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SENSITIVITY ANALYSIS AS A METHOD OF ABSORBER TUNING FOR REDUCTION OF STEADY STATE RESPONSE OF LINEAR PARAMETRIC SYSTEMS

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The aim of this paper is to determine whether one dynamic absorber can reduce the amplitude of the steady state vibration of a parametric system for natural and parametric resonance frequencies simultaneously. The efficiency of both the conventional dynamic absorber and the parametric absorber is analyzed. The first order sensitivity analysis of parametric periodic systems in the time domain is applied to obtain logarithmic sensitivity functions in the frequency domain. The first order sensitivity logarithmic functions are used to tune the conventional absorber and the parametric absorber.

1. INTRODUCTION

A linear dynamical system subjected to combined parametric and forcing excitations of a periodic nature is governed by a system of inhomogeneous differential equations with periodic coefficients. A steady state periodic response of such a system was analyzed by Hsu and Cheng [1] and Klasztorny and Wójcicki [2]. The amplitudes of the steady state response of the system become unlimited on the stability boundaries of a homogeneous system corresponding to the non-homogeneous one. There are three sets of boundary curves separating stability regions from regions of instability. The first set (an even order one when the greatest multiplier is equal to 1 and a T-periodic solution exists) is characterized by the fact that near its boundary curves the steady state response amplitude of the system is amplified (unlimited) only when the period of the forcing excitation is an odd multiple of the period of the parametric excitation. For the second set (an odd order one when the greatest multiplier is equal to -1 and a $2T$-periodic solution exists) the steady state response amplitude of the structure is amplified (unlimited) only when the period of the forcing excitation is an even multiple of the period of the parametric excitation [1, 2]. The third set of boundary curves (the absolute value of the greatest multiplier is equal to 1 separates stability regions from regions of combined parametric instability. The steady state response amplitude of the system is not amplified (is always limited) near these boundaries.

The problem of optimal tuning of the absorber for an undamped single-degree-of-freedom system subjected to harmonic forcing excitation was formulated and solved by Den Hartog [3]; see also Harris [4]. The absorber's stiffness and damping are the parameters which change during the optimization procedure.

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The most effective reduction in the steady state response of the parametric system near the resonance areas is a more complicated problem, especially as there are no analytic solutions. The main difficulty is that the dynamic absorber must be tuned for more than one parametric resonance frequency. Hence, it is difficult to introduce a suitable optimization method. Optimal tuning for one frequency (force to parametrical excitation) ratio cannot be (usually is not) optimal for another one. Moreover, unlike any other kind of vibration, the resonance response amplitude in a linear parametric system may increase boundlessly in spite of the damping in the primary system [5]. Thus, damping must be involved to eliminate the unstable regions [5]. It therefore also becomes necessary to investigate the instability regions according to the Floquet theory [6]. This stability analysis is necessary as it makes little sense to talk about a steady state response of the system if the system is unstable [1, 2, 6]. Therefore, the optimization method ought to contain an effective stabilization method, for instance, similar to the one proposed by Seyranian *et al.* [7]. Thus, the stability of the system has to be checked continuously and an objective optimization function is difficult to define. Moreover, many parameters have to be changed. The interactive participation of a designer in the tuning procedure seems to be a better solution. A decision about which parameter and how much it should be changed to obtain better efficiency of the absorber is quite simple if the sensitivity function values are known.

When a harmonic force is the only one to excite the primary undamped system, an auxiliary mass damper in the resonance zone (close to the natural frequency of the primary system) is effective [3, 8]. In the present paper, the efficiency of a vibration absorber, when parametric excitation appears beside the forcing excitation, is analyzed. Moreover, a case of parametric excitation under a constant load is also examined [9, 10].

It is assumed that parametric instability resonance does not occur but excitation is very close to an instability region. Then the steady state response amplitude values may be very high [1, 2]. The most important question to examine is whether one dynamic absorber can reduce the amplitude of vibration of the primary system for both natural and parametric resonance frequencies. From this point of view, the application of a dynamic absorber to reduce the vibration of a single-degree-of-freedom parametric system is similar to the problem of a multi-degree-of-freedom system with many resonant peaks [8].

A numerical verification of the above question is the aim of this paper. The first order logarithmic sensitivity functions [11–13] are used to tune the dynamic absorber. The numerical calculations are aided by the *Mathematica* computer software [14].

2. STABILITY AND SENSITIVITY ANALYSIS

2.1. MATRIX EQUATION OF MOTION

The matrix equation of motion has the form

$$\mathbf{B}(t)\,\ddot{\mathbf{q}} + \mathbf{C}(t)\,\dot{\mathbf{q}} + \mathbf{K}(t)\,\mathbf{q} = \mathbf{f}(t), \tag{1}$$

where overdots refer to differentiation with respect to time t; $\mathbf{q}(t)$ is an n-dimensional vector of the generalized co-ordinates; $\mathbf{f}(t)$ is an n-dimensional force vector periodic in t with period T_f; and $\mathbf{B}$, $\mathbf{C}$, $\mathbf{K}$ are square n-dimensional real matrices of inertia, damping and stiffness respectively. Matrices $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{K}(t)$ are also periodic in t but with period T_0. Let $T_f = T_0 m_0/m_f$, where m_0 and m_f are assumed to be positive integers and m_0/m_f is an irreducible rotational number. Thus, the forcing period of $\mathbf{f}(t)$ is commensurate with the period of coefficient matrices $\mathbf{B}$, $\mathbf{C}$, $\mathbf{K}$. Let T_c be a common period between parametric and forcing excitation so that

$$T_c = T_0 m_0 = T_f m_f. \tag{2}$$

The excitation angular frequency of the system is $\nu_c = 2\pi/T_c$. For simplicity, instead of "angular frequency" the term "frequency" will be used in further analysis.

2.2. STABILITY ANALYSIS

Provided that eigenvalues ρ_i of monodromy matrix $\mathbf{D}$ lie within a unit circle on a complex plane, the trivial solution of the parametric periodic linear systems is (asymptotically) stable in the Lyapunov sense [6, 7]. Alternatively, stability can be expressed in terms of characteristic system exponents $\lambda_i = \ln|\rho_i|/T_0$. In this case, stability requires that the highest Lyapunov's characteristic exponent λ of the system has a negative value [6, 7].

2.3. STEADY STATE RESPONSE OF LINEAR PARAMETRIC SYSTEM

2.3.1. *Combined parametric and forcing periodic excitation*

Let $\mathbf{R}(t)$ with $\mathbf{R}(0) = \mathbf{I}$ be the fundamental matrix for the solution of a homogeneous equation corresponding to equation (1). The steady state response, i.e., the periodic solution of non-homogeneous equation (1) is given by [1, 2]

$$\mathbf{r}_f(t) = \mathbf{r}_w(t) - \mathbf{R}(t)(\mathbf{I} - \mathbf{H})^{-1}\mathbf{b} = \mathbf{r}_f(t + T_c), \tag{3}$$

where

$$\mathbf{r}_w(t) = \mathbf{R}(t)\int_0^t \mathbf{R}^{-1}(s)\mathbf{g}(s)\,\mathrm{d}s \tag{4}$$

is the particular integral which satisfies the initial condition $\mathbf{q}(0) = \dot{\mathbf{q}}(0) = \mathbf{0}$; $\mathbf{r} = [\mathbf{q}, \dot{\mathbf{q}}]^{\mathrm{T}}$ and $\mathbf{g}(t) = [\mathbf{0}, \mathbf{f}]^{\mathrm{T}} = \mathbf{g}(t + T_f)$ are $2n$-vectors of state and forcing excitation respectively; $\mathbf{H} = \mathbf{D}^{m_0}$ where $\mathbf{D} = \mathbf{R}(T_0)$ is a monodromy matrix and $\mathbf{b} = \mathbf{r}_w(T_c)$.

