



NON-LINEAR DYNAMICS AND CONTROL OF CHAOS FOR A TACHOMETER

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(Received 8 July 2000, and in final form 4 April 2001)

The dynamic behaviors of a rotational tachometer with vibrating support are studied in the paper. Both analytical and computational results are used to obtain the characteristics of the system. The Lyapunov direct method is applied to obtain the conditions of stability of the equilibrium position of the system. The center manifold theorem determines the conditions of stability for the system in a critical case. By applying various numerical analyses such as phase plane, Poincaré map and power spectrum analysis, a variety of periodic solutions and phenomena of the chaotic motion are observed. The effects of the changes of parameters in the system can be found in the bifurcation diagrams and parametric diagrams. By using Lyapunov exponents and Lyapunov dimensions, the periodic and chaotic behaviors are verified. Finally, various methods, such as the addition of a constant torque, the addition of a periodic torque, delayed feedback control, adaptive control, Bang–Bang control, optimal control and the addition of a periodic impulse are used to control chaos effectively.

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1. INTRODUCTION

Most mechanical dynamical systems are non-linear in nature, and can be described by the non-linear equations of motion. It has been realized that the responses of many non-linear dynamical systems do not follow simple, regular and predictable trajectories. A large number of studies have shown that chaotic phenomena are observed in many non-linear dynamical systems [1–4]. This paper will study the non-linear behavior of a tachometer with vibrating support. Both analytical and computational methods are used to obtain the characteristics of the system. Further, seven methods are used to control the chaotic behavior of the system.

At first, the governing equation of motion will be established. The analytical analyses for the non-linear dynamical system are obtained by Lyapunov direct method and center manifold theorem. Lyapunov direct method is applied to obtain the conditions of stability of the relative equilibrium position [5–9]. The Jacobi energy integral of the dynamical system is used to construct a Lyapunov function. The center manifold theorem determines the conditions of stability for the autonomous system in a critical case. Further, the sufficient condition for stability with physical meaning is obtained.

A number of numerical analyses is used to study the dynamical behaviors of the tachometer system. The bifurcation diagram, parametric diagrams, phase portraits, Poincaré maps, power spectrum analysis, Lyapunov exponents and Lyapunov dimension, all of them present a variety of periodic solutions and the phenomena of chaotic motion. As the parameter is changed, the equilibrium points are created or destroyed; or their stability may be changed. The effect of the change of parameters in the system can be found in the

bifurcation and parametric diagrams. The phase portrait is a collection of trajectories that represent the solutions of these equations of motion in the phase space. A more informative representation of the periodic solutions is Poincaré map method. For different specific parameters, a variety of the periodic solutions can be clearly presented in the phase portrait, Poincaré map and power spectrum analysis. For a broad range of parameters, the Lyapunov exponent test is the most powerful method to measure the sensitivity of the dynamical system to change in initial conditions. It can help us to examine as to whether the system is in chaotic motion or not.

Finally, in order to improve the performance of a dynamic system or avoid the chaotic phenomena, sometimes we have to convert a chaotic behavior into a periodic motion. Various methods are presented to control chaos: the addition of a constant torque, the addition of a periodic torque, delayed feedback control, adaptive control, Bang–Bang control, optimal control and the addition of a periodic impulse.

2. DESCRIPTION OF THE SYSTEM MODEL AND DIFFERENTIAL EQUATIONS OF MOTION

The tachometer system considered here is shown in Figure 1. The vertical support O_1O_2 of the system is subjected to a vertical vibration $A \sin \omega t$. If the masses of the rods and the vertical support are neglected, and the balls A and B can be assumed as particles and have the same mass, then the Lagrange equations of motion for ϕ and θ can be expressed as follows:

$$2m_1l^2\ddot{\phi} + 4m_2l^2 \sin^2 \phi \ddot{\phi} + 4m_2l^2 \sin \phi \cos \phi \dot{\phi} - 2m_1l^2 \sin \phi \cos \phi \dot{\theta}^2 + 2m_2l(g + A\omega^2 \sin \omega t)\sin \phi + k_1\phi + k_2\dot{\phi} = 0, \tag{2.1}$$

$$2 \cos \phi \dot{\phi} \dot{\theta} + \sin \phi \ddot{\theta} = 0, \tag{2.2}$$

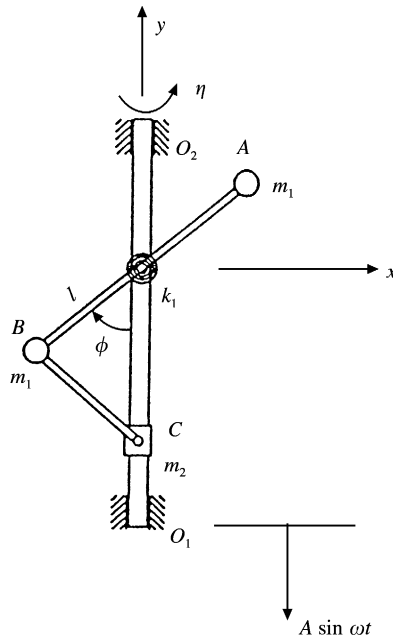


Figure 1. The tachometer system.

where m_1 is the mass of fly balls, m_2 is the mass of the sleeve, l is the length of the rod, ϕ is the angle between the rod AB and the y -axis, $\dot{\theta}$ is the angular velocity of vertical support O_1O_2 , k_1 is the spring constant of a restoring spiral spring which is used to restrain the angle ϕ caused by centrifugal forces of A and B, k_2 is the viscous damping coefficient.

