. Free vibration of multi-layered spherically isotropic hollow spheres.

7. G. R. BUCHANAN and B. S. RICH *Journal of Vibration and Control*, in press. Effect of boundary conditions on free vibration of thick spherical shells.
8. R. G. PAYTON 1983 *Elastic Wave Propagation in Transversely Isotropic Media*. The Hague: Martinus Nijhoff.
9. S. HOSSEINI-HASHEMI and J. S. ANDERSON 1988 *Journal of Sound and Vibration* **121**, 511–524. Orthogonality and normalization of torsional modes of vibration of solid elastic spheres.
10. P. J. SCHAFBUCH, F. J. RIZZO and R. B. THOMPSON 1992 *Journal of Applied Mechanics* **59**, 458–459. Eigenfrequencies of an elastic sphere with fixed boundary conditions.
11. W. Q. CHEN, J. B. CAI, G. R. YE and H. J. DING 2000 *Journal of Applied Mechanics* **67**, 422–424. On eigenfrequencies of an anisotropic sphere.
12. H. COHEN, A. H. SHAH and C. V. RAMAKRISHNAN 1972 *Acustica* **26**, 329–333. Free vibrations of a spherically isotropic hollow sphere.