The amplitudes of this steady state response become unlimited [1, 2] on the boundaries of the even or odd order periodic parametric instability regions of the homogeneous system corresponding to non-homogeneous equation (1). The only interesting case from a practical point of view is when the homogeneous system is asymptotically stable. Otherwise, the homogeneous system's response is unlimited and it does not make much sense to talk about a steady state response, even though the mathematical expression (3) for $\mathbf{r}_f(t)$ is still valid [1, 2]. Therefore, the stability of zero solution $\mathbf{q} \equiv \mathbf{0}$ of the homogeneous equation corresponding to equation (1) must be investigated initially.

2.3.2. *Parametric excitation under constant load*

All the theoretical nuances of the theory which is contained in this section are not easy to explain simply in this paper. Proofs of the statements and a detailed analysis would take many pages. Since it may be difficult to follow all aspects of the theory, the author will try to explain all the details in a separate paper in the future.

The algorithm presented in section 2.3.1 allows the steady state response of a dynamic system to be determined under combined parametric and forcing periodic excitation [1, 2]. It is known, however, that when a parametric periodic system is under a constant load, an additional family of parametric vibrations occurs [5, 9, 10]. The simple modification of the algorithm presented in section 2.3.1 ought to lead to useful formulae describing this additional family of parametric vibrations. This can be done through a simple change in equation (4). Since every constant function can be treated as a periodic function, with a period that can be arbitrarily assumed, it is sufficient to substitute constant functions (constant loads) for the periodic ones in vector $\mathbf{g}(t)$ and then to calculate $\mathbf{r}_w(t)$ according to equation (4). There are now two cases possible: (i) the period of the constant function is assumed to be an odd multiple of the parametric excitation period (m_0 is an odd number), and (ii) the period of the constant function is assumed to be an even multiple of the parametric excitation period (m_0 is an even number). In order not to complicate the problem, it is enough to consider the cases: $m_0 = 1$ and 2.

(I) For $m_0 = 1 - T_c = T_0$, $\mathbf{H} = \mathbf{D}^{m_0} = \mathbf{D}$ where $\mathbf{D} = \mathbf{R}(T_0)$ is a monodromy matrix and $\mathbf{b} = \mathbf{r}_w(T_0)$. The amplitude of steady state periodic response (3) may become unbounded if factor $(\mathbf{I} - \mathbf{D})^{-1}\mathbf{b}$ is unbounded, i.e., when matrix $(\mathbf{I} - \mathbf{D})$ is singular (a necessary condition). This happens if at least one multiplier ρ_i of monodromy matrix $\mathbf{D}$ is equal to 1. The multiplier is equal to 1 on the boundary of the even order periodic parametric instability regions. On this boundary a T_0-periodic solution of the homogeneous system exists, but then, according to equation (4), vector $\mathbf{b} = \mathbf{r}_w(T_0) \neq \mathbf{0}$. Finally, the amplitude of steady state response (3) becomes unbounded because factor $(\mathbf{I} - \mathbf{D})^{-1}\mathbf{b}$ may be unbounded when vector $\mathbf{b} \neq \mathbf{0}$ and matrix $(\mathbf{I} - \mathbf{D})$ is singular.

(II) For $m_0 = 2 - T_c = 2T_0$, $\mathbf{H} = \mathbf{D}^{m_0} = \mathbf{D}^2$ and $\mathbf{b} = \mathbf{r}_w(2T_0)$. The amplitude of steady state response (3) becomes unbounded if factor $(\mathbf{I} - \mathbf{D}^2)^{-1}\mathbf{b}$ is unbounded. This factor may become unbounded if matrix $(\mathbf{I} - \mathbf{D}^2)$ is singular (a necessary condition) and simultaneously vector $\mathbf{b} \neq \mathbf{0}$. There are two cases in which matrix $(\mathbf{I} - \mathbf{D}^2)$ is singular:
   (1) at least one multiplier ρ_i of monodromy matrix $\mathbf{D}$ is equal to 1 (a point on the boundary of the even order periodic parametric instability regions for which a T_0-periodic solution exists), but vector $\mathbf{b} = \mathbf{r}_w(2T_0) \neq 0$ and then the amplitude of steady state solution (3) may become unbounded;
   (2) at least one multiplier ρ_i of monodromy matrix $\mathbf{D}$ is equal to -1 (a point on the boundary of the odd order periodic parametric instability regions for which a $2T_0$-periodic solution exists), but then vector $\mathbf{b} = \mathbf{r}_w(2T_0) = \mathbf{0}$ (see equation (4)) and the amplitude of steady state solution (3) must stay limited.

A general conclusion can now be formed. If the parametric periodic system is under a constant load, an additional family of parametric vibrations appears. The amplitudes of the steady state vibrations can become theoretically unbounded on the boundaries of the parametric instability regions but only on the even order ones. This means that when the parametric excitation frequency is equal to the doubled natural vibration frequency, the main parametric resonance (the first odd order one) does not occur under a constant load. Basic resonance under a constant load can appear when the parametric excitation frequency is equal to the frequency which corresponds to the boundary of the first even order parametric periodic instability region. This boundary is close to the natural frequency of the system [9, 10]. All the parametric even order instability regions (at slight damping) are close to the points (in the frequency domain) which are multiples of the natural frequency of the system (see, for instance, the stability chart for Mathieu's equation [4]). Thus, the statement that the resonances of the parametric periodic system under a constant load occur near these points seems to be reasonable. Actually, resonance occurs either on the left boundary or the right boundary, or on both boundaries of the parametric instability regions which are near the natural frequency of the system. The amplitudes of these resonance vibrations can be theoretically unbounded despite the damping in the system. This phenomenon is not observed in constant parameter systems with harmonic forcing excitations, which proves that the above resonance vibrations are of a parametric nature. Only if damping is large enough to limit the parametric periodic even order instability regions, the vibrations have a character analogous to the resonance vibrations of harmonically excited constant parameter systems. Then the parametric resonance vibrations are observed (if they occur) close to the integer multiple of the natural frequency of the system. Damping, however, can effectively reduce the amplification ratio of the resonance vibrations. The amplitudes of the responses are then proportional to the values of the loading force (see also reference [9]).

2.4. FIRST-ORDER SENSITIVITY FUNCTIONS

It is now assumed that each component of matrices **B**, **C**, **K** depends on the parameter vector

$$\mathbf{a} = [a_1, a_2, \ldots, a_j]^{\mathrm{T}}. \tag{5}$$

The parameter vector, after variation called the actual parameter vector, is $\mathbf{a} = \mathbf{a}_0 + \Delta\mathbf{a}$ where $\Delta\mathbf{a}$ is a small parameter variation and $\mathbf{a}_0$ stands for the nominal parameter vector. Solution (3) of equation (1) depends on time t and parameter vector $\mathbf{a}$, i.e., $\mathbf{q} = \mathbf{q}(t, \mathbf{a})$. Let a be any element of vector $\mathbf{a}$. The equation of sensitivity for this element can be written in the same form as equation (1), i.e.,

$$\mathbf{B}(t)\,\ddot{\mathbf{s}}_a(t) + \mathbf{C}(t)\dot{\mathbf{s}}_a(t) + \mathbf{K}(t)\mathbf{s}_{\mathbf{a}}(t) = \mathbf{p}(t), \tag{6}$$

where

$$\mathbf{s}_a(t) = \left.\frac{\partial \mathbf{q}(t, \mathbf{a})}{\partial a}\right|_{\mathbf{a}_0} \tag{7}$$

is a vector of the sensitivity functions in the time domain.

Subscript "$\mathbf{a}_0$" in equation (7) indicates that the partial derivative is evaluated at the nominal parameter value. For simplicity, this evaluation indication will be omitted in further analysis, the evaluations thus always being implicitly understood.