Let $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = \dot{\theta}$ be the state variables, and by changing time scale to dimensionless time $\tau = \omega t$, the state equations can be written as

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{2m_1 + 4m_2 \sin^2 x_1} \left(\frac{-2m_2 g \sin x_1}{\omega^2 l} - \frac{2m_2 A \omega^2 \sin \tau \sin x_1}{\omega^2 l} \right. \\ &\quad \left. - 4m_2 \sin x_1 \cos x_1 x_2^2 + 2m_1 \sin x_1 \cos x_1 x_3^2 - \frac{k_1}{\omega^2 l^2} x_1 - \frac{k_2}{\omega l^2} x_2 \right), \\ \dot{x}_3 &= \frac{-2 \cos x_1 x_2 x_3}{\sin x_1}.\end{aligned}\tag{2.3}$$

3. STABILITY ANALYSIS BY LYAPUNOV DIRECT METHOD

The stability of the system will be investigated by Lyapunov direct method in this section, when the vertical vibration $A \sin \tau$ is absent. The system becomes an autonomous system and the generalized energy of the system is used to construct a Lyapunov function. The equilibrium of the system can be found from equation (2.3) as $(0, 0, c)$

The generalized energy $E^* = T_2 - T_0 + \Pi$ is varied as [10]

$$\frac{dE^*}{dt} = -\frac{\partial L}{\partial t} - 2R,\tag{3.1}$$

where R is the Rayleigh dissipation function. Construct a Lyapunov function as

$$\begin{aligned}V &= E^* + \int \frac{\partial L}{\partial t} dt \\ &= m_1 l^2 x_2^2 + 2m_2 l^2 \sin^2 x_1 x_2^2 + m_1 l^2 \sin^2 x_1 (c + x_3)^2 \\ &\quad - 2m_2 g l \cos x_1 + \frac{1}{2} k_1 x_1^2 \\ &= m_1 l^2 x_2^2 + m_1 l^2 x_1^2 c^2 + m_2 g l x_1^2 + \frac{1}{2} k_1 x_1^2 + \dots\end{aligned}\tag{3.2}$$

V is a positive-definite function, and the derivative of V from equation (2.3) is

$$\begin{aligned}\frac{dV}{dt} &= -2R \\ &= -k_2 x_2^2.\end{aligned}\tag{3.3}$$

So the derivative of V is negative semi-definite. By Lyapunov stability theorem, this solution $(0, 0, c)$ is stable for x_1 and x_2 .

4. APPLICATION OF THE CENTER MANIFOLD

In this section, the application of the center manifold will be introduced. We use it to study the stability if an autonomous system is in the critical case where linearization fails [11, 12].

Consider the autonomous system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{2m_1l^2 + 4m_2l^2 \sin^2 x_1} (-2m_2gl \sin x_1 - 4m_2l^2 \sin x_1 \cos x_1 x_2^2 \\ &\quad + 2m_1 \sin x_1 \cos x_1 x_3^2 - k_1 x_1), \\ \dot{x}_3 &= \frac{1}{2m_1l^2 \sin^2 x_1 + I} (-4m_1l^2 \sin x_1 \cos x_1 x_2 x_3 - k_3 x_3 + M), \end{aligned} \tag{4.1}$$

where I is the moment of inertia of the sleeve C about y -axis, k_3 is the viscous damping coefficient for the bearings O_1 and O_2 , M is a constant driving torque.

The system has an equilibrium point $(0, 0, M/k_3)$, equation (4.1) can be rewritten in matrix form

$$\dot{X} = AX + f(X) + O(4), \quad X \in R^3, \tag{4.2}$$

where

$$X = [x_1, x_2, x_3]^T, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & 0 \\ 0 & 0 & \frac{-k_3}{I} \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ f_1(x) \\ f_2(x) \end{bmatrix}$$

and where

$$\begin{aligned} \Delta &= \sqrt{\frac{2m_2glk_3^2 - 2m_1M^2 + k_1k_3^2}{2m_1l^2k_3^2}}, \\ f_1(x) &= \frac{Mx_1x_3}{l^2k_3} + \frac{m_2gl(m_1 + 12m_2) - 4m_1M^2(m_1 + 3m_2)/k_3^2 + 6k_1m_2}{m_1^2l^2} x_3^3, \\ &\quad - \frac{2m_2x_1x_2^2}{3m_1} + \frac{x_1x_3^2}{3l^2}, \\ f_2(x) &= \frac{-2m_2l^2Mx_1x_2}{Ik_3} - \frac{2m_1l^2x_1x_2x_3}{3I^2}. \end{aligned}$$