On the right side of equation (6) there is a vector which can be defined as

$$\mathbf{p}(t) = \frac{\partial \mathbf{f}(t)}{\partial a} - \left(\frac{\partial \mathbf{B}(t)}{\partial a}\ddot{\mathbf{q}} + \frac{\partial \mathbf{C}(t)}{\partial a}\dot{\mathbf{q}} + \frac{\partial \mathbf{K}(t)}{\partial a}\mathbf{q}\right). \tag{8}$$

Each element of vector $\mathbf{p}(t)$ is a periodic (but non-harmonic) function of time with period T_c, since steady state response $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$, $\ddot{\mathbf{q}}(t)$ of equation (1) and matrices $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{K}(t)$ are periodic in t with period T_c . The solution of equation (6) is given by the same algorithm (formulae (3) and (4)) as that for equation (1). The only difference is that vector $\mathbf{p}(t)$ must be substituted for vector $\mathbf{f}(t)$. Unfortunately, functions (7) do not depend on frequency v (they do so only implicitly), but on time t. To obtain the logarithmic sensitivity functions in the frequency domain, time must be eliminated. This can be done by substituting the "amplitude" of function $q(t)$ for this function in equation (7) [13, 15, 16]. The first order absolute dimensional sensitivity function in the frequency domain of general co-ordinate $q(t)$ can be calculated from the following formula:

$$s_a = s_a(t_q) = \left.\frac{\partial q(t, \mathbf{a})}{\partial a}\right|_{\mathbf{a}_0, t_q}, \tag{9}$$

where $t_q \in \langle 0, T_c \rangle$ denotes the time for which function $q(t)$ has the greatest absolute value q_{max} (in period T_c), where

$$q_{max} = \max_t (|q(t, \mathbf{a})|\Big|_{\mathbf{a}_0} = |q(t_q, \mathbf{a}_0)|. \tag{10}$$

Finally, the first order non-dimensional relative (logarithmic) sensitivity function in the frequency domain [9, 12, 13] has the form

$$w_a = \frac{\partial \ln q(t, \mathbf{a})}{\partial \ln a}\Bigg|_{\mathbf{a}_0, t_q} = \frac{\partial q(t, \mathbf{a})/q(t, \mathbf{a})}{\partial a/a}\Bigg|_{\mathbf{a}_0, t_q} = \frac{a}{q(t, \mathbf{a})}\frac{\partial q(t, \mathbf{a})}{\partial a}\Bigg|_{\mathbf{a}_0, t_q} = \frac{a}{q_{max}} s_a. \tag{11}$$

Sensitivity functions (7), (9), and (11) are calculated according to the algorithm described in this section. The basis of the calculation is the numerical integration of equations (1) and (6) in agreement with relations (3) and (4). This numerical integration must be done merely over one period T_c.

Theoretically, there is a slight risk of the jumping of the logarithmic sensitivity function values in equation (11). This can occur when the parametric periodic response has almost the same absolute values of maximum positive and maximum negative total deflection from the equilibrium state. Because of the numerical integration (discretization in time) of equation of motion (1) the values of the maxima are only calculated approximately and can be exchanged (theoretically). The positive and negative maximum deflections occur at different time points, thus the sensitivity functions are also calculated at different time points. The jumping of the sensitivity functions may also occur then. As regards deflection, this situation may arise sporadically for two reasons. Firstly, the accuracy of deflection calculations is higher than that of velocity and acceleration calculations. Secondly, a slight difference between the positive and negative maxima for steady state response parametric systems occurs very rarely, the more so as the static load involves the asymmetry of the total (a sum of the static and dynamic displacement) periodic steady state displacement time functions. Therefore, the jumping of the logarithmic sensitivity functions did not occur during the calculations.

3. EFFICIENCY ANALYSIS OF VIBRATION ABSORBER

3.1. MODEL OF SYSTEM

The viscously damped two-degree-of-freedom parametric system is shown in Figure 1.

Auxiliary mass m is coupled by spring k and viscous damper c to primary system M, which is coupled by spring K and viscous damper C to a ground base (the lower lined block in Figure 1). The primary system is excited by force $f(t)$, which is harmonically changing in time with period T_f, and simultaneously by parametric excitation — the characteristic of spring $K(t)$ changes periodically in time with period T_0. Viscous damper characteristic c can also change periodically in time with period T_0. Thus,

$$K(t) = K_0 + K_1 \cos(\nu_0 t),$$

$$c(t) = c + c_1 \cos(\nu_0 t), \qquad f(t) = f_0 + f_1 \cos(\nu_f t). \tag{12}$$

The elements of matrix equation of motion (1) and equation of sensitivity (6) have the form

$$\mathbf{B} = \begin{bmatrix} m & 0 \\ 0 & M \end{bmatrix}, \tag{13}$$

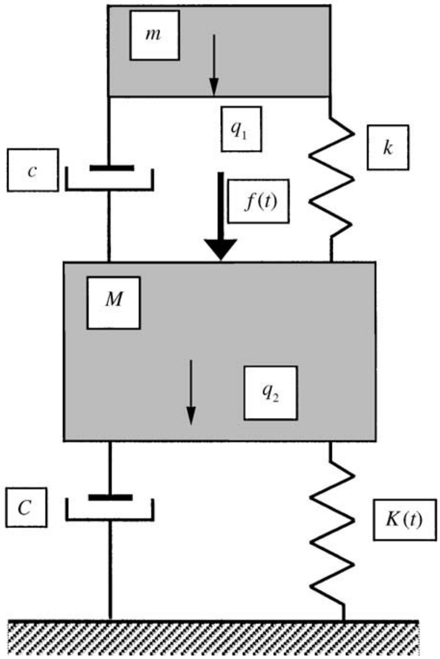

Figure 1. Schematic diagram of auxiliary mass m coupled by spring k and viscous damper c to primary parametric system M which is coupled by spring K and viscous damper C to ground base (lower lined block). Primary system is excited simultaneously by force $F(t)$ and parametric excitation $K(t)$.

$$\mathbf{C}(t) = \begin{bmatrix} c & -c \\ -c & c+C \end{bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} \cos(\nu_0 t), \tag{14}$$

$$\mathbf{K}(t) = \begin{bmatrix} k & -k \\ -k & k+K_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix} \cos(\nu_0 t), \tag{15}$$

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \end{bmatrix} + \begin{bmatrix} 0 \\ f_1 \end{bmatrix} \cos(\nu_f t), \tag{16}$$

where according to relation (2)

$$\nu_c = \frac{\nu_0}{m_0} = \frac{\nu_f}{m_f}, \quad m_0, m_f = 1, 2, \ldots . \tag{17}$$

In order not to complicate the algebraic analysis which is to follow, the most general case will not be considered and attention will be focussed on the cases where $m_f = 1$.

3.2. UNDAMPED NON-PARAMETRIC SYSTEM

Nominal parameter values are assigned as follows: $M = 3 \times 10^4$ kg, $K_0 = 4 \times 10^8$ N/m, $C = 0$, $f_0 = 0$, $f_1 = 1$ N, $K_1 = 0$, $c_1 = 0$. The components of the parameter vector are $\mathbf{a} = [c, k]^{\mathrm{T}}$. The natural frequency of the primary system is $\Omega = \sqrt{K_0/M} = 115{\cdot}5$ rad/s. Assuming mass ratio $\mu = m/M = 1/20$, the mass parameter of the vibrating absorber is $m = 1500$ kg. The problem of the optimal tuning frequency and damping absorber