The eigenvalues of matrix A are $\pm i\Delta$, $-k^3/I$, this is a critical case. It failed to determine the stability of the equilibrium point by linearized equations and it becomes necessary to consider the higher order terms to analyze the three-dimensional non-linear system. The center manifold theorem will be applied to reduce the dimension of the state space. Let

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix},$$

where

$$T = \begin{bmatrix} 1 & 1 & 0 \\ \Delta & -\Delta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then equation (4.2) is transformed into Jordan form

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & \frac{-k_3}{I} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + T^{-1}f(Tq), \tag{4.3}$$

where

$$T^{-1}f(Tq) = \begin{bmatrix} f_1(Tq) \\ f_2(Tq) \\ f_3(Tq) \end{bmatrix} = \begin{bmatrix} \frac{M}{2l^2k_3\Delta}(q_1q_3 + q_2q_3) \\ -M \\ \frac{2l^2k_3\Delta}{-2m_2l^2M\Delta}(q_1q_3 + q_2q_3) \\ \frac{Ik_3}{(q_1^2 - q_2^2)} \end{bmatrix}.$$

According to the center manifold theory, it is found that there exists a center manifold tangent to the two-dimensional center eigenspace which was formed by eigenvectors corresponding to the pure imaginary pair eigenvalues. The behavior of the original system in close vicinity to the degenerate point can be determined by a two-dimensional system restricted within the center manifold $h(q_1, q_2)$.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\Delta \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{M}{2l^2k_3\Delta} [aq_1^3 + bq_2^3 + (b+c)q_1q_2^2 + (a+c)q_1^2q_2] \\ -M \\ \frac{2l^2k_3\Delta}{2l^2k_3\Delta} [aq_1^3 + bq_2^3 + (b+c)q_1q_2^2 + (a+c)q_1^2q_2] \end{bmatrix}, \tag{4.4}$$

where

$$h(q_1, q_2) = aq_1^2 + bq_2^2 + cq_1q_2$$

and where

$$a = \frac{-2m_2l^2M\Delta}{k_3^2 + 4\Delta^2I^2}, \quad b = \frac{2m_2l^2M\Delta}{k_3^2 + 4\Delta^2I^2}, \quad c = \frac{-8m_2l^2M^2\Delta^2I}{k_3^3 + 4\Delta^2I^2k_3}.$$

The stability criterion for a general two-dimensional system of equation (4.3) is

$$e = \frac{1}{16} \left(\frac{\partial^3 f_1}{\partial q_1^3} + \frac{\partial^3 f_1}{\partial q_1 \partial q_2^2} + \frac{\partial^3 f_2}{\partial q_1^2 \partial q_2} + \frac{\partial^3 f_2}{\partial q_2^3} \right) + \frac{1}{16\Delta} \left[\frac{\partial^2 f_1}{\partial q_1 \partial q_2} \left(\frac{\partial^2 f_1}{\partial q_1^2} + \frac{\partial^2 f_1}{\partial q_2^2} \right) - \frac{\partial^2 f_2}{\partial q_1 \partial q_2} \left(\frac{\partial^2 f_2}{\partial q_1^2} + \frac{\partial^2 f_2}{\partial q_2^2} \right) - \frac{\partial^2 f_1}{\partial q_1^2} \frac{\partial^2 f_2}{\partial q_1^2} + \frac{\partial^2 f_1}{\partial q_2^2} \frac{\partial^2 f_2}{\partial q_2^2} \right]. \tag{4.5}$$

If $e < 0$ the periodic orbits are stable, while if $e > 0$ they are unstable.

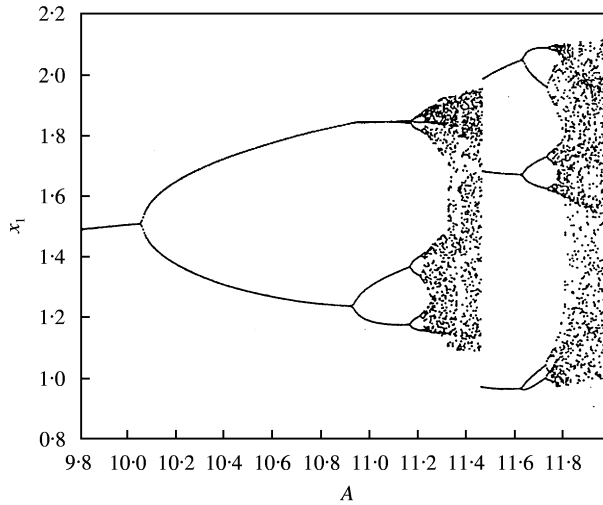


Figure 2. Bifurcation diagram of the amplitude of the vibration of support A versus angle x_1 .

By substituting equation (4.4) into equation (4.5) we obtain

$$e = \frac{-m_2 M^2}{2k_3(k_3^2 + 4A^2 I^2)} < 0,$$

therefore, the equilibrium point $(0, 0, M/k_3)$ is stable.

5. BIFURCATION DIAGRAM AND PARAMETRIC DIAGRAMS

As the parameter is changed, the equilibrium points are created or destroyed, or their stability may be changed. If the phase portrait changes its topological structure as a parameter is varied, we say that a bifurcation has occurred, and the parameter values at which they occur are called bifurcation points. Let us vary one of the control parameters in the system, and record the data of Poincaré map corresponding to every different parameter value. Then the steady state behavior of the system versus the range of control parameters will be plotted. This is called a bifurcation diagram. It is a widely used technique to describe a transition from periodic motion to chaotic motion for a dynamical system. The dynamic behavior may be observed more completely over a range of parameter values by the bifurcation diagram. The bifurcation diagram of the non-linear system of equation (2.3) is shown in Figure 2. It is calculated by the fourth order Runge–Kutta numerical integration and the angle x_1 is plotted against the amplitude of the vibration $A \in [9.8, 12]$. At each A , the points of Poincaré map in the transient state of motion are discarded. It is observed that the periodic motions transform to the chaotic behavior with an increase in amplitude A .

Further, the parameter values, damping coefficient and forcing frequency, will also be varied to observe the behaviors of bifurcation of the system. By varying simultaneously any two of the three parameters, amplitude of the vibration, damping coefficient and frequency, the parametric diagrams are described and shown as Figures 3 and 4. The enriched information of the behaviors of the system can be obtained from the diagrams.

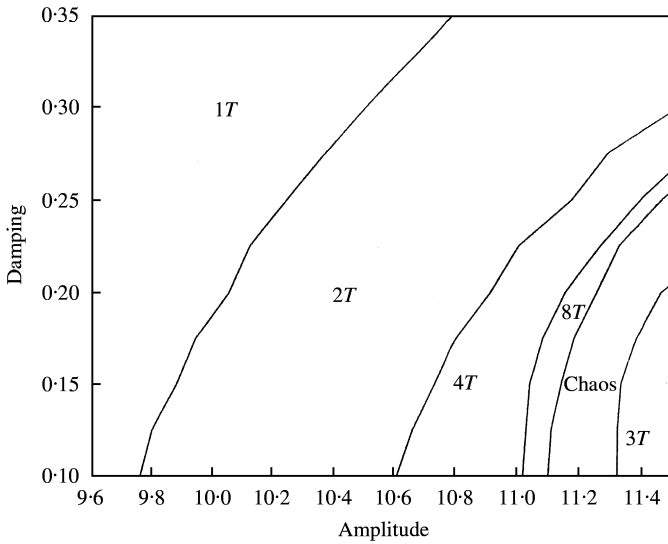


Figure 3. Parametric diagram of A versus damping k_2 for $\omega = 1$.

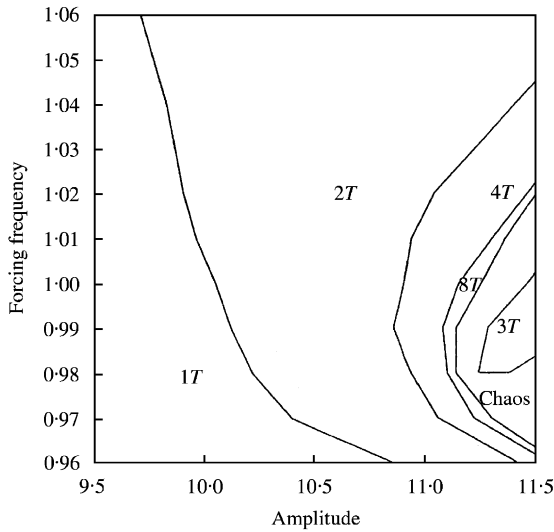
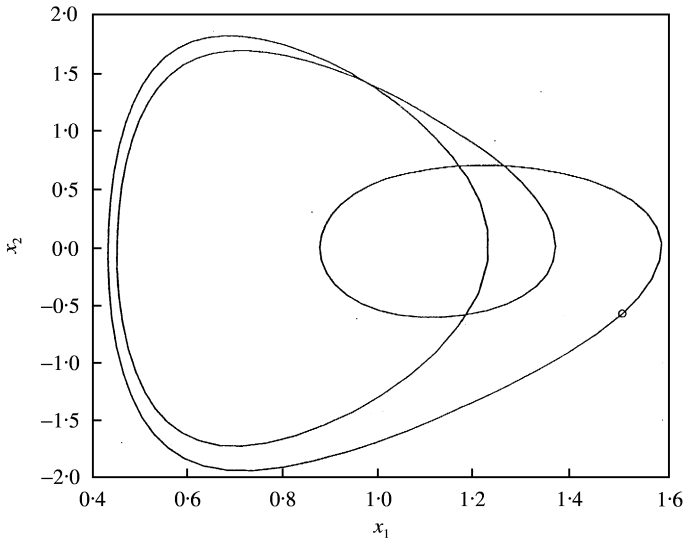
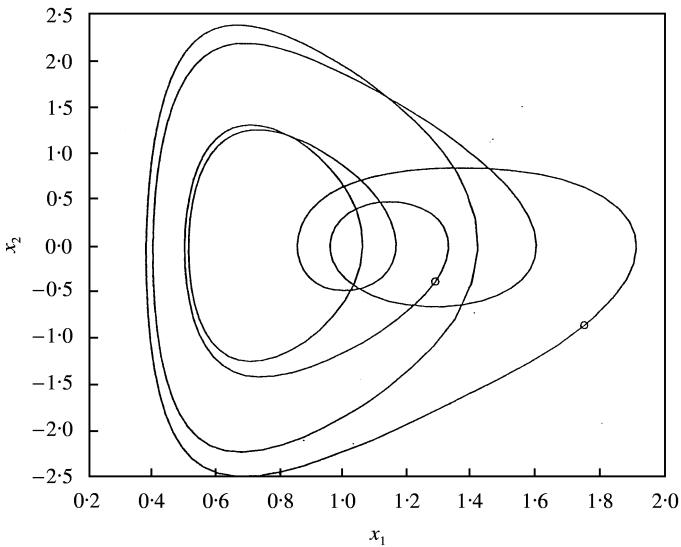