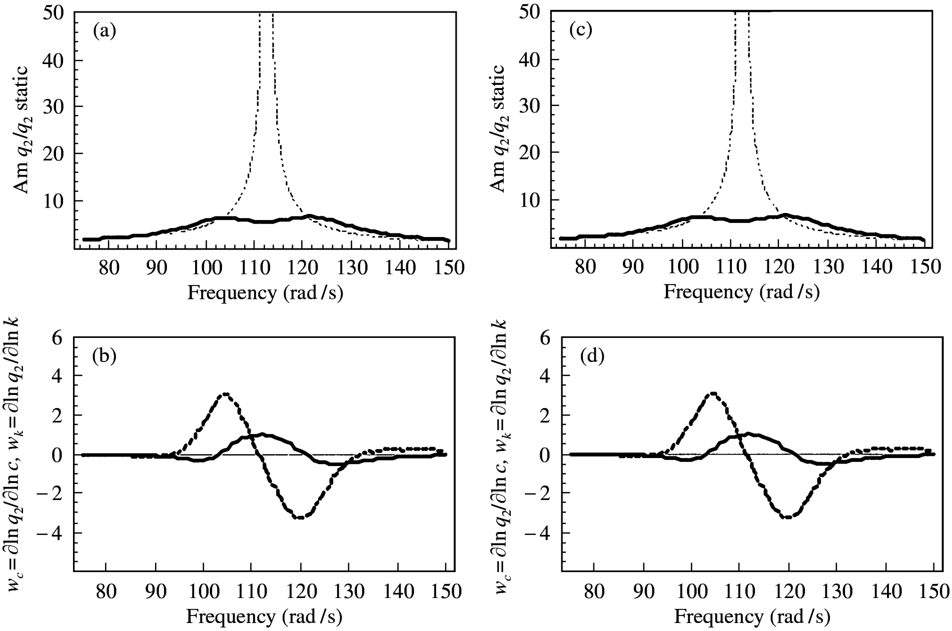

Figure 2. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for undamped non-parametric system with optimally tuned classical absorber (identical left and right columns): (a) …, $c=\infty$; $\alpha=\alpha_{opt}$; $C=0$; ——, $K_1=0$; $c_1=0$; $f=f_0+f_1(t)$; (b) · · ·, w_c; ——, w_k; (c) …, $c=\infty$; $\alpha=\alpha_{opt}$; $C=0$; ——, $K_1=0$; $c_1=0$; $f=f_0+f_1(t)$; (d) · · ·, w_c; ——, w_k.

parameters for an undamped single-degree-of-freedom system subjected to harmonic forcing excitation was solved and formulated by Den Hartog [3]. According to references [3, 15], optimum frequency ratio $\alpha=\omega/\Omega$ ($\omega=\sqrt{k/m}$ is the natural frequency of the absorber) and optimum damping c of the absorber are given by

$$\alpha_{opt}=\frac{1}{1+\mu}\approx 0{\cdot}95, \qquad c=\sqrt{\frac{K_0 M\mu^3}{2(1+\mu)^3}}. \tag{18}$$

Thus, parameter k can be calculated from formula $k=\mu\alpha^2K_0$ and the natural frequency from formula $\omega=\sqrt{k/m}=112{\cdot}7$ rad/s. The efficiency of the auxiliary mass damper with optimum tuning is shown in Figure 2.

The amplitude–frequency characteristic of displacement q_2 of primary system mass M as a function of v_f is shown in Figure 2(a, c) and at the top of all the next composite figures. The logarithmic sensitivity functions of displacement q_2 are shown in the lower figures. Functions w_c and w_k (dashed line) are first order logarithmic sensitivity functions with respect to the appropriate design variable c or k as can be seen in Figure 2(b, d) and at the bottom of all the following composite figures. All the left-hand figures are for $m_0=1$ and all the right-hand ones for $m_0=2$. The amplitude–frequency characteristics, when the characteristic of damper constrain $c=\infty$, are shown in the background of the upper figures (dashed line).

3.3. UNDAMPED PARAMETRIC SYSTEM

The only change involving parametric excitation parameter $K_1=0{\cdot}5\,K_0$ is now introduced. The amplitude–frequency characteristics of q_2 and first order logarithmic

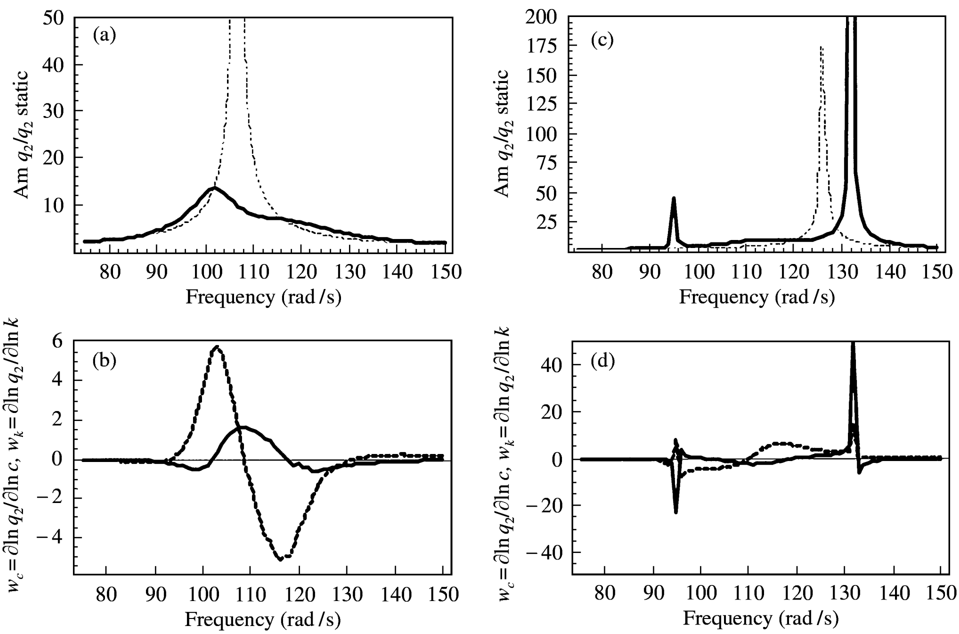

Figure 3. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for undamped parametric system with classical absorber. Left column for parametric excitation frequency equal to force frequency, right column for parametric excitation frequency twice as high as force frequency: (a) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 0$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 1$; (b) · · ·, w_c; ——, w_k; (c) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 0$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 2$; (d) · · ·, w_c; ——, w_k.

sensitivity w_c and w_k are shown in Figure 3. The left-hand figures are for $m_0 = 1$ and the right-hand ones for $m_0 = 2$.

When parametric excitation occurs, damping and the frequency ratio of the absorber for $m_o = 1$ are not optimum. The amplitude of the response close to the natural frequency of the primary system increases almost 3 times in comparison with the one shown in Figure 2. The sensitivity functions are useful when carrying out the tuning procedure. Sensitivity functions show the influence of damping parameter c and stiffness parameter k on the amplitude–frequency characteristic. For example, the results presented in Figure 3 lead to the following conclusions: the stiffness coefficient k is the only important parameter, changing the value of the primary mass M amplitude. A decrease in the value of the stiffness parameter k leads to a decrease in the amplitude of the vibration close to the frequency of 100 rad/s (the positive value of the sensitivity function in this area) but at the same time it leads to an increase in the amplitude of the vibration close to the frequency of 120 rad/s (the negative value of the sensitivity function in this area). By changing parameter c appropriately the optimum tuning of the absorber is obtained. The optimum frequency ratio is $\alpha = 0{\cdot}92 < \alpha_{opt} = 1/(1 + \mu) \approx 0{\cdot}95$. The results are shown in Figure 4.