Figure 4. Parametric diagram of A versus forcing frequency ω for $k_2 = 0.2$.

6. PHASE PORTRAITS, POINCARÉ MAP AND POWER SPECTRUM

A valuable description of a solution is obtained by examining its behavior in the phase plane. The phase portrait is a collection of trajectories that represent the solutions of these equations of motion in the phase space. When the solution becomes stable, the asymptotic behaviors of the phase trajectories are of particular interest and the transient behaviors in the system are neglected. The Poincaré map is a method introduced by Henri Poincaré [13] as a means of simplifying the phase space diagram of complicated systems. One of many advantages of Poincaré map is to reduce the dimension of the dynamical system. The

Figure 5. Poincaré maps and phase portraits for $A = 10$.Figure 6. Poincaré maps and phase portraits for $A = 10.5$.

solution of period- $1T$ in the phase plane will become one point in the Poincaré map. The phase portraits and the Poincaré maps of equation (2.3) are shown in Figures 5–7 for $A = 10, 10.5$ and 11.44 . Figures 5 and 6 present clearly the periodic motion of the system. Figure 7 shows the chaotic behavior of the system, where the Poincaré map is neither a finite set of points nor a closed orbit.

A variable technique for the identification and characterization of the system is power spectrum. It is often used to distinguish between periodic and chaotic behaviors for a dynamical system. Any function $x(t)$ may be represented as a superposition of different periodic components. The determination of their relative strength is called spectral analysis.

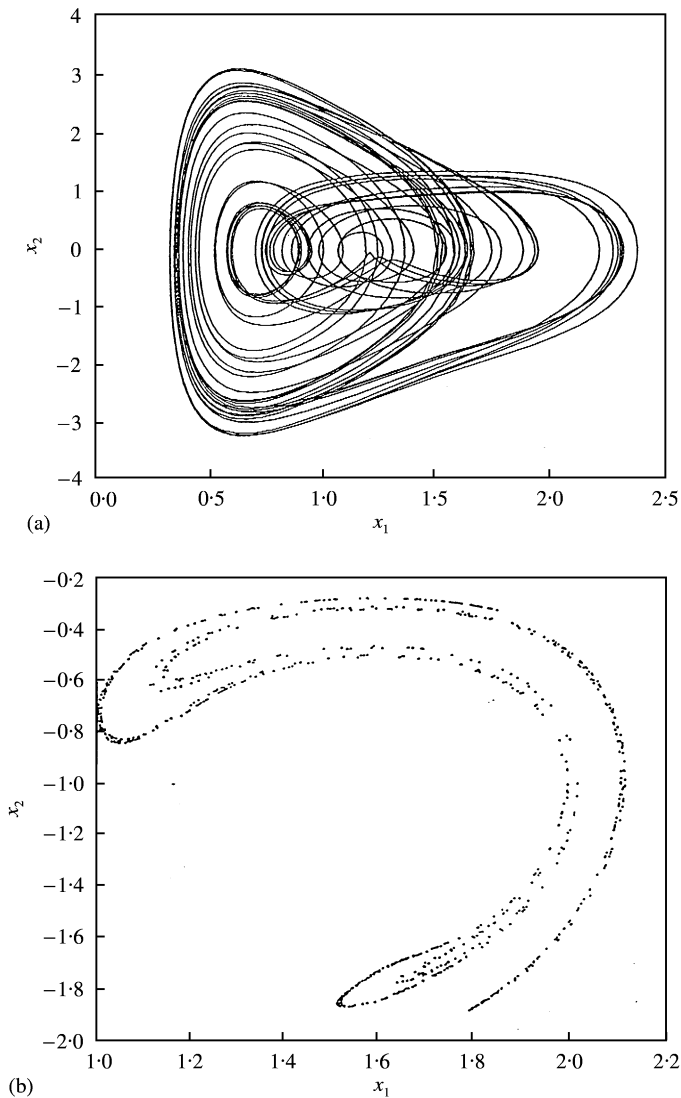
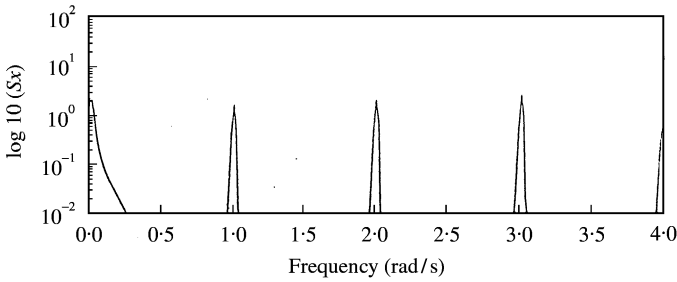
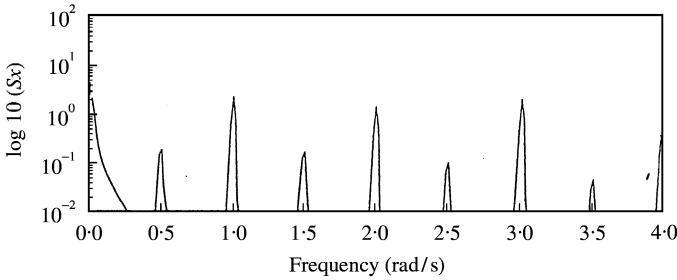
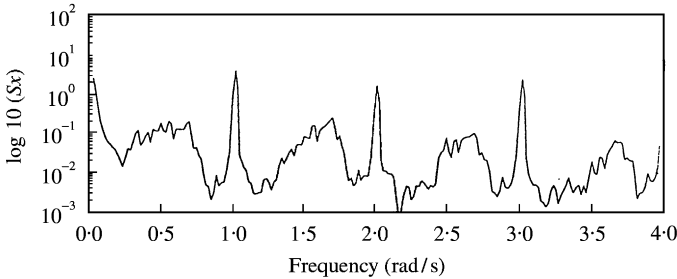


Figure 7. (a) Phase portraits for $A = 11.44$; (b) Poincaré maps for $A = 11.44$.

If it is periodic, the spectrum may be a linear combination of oscillations whose frequencies are integer multiples of basic frequency. The linear combination is called a Fourier series. If it is not periodic, the spectrum must then be in terms of oscillations with a continuum of frequencies. Such a representation of the spectrum is called Fourier integral of $x(t)$. The representation is useful for dynamical analysis. The periodic motion and chaotic behavior of the system are observed by the portraits of power spectrum in Figures 8–10 for $A = 10$, 10.5 and 11.44, where in Figure 10 a continuous spectrum is obtained by chaos.

7. LYAPUNOV EXPONENT AND LYAPUNOV DIMENSION

In previous studies, a variety of observations from time history, Poincaré maps and power spectral density diagrams prove that chaotic systems have non-periodic complex

Figure 8. Power spectrum $A = 10$, period-1.Figure 9. Power spectrum $A = 10.5$, period-2.Figure 10. Power spectrum $A = 11.4$, chaos.

motion. However, these methods sometimes do not clearly reveal as to whether the system is in non-periodic or chaotic motion. Therefore, the criterion of detecting and characterizing chaos has become an important one. Lyapunov exponents provide qualitative and quantitative characterization of dynamic behavior. The Lyapunov exponent calculation algorithm was proposed by Wolf *et al.* [14, 15]. It can be used to obtain a measure of how sensitive this behavior is to the initial condition, and can serve as criteria for detecting the chaotic motion: a system is chaotic when there exists at least one positive Lyapunov exponent. It is an index for chaotic behavior. Different solutions of dynamical system, such as fixed points, periodic motions and chaotic motion can be distinguished by it. If two trajectories start close to one another in phase space, they will move exponentially away from each other for small times on the average. Thus, if d_0 is a measure of the initial distance between the two starting points, the distance is $d(t) = d_0 2^{\lambda t}$. The symbol λ is called

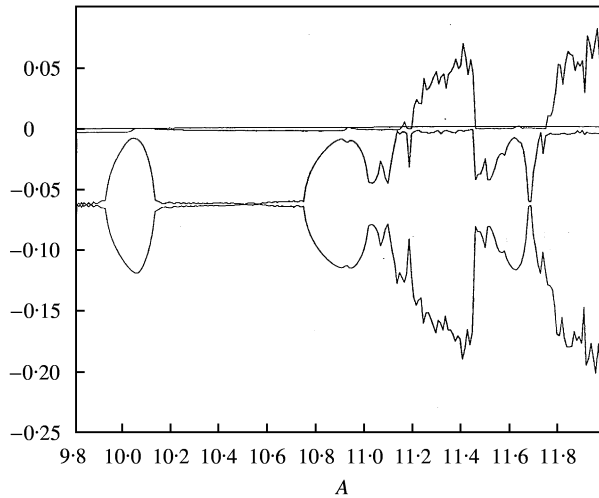


Figure 11. Lyapunov exponents for $A = 9.8-12$.