Additionally, in Figure 4(a, b) it can be seen that for $m_0 = 1$ optimum tuning occurs where the displacement function has its maximum and at the same time the first order logarithmic sensitivity function with respect to c has zero value. This means that a change of the damping value cannot decrease the response of the system. The damper of the parametric system for $m_0 = 1$ seems to be well founded. The amplitude value of the response of the

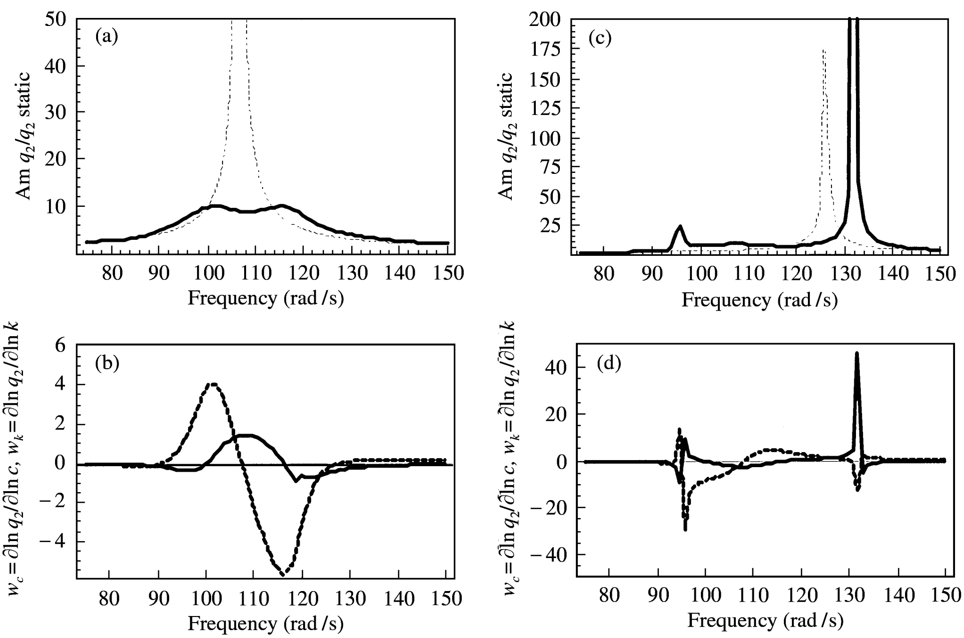

Figure 4. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for undamped parametric system with classical absorber optimally tuned for parametric excitation frequency equal to force frequency. Left column for parametric excitation frequency equal to force frequency, right column for parametric excitation frequency twice as high as force frequency: (a) …, $c = \infty$; $\alpha = \alpha_{opt}$; $C = 0$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = \nu_f/\nu_0 = 1$; (b) · · ·, w_c; ——, w_k; (c) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 0$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = \nu_f/\nu_0 = 2$; (d) · · ·, w_c; ——, w_k.

primary system only approximately doubles (the effect of the parametric excitation) in comparison with the one shown in Figure 2(a) and decreases about one-third in comparison with that shown in Figure 4(a).

For $m_0 = 2$ the situation is substantially different. High peaks are observed close to the frequencies of $\nu_f = 95$ rad/s ($\nu_c = 2\nu_f = 170$ rad/s) and $\nu_f = 135$ rad/s ($\nu_c = 2\nu_f = 270$ rad/s). Theoretically, there is only one possible explanation of such resonance amplifications. An odd order instability region must occur for a frequency of 95–135 rad/s. And then

- if the frequency of the parametric excitation is an odd multiple: $m_0 = 1, 3, 5, \ldots$ of the forcing excitation (for instance they are equal), the amplitude of the steady state response of the system should not be increased (Figures 3(a) and 4(a));
- if the frequency of the parametric excitation is an even multiple: $m_0 = 2, 4, 6, \ldots$ of the frequency of the forcing excitation (for instance twice as higher), the amplitude of the steady-state response of the system should be increased (theoretically unlimitedly) near the boundary curves separating the stability regions from the instability ones (Figures 3(c) and 4(c)).

According to section 2.2, the stability can be expressed in terms of characteristic system exponents $\lambda_i = \ln|\rho_i|/T_0$, where T_0 is the period of the parametric excitation. The instability requires the highest Lyapunov's characteristic exponent λ to have a positive value [2, 6]. Lyapunov's characteristic exponent λ as a function of the forcing excitation frequency (not the parametric excitation frequency) is shown in Figure 5.

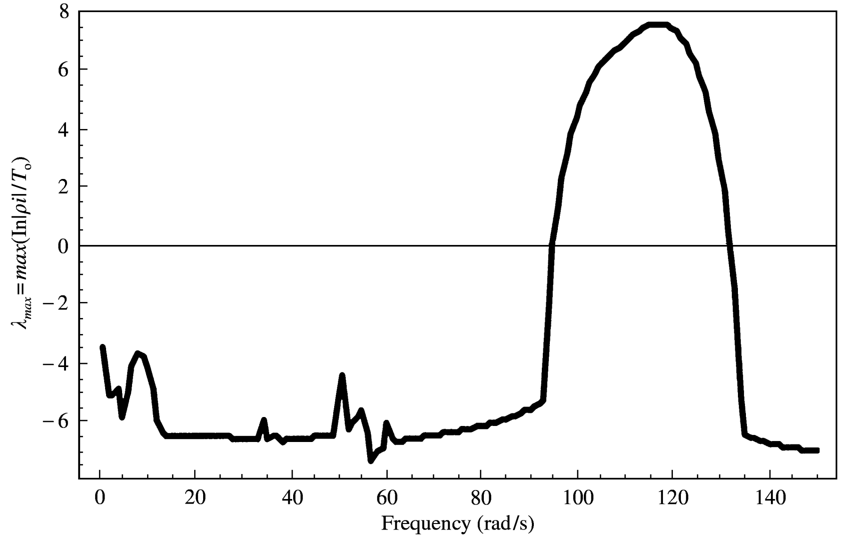

Figure 5. Lyapunov's characteristic exponent function for undamped parametric system (positive values of function mean that system is unstable).

The even order instability region in the frequency range $\nu_f = 95\text{–}135$ rad/s ($\nu_0 =$ 190–270 rad/s) indeed occurs. Moreover, the natural frequency of the primary system, which is 115·5 rad/s, is within this parametric instability region. Consequently, the parametric resonance and the forcing resonance occur simultaneously. This is the most dangerous resonance combination for this system. The best way to stabilize the system could be to apply the stabilization method presented in reference [7]. Another way to stabilize the system is to introduce damping for mass M of the primary parametric system (Figure 1).

3.4. DAMPED PARAMETRIC SYSTEM

Value C of the characteristic damping constraint must be high enough to eliminate the unstable regions. This happens when damping characteristic $C = 4{\cdot}4 \times 10^5$ N s/m is assumed (it amounts to about $10c$). It can be seen (Figure 6) that Lyapunov's characteristic exponent function for this value of the damping coefficient has only negative values. This means that the parametric system is stable.

Analogous amplitude–frequency characteristics of q_2 and the first order logarithmic sensitivity functions for $m_0 = 1$ and 2 are shown in Figure 7.

Although the regions of the parametric instability are reduced slightly, the boundary curves are very close for $\nu_f = 100$ rad/s ($\nu_0 = 200$ rad/s) and $\nu_f = 122{\cdot}5$ rad/s ($\nu_0 = 245$ rad/s) (Figure 6). The amplitudes of the steady state response of the system for $m_0 = 2$ are still very large (Figure 7(c)), and the tuning of the absorber is not optimal. Since the amplitude of the response for $m_0 = 2$ (Figure 7(c)) is larger than that of the response for $m_0 = 1$ (Figure 7(a)), the tuning procedure was carried out for $m_0 = 2$ as a more important event. In Figure 7(c), the amplitude is much greater for $\nu_f = 122{\cdot}5$ rad/s than for $\nu_f = 100$ rad/s. Sensitivity function w_k has negative values close to the higher peak and positive values close to the lower one (Figure 7(d)). This means that an increase in the

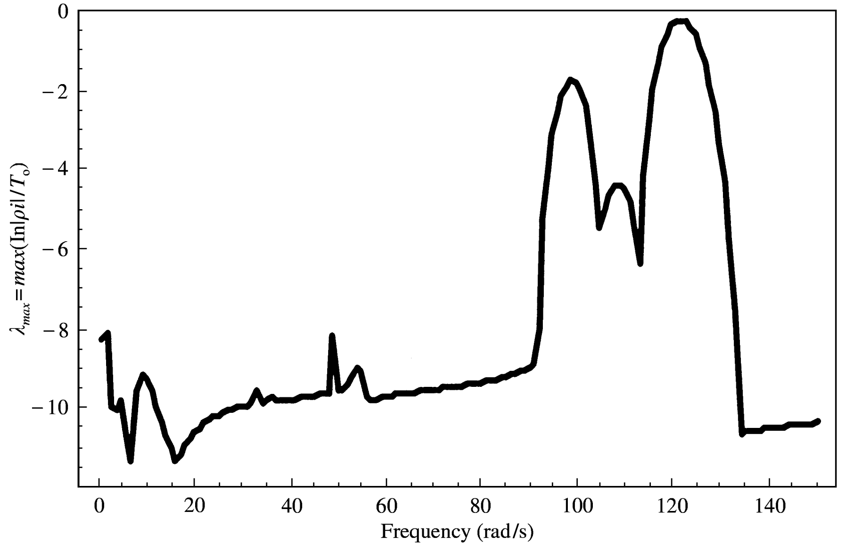

Figure 6. Lyapunov's characteristic exponent function for damped parametric system (negative values of function mean that system is stable).

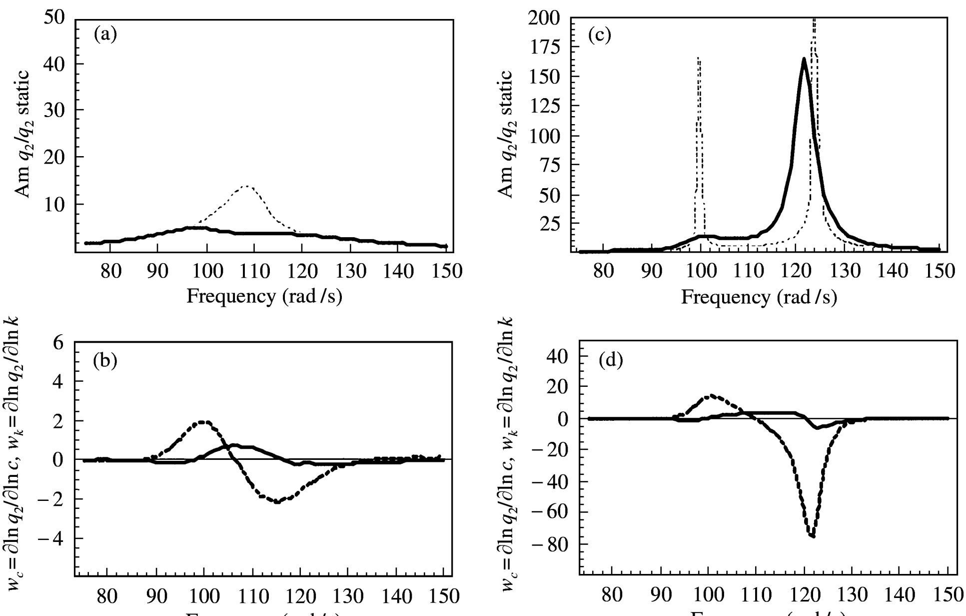

Figure 7. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for damped parametric system with classical absorber. Left column for parametric excitation frequency equal to force frequency, right column for parametric excitation frequency twice as high as force frequency: (a) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 1$; (b) $\cdot\cdot\cdot$, w_c; ——, w_k; (c) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 2$; (d) $\cdot\cdot\cdot$, w_c; ——, w_k.

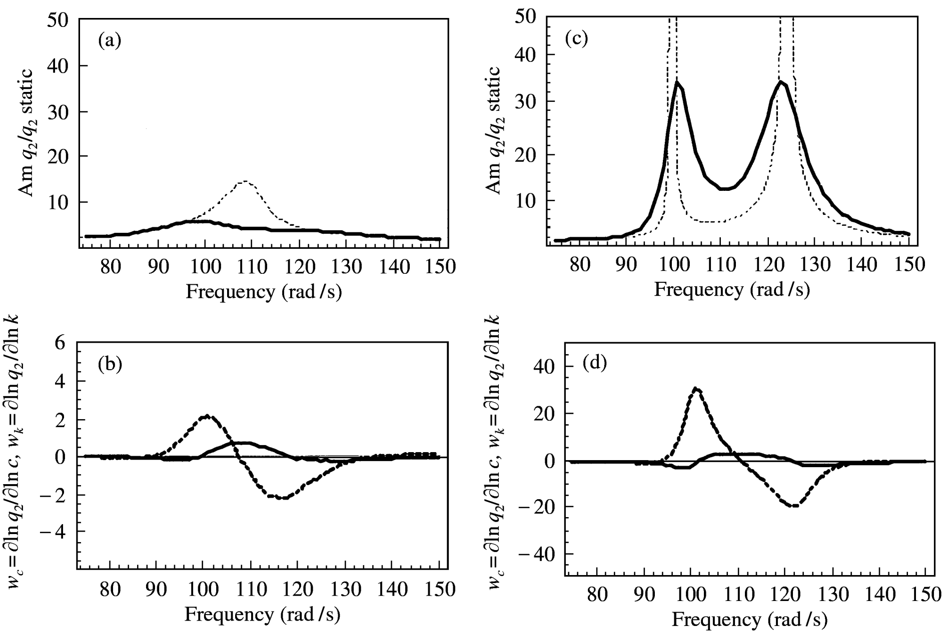

Figure 8. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for damped parametric system with classical absorber optimally tuned for parametric excitation frequency which is twice as high as force frequency. Left column for parametric excitation frequency equal to force frequency, right column for parametric excitation frequency twice as high as force frequency: (a) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 1$; (b) · · ·, w_c; ——, w_k; (c) …, $c = \infty$; $\alpha = \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = 0$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 2$; (d) · · ·, w_c; ——, w_k.

stiffness k of the absorber should decrease the higher peak and increase the lower one. Therefore, the value of the stiffness k of the absorber spring must be increased, and consequently the frequency ratio α of the absorber must be increased too. Optimum tuning can be achieved for absorber stiffness $k = 1{\cdot}77 \times 10^7$ N/m ($\alpha = 0{\cdot}9407$). The results are shown in Figure 8.

The parameters of the absorber are tuned optimally in such a way as to obtain a minimum response near the parametric resonance zone. The peaks in Figure 8(c) for $m_0 = 2$ have the same height when $v_f = 100$ rad/s ($v_0 = 200$ rad/s) and $v_f = 122{\cdot}5$ rad/s ($v_0 = 245$ rad/s), which proves that the tuning really is optimal. This is additionally confirmed by the logarithmic sensitivity functions. The amplitude–frequency characteristic of displacement function q_2 has maximum values at the same points at which the first order logarithmic sensitivity function with respect to c has zero value (Figure 8(d)).

For the other values of the excitation frequency the results of applying the absorber are also significant. For $m_0 = 1$ the response of the system for which the force excitation is close to the natural frequency of the system (about 115·5 rad/s) is still reduced by more than 50% (Figure 8(a)).

The tuning capabilities of the conventional absorber cannot be improved further. This means that changes of the absorber parameters' values will not decrease the response of the parametric system. Therefore, it seems sensible to introduce another kind of absorber such as a parametric one.