TABLE 1

A	λ_1	λ_2	λ_3	λ_4	$\sum \lambda_i$	d_L	
10	-0.00317	0	-0.01462	-0.10958	-0.12738	1	Period-1
10.5	-0.00251	0	-0.06262	-0.06274	-0.12788	1	Period-2
11.1	-0.00223	0	-0.04641	-0.07918	-0.12784	1	Period-4
11.44	0.05505	0	-0.00274	-0.17967	-0.12736	3.2911	Chaos
11.5	-0.00198	0	-0.04399	-0.08190	-0.12788	1	Period-3

Lyapunov exponent. The divergence of chaotic orbits can only be locally exponential, because if the system is bounded, $d(t)$ cannot grow to infinity. A measure of this divergence of orbits is that the exponential grown at many points along a trajectory has to be averaged. When $d(t)$ is too large, a new “nearby” trajectory $d_0(t)$ is defined. The Lyapunov exponent can be expressed as

$$\lambda = \frac{1}{t_N - t_0} \sum_{k=1}^N \log_2 \frac{d(t_k)}{d_0(t_k - 1)}. \tag{7.1}$$

The signs of the Lyapunov exponents provide a qualitative picture of a system dynamics. The criteria are

$$\lambda > 0 \text{ (chaotic),} \quad \lambda \leq 0 \text{ (regular motion).}$$

The Lyapunov exponents of the solutions of the dynamical system, are plotted in Figure 11 as $A = 9.8-12$. If the value of Lyapunov exponent is greater than zero, it is chaos, otherwise periodic solution.

The Lyapunov dimension is a measure of the complexity of the attractor. The Lyapunov dimension d_L which has been developed by Kaplan and Yorke [16], is introduced as

$$d_L = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}, \tag{7.2}$$

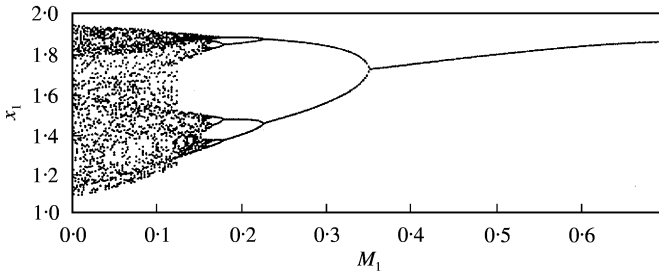


Figure 12. Bifurcation diagram of M_1 versus angle x_1 .

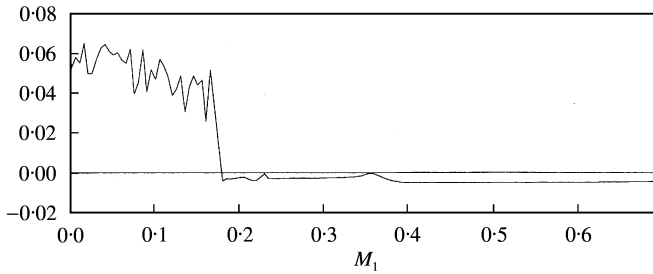


Figure 13. Maximum Lyapunov exponent against M_1 for $A = 11.4$.

where j is defined by the condition that

$$\sum_{i=1}^j \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^{j+1} \lambda_i < 0.$$

The Lyapunov dimension for a strange attractor is a non-integer number. The Lyapunov dimension and the Lyapunov exponent of the non-linear system are listed in Table 1.

8. CONTROLLING OF CHAOS

In the previous sections, regular and chaotic behaviors of the system have been found. Analyzing and predicting the behaviors of a chaotic system is beneficial, but to maximize the benefit, one has to be able to control it. In order to improve the performance of a dynamic system or avoid the chaotic phenomena, we must convert a chaotic behavior into a desired stable periodic motion.

In this section, we discuss how chaos can be controlled to periodic motion by minimal efforts. Seven methods which are used to control chaos will be presented [17–19]: the addition of constant torque, the addition of periodic torque delayed, feedback control, adaptive control, Bang–Bang control, optimal control and the addition of periodic impulse.

8.1. CONTROLLING OF CHAOS BY THE ADDITION OF A CONSTANT TORQUE

To control the chaotic behavior to a desired periodic one in the system, we can just add a constant torque. It ensures effective controlling in a very simple manner. A constant

torque M_1 is added to the right side of the first equation of equation (2.1). The state equations in equation (2.3) become

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{2m_1 + 4m_2 \sin^2 x_1} \left(\frac{-2m_2 g \sin x}{\omega^2 l} - \frac{2m_2 A \omega^2 \sin \tau \sin x_1}{\omega^2 l} - 4m_2 \sin x_1 \cos x_1 x_2^2 \right. \\ &\quad \left. + 2m_1 \sin x_1 \cos x_1 x_3^2 - \frac{k_1}{\omega^2 l^2} x_1 - \frac{k_2}{\omega l^2} x_2 + M_1 \right), \\ \dot{x}_3 &= \frac{-2 \cos x_1 x_2 x_3}{\sin x_1}. \end{aligned} \tag{8.1}$$

The bifurcation diagram in Figure 12 for constant torque M_1 versus angular displacement presents the effect of constant torque. By increasing it from zero upward the chaotic behavior is converted to stable periodic motion.