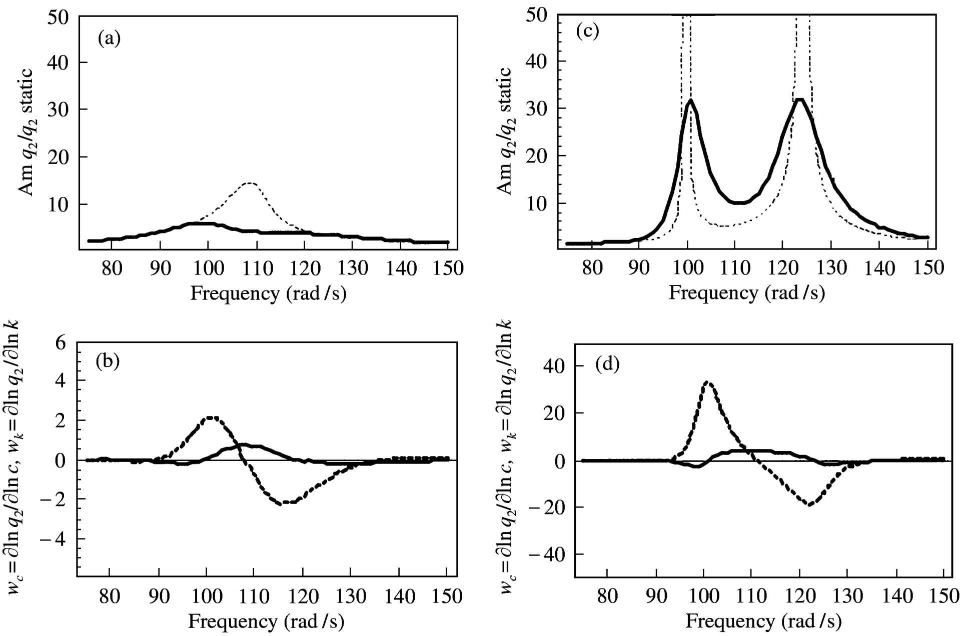

Figure 9. Normalized amplitude–frequency characteristics of primary mass and logarithmic sensitivity functions of displacement of primary mass w_c (dashed line) and w_k (continuous line) with respect to parameter c or k for damped parametric system with parametric absorber optimally tuned for parametric excitation frequency which is twice as high as force frequency. Left column for parametric excitation frequency equal to force frequency, right column for parametric excitation frequency twice as high as force frequency: (a) …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = c/2$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 1$; (b) · · ·, w_c; ——, w_k; (c) …, $c = \infty$; $\alpha = \alpha_{opt}$; $C = 10c$; $K_1 = K_0/2$; ——, $c_1 = c/2$; $f = f_0 + f_1(t)$; $m_0 = v_f/v_0 = 2$; (d) · · ·, w_c; ——, w_k.

3.5. PARAMETRIC ABSORBER

In this case, periodically changing characteristic $c(t)$ of the absorber is involved and the latter becomes parametric [16]. The amplitude–frequency characteristics of q_2 and the first order logarithmic sensitivity functions for the parametric damper are shown in Figure 9. It is assumed that $K_1 = 0{\cdot}5\,K_0$, $k = 1{\cdot}77 \times 10^7$ N/m, $c_1 = c/2$.

The tuning remains optimal at $m_0 = 2$. When the parametric absorber is used, the maximum response of the primary system (max. amplitude q_2 of mass M) (Figure 9(c)) is over 10% lower than that obtained for the conventional absorber (Figure 8(c)). For $m_0 = 1$ the steady state response of the system is still very low (Figure 8(a)) in comparison with that obtained for $m_0 = 2$ (Figure 8(c)). The system is still stable, even though the arrangement of the instability regions changes slightly (Figure 10).

3.6. PARAMETRIC EXCITATION UNDER CONSTANT LOAD

An efficiency analysis of the classical absorber and the parametric absorber for parametric excitation under a constant load is also carried out. Two cases are analyzed: (1) damping is low and parametric instability regions may occur (see section 3.3.—undamped parametric system) (Figure 11(a)); and (2) damping is high enough to eliminate parametric instability regions (see section 3.5—parametric absorber) (Figure 11(b)), but this is close to

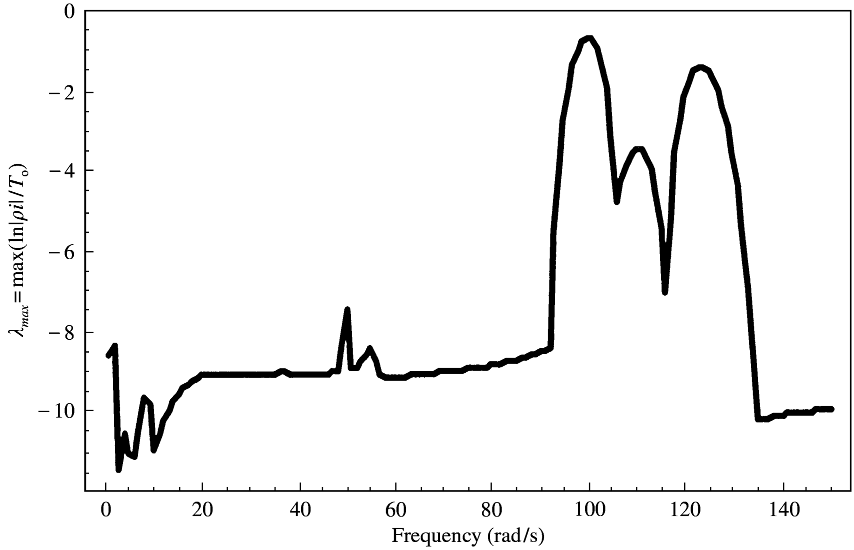

Figure 10. Lyapunov's characteristic exponent function for damped parametric system with optimally tuned parametric absorber (positive values of function mean that system is unstable).

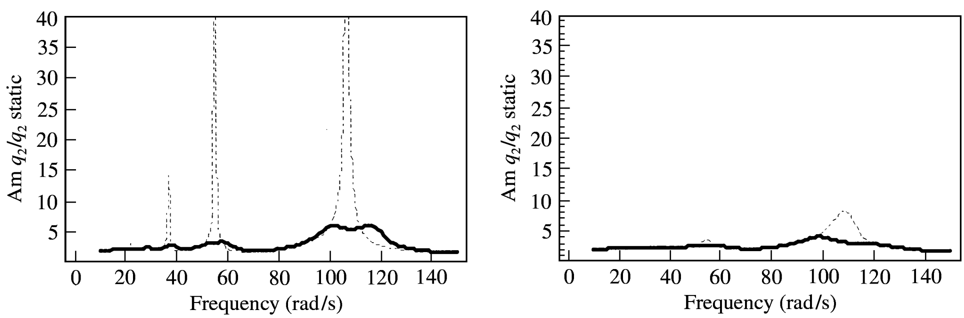

Figure 11. Normalized amplitude–frequency characteristics of primary mass for undamped: ..., $c = \infty$; $\alpha = \alpha_{opt}$; $C = 0$; ——, $K_1 = K_0/2$; $c_1 = c/2$; $f = f_0 = const.$; (a) and damped: ..., $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; ——, $K_1 = K_0/2$; $c_1 = c/2$; $f = f_0 = const.$; (b) parametrically excited system under constant load.

the boundaries of the regions. The efficiency of the absorbers in reducing the steady state response of linear parametric systems under a constant load is quite good.

3.7. ESTIMATION OF PARAMETRIC ABSORBER EFFICIENCY

The efficiency of the parametric absorber in some cases of excitation of the damped primary system can be estimated on the basis of the results presented in Figure 12. In all cases the system remains the same and only the excitations change. The dashed line in the background refers to a situation when the parametric absorber is not working (absorber damping characteristic $c = \infty$). The bold line represents the amplitude–frequency characteristic in the case when the parametric absorber is working.

The amplitude–frequency characteristics for the harmonic forcing excitation of a non-parametric system are shown in Figure 12(a). The tuning of the absorber is not

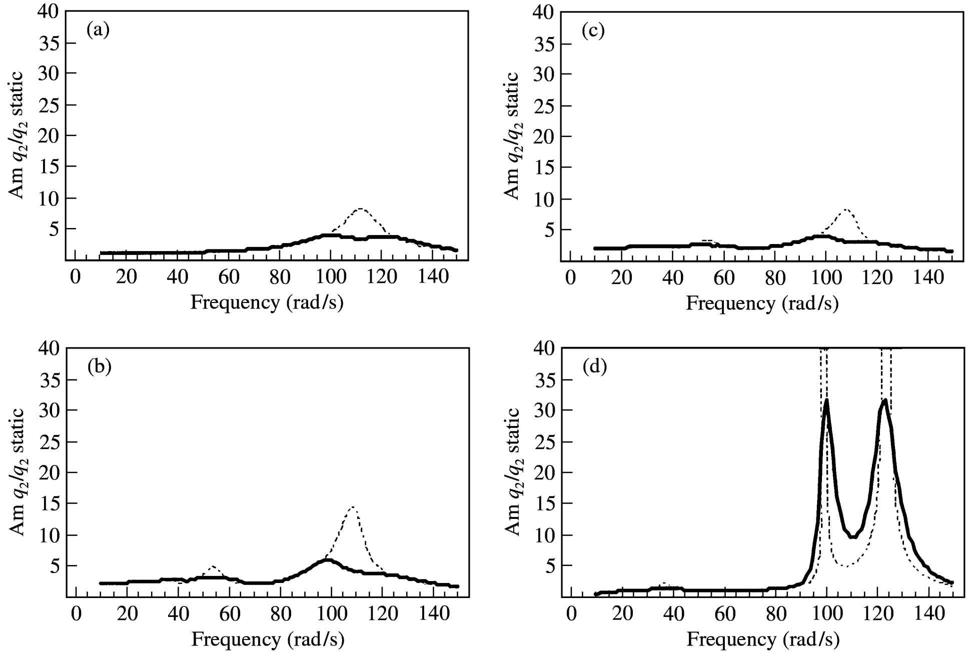

Figure 12. Efficiency of parametric absorber in some cases of excitation of damped primary system: (a) harmonic forcing excitation of non-parametric system: …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; ——, $K_1 = 0$; $c_1 = 0$; $f = f_0 + f_1(t)$; (b) parametrically excited system under constant load: …, $c = \infty$; $\alpha \neq \alpha_{opt}$; $C = 10c$; ——, $K_1 = K_0/2$; $c_1 = c/2$; $m_0 = v_f/v_0 = 1$; (c) harmonic forcing excitation of parametric system when parametric excitation frequency is equal to force frequency: …, $c = \infty$; $\alpha = \alpha_{opt}$; $C = 10c$; ——, $K_1 = K_0/2$; $c_1 = c/2$; $f = f_0 = const.$; (d) harmonic forcing excitation of parametric system when parametric excitation frequency is twice as high as force frequency (optimally tuned absorber): …, $c = \infty$; $\alpha = \alpha_{opt}$; $C = 10c$; ——, $K_1 = K_0/2$; $c_1 = c/2$; $m_0 = v_f/v_0 = 2$.

optimal but its efficiency near the natural frequency of the system is quite good, being approximately 50%.

The amplitude–frequency characteristics for the parametric excitation system under a constant load are shown in Figure 12(c). The tuning of the absorber is optimal and its efficiency near the natural frequency of the system is still quite good (about 50%). The resonance amplitudes are almost the same as in the previous case, the only difference being that the width of the resonance zone is greater.

The amplitude–frequency characteristics for the harmonic forcing excitation of the parametric system are shown in Figure 12(b, d).

The case when the frequency of the parametric excitation is equal to the frequency of the forcing excitation ($m_0 = 1$) is shown in Figure 12(b). The tuning of the parametric absorber is not optimal. The efficiency of the absorber in the resonance frequency zone is a little higher than in the previous cases, but it is still about 50%. The resonance amplitudes reduced by the absorber are almost twice as large as the previous ones.

The case when the frequency of the parametric excitation is double the frequency of the forcing excitation ($m_0 = 2$) is shown in Figure 12(d). The tuning of the parametric absorber is now optimal. The efficiency of the absorber in the resonance frequency zone is much higher, being about 90%. The resonance amplitudes reduced by the absorber are almost seven times larger than for the harmonic forcing excitation of the constant parameter system (Figure 12(a)).

The application of a dynamic parametric absorber to reduce the vibration of a single-degree-of-freedom primary parametric periodic system is similar to the problem of multi-degree-of-freedom systems. The general conclusion is that one dynamic absorber can reduce effectively the amplitude of vibration of a parametric system for non-parametric and many parametric resonance frequencies simultaneously. Also the efficiency with which the absorber reduces the steady state response of linear parametric systems under a constant load is sufficient.

4. CONCLUSIONS

The study presents a methodology for designing a dynamic parametric absorber for the reduction of the steady state response of a single-degree-of-freedom system subjected to parametric and forcing excitations.

The study incorporates the following results:

- For the parametric system, the parametric absorber is more efficient than the conventional one. The maximum response of the primary mass can be reduced by about 10% more when the parametric damper is employed (Figures 8(c) and 9(c)).
- The parametric resonance is much more dangerous and the amplitudes of the steady state response of such a system are greater than those obtained by forcing excitation only (Figures 9(c) and 2(c)).
- Damping in the primary system is necessary to reduce the parametric resonance regions (Figures 5 and 6).
- One dynamic absorber can reduce the amplitude of vibration of a parametric system for non-parametric and many parametric resonance frequencies simultaneously (Figure 12).
- The efficiency of the absorber in reducing the steady state response of linear parametric systems under a constant load is also sufficient.
- The application of a dynamic absorber to reduce the vibration of a single-degree-of-freedom primary parametric system is similar to the problem of multi-degree-of-freedom systems.
- First order logarithmic sensitivity functions with respect to appropriate design variables k (the stiffness constraint of the absorber) and c (the damping constraint of the absorber) are very useful tools for the absorber tuning procedure.

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APPENDIX A: NOMENCLATURE

$\mathbf{B}$, M, m	mass matrix and masses of primary system (capital letter) and absorber (small letter) respectively
$\mathbf{C}$, C, c, c_0, c_1	damping matrix and damping coefficients of primary system (capital letter) and absorber (small letters) respectively
$\mathbf{D}$	monodromy matrix
$\mathbf{H}$	integer power of monodromy matrix
$\mathbf{I}$	identity matrix
$\mathbf{K}$, K, K_0, K_1, k	stiffness matrix and primary system (capital letters) and absorber (small letter) spring stiffnesses
$\mathbf{R}$	fundamental matrix of solutions
$\mathbf{a}$, $\mathbf{a}_0 a$, a_i	parameter vector, nominal parameter vector and parameters
$\Delta\mathbf{a}$	small parameter variation vector
$\mathbf{b}$	particular integral vector in state space (evaluated at end of period)
$\mathbf{f}$, f_0, f_1	force excitation vector, static and dynamic force amplitudes
$\mathbf{g}$	force excitation vector in state space
$\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$	vectors of dynamic displacements (general co-ordinates), velocities and accelerations respectively
$\mathbf{r}$, $\mathbf{r}_f$	vector of state and steady state (periodic) response vector in state space (particular integral vector which satisfies non-zero initial conditions)
$\mathbf{r}_w$	particular integral vector which satisfies zero initial condition in state space
$\mathbf{s}_a$, $\dot{\mathbf{s}}_a$, $\ddot{\mathbf{s}}_a$	absolute sensitivity functions vector and its time derivatives
m_0, m_f	positive integers
q_{max}	highest absolute value of total (static and dynamic) displacement of primary mass
s_a, w_a	absolute sensitivity function and relative (logarithmic) sensitivity function;
t	time variable
t_q	time point for which highest absolute value of total displacement of primary mass occurs
T_c, T_0, T_f	common, parametric and force period
α, α_{opt}	frequency ratio and optimum frequency ratio
λ_i, λ_{max}	Lyapunov's characteristic exponent and highest Lyapunov's characteristic exponent
ν_c, ν_0, ν_f	common, parametric and force frequency
μ	mass ratio
ρ_i	eigenvalue of monodromy matrix
ω	angular undamped natural frequency of absorber
Ω	angular undamped natural frequency of primary system