By using the analysis of the Lyapunov exponents, the range of the parameter which can stabilize the dynamics of the system will be estimated. When the maximal Lyapunov exponents $\lambda_i \leq 0$, it is clear that the system returns to a regular one. The Lyapunov exponent versus the constant torque is shown in Figure 13 for amplitude $A = 11.4$.

8.2. CONTROLLING OF CHAOS BY THE ADDITION OF A PERIODIC TORQUE

The method of controlling chaos by the addition of periodic torque is similar to the method used in section 8.1. We add the periodic torque $M_2 \sin \omega t$ instead of constant torque M_1 in equation (8.1) to the system. Similarly, the bifurcation diagram in Figure 14 shows the value of periodic torque M_2 versus angular displacement, and Figure 15 for the Lyapunov exponent versus M_2 . It also presents the return of the chaotic behavior of the system to stable periodic motion while the value of M_2 increases from zero upward.

8.3. CONTROLLING OF CHAOS BY DELAYED FEEDBACK CONTROL

In this section, we will introduce the application of delayed feedback control [20]. The difference $D(t)$ between the delay output signal $x(t - \tau)$ and the output signal $x(t)$ is used as a control signal:

$$F(t) = -K[x(t - \tau) - x(t)] = KD(t). \tag{8.2}$$

Here K is an experimentally adjustable weight of the control signal, and τ is a delay time. We add equation (8.2) instead of the constant torque M_1 in equation (8.1). Depending on

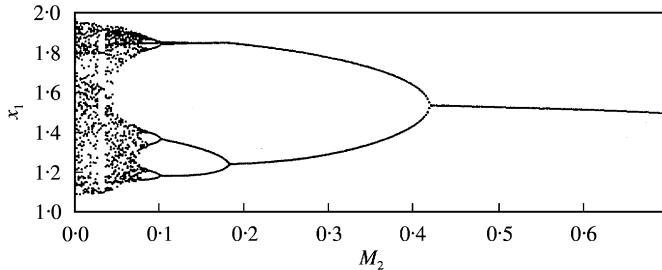


Figure 14. Bifurcation diagram of M_2 versus angle x_1 .

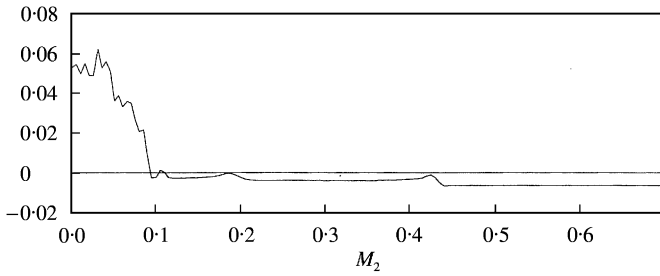


Figure 15. Maximum Lyapunov exponent against M_2 for $A = 11.4$.

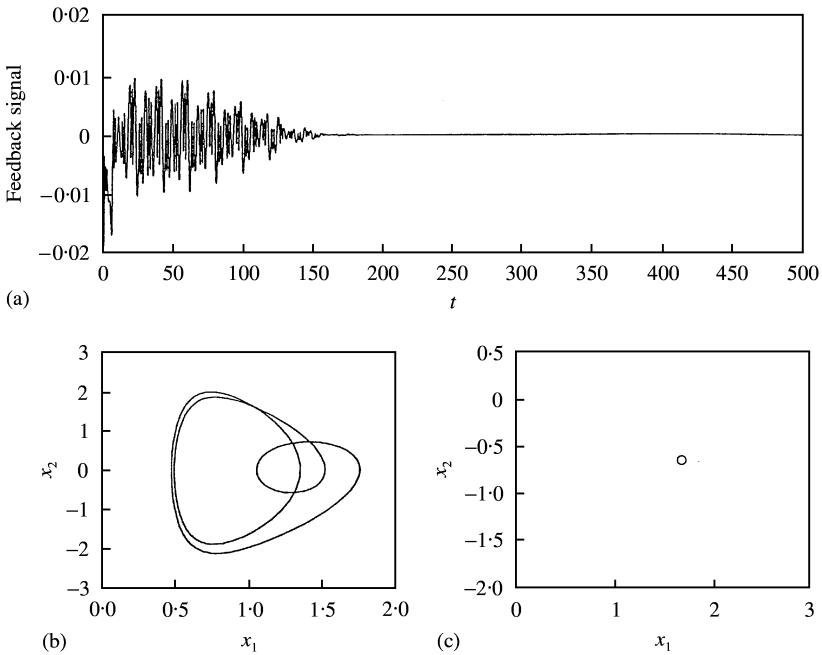


Figure 16. (a)–(c) Feedback control for $K = 0.01$: (a) feedback signal; (b) phase portrait; (c) Poincaré map.

the delay constant τ and weight K , various kinds of periodic behaviors can be observed in the chaotic system. The result, Figures 16(a–c) and 17(a–c) show the feedback signal, phase portrait, Poincaré map for $\tau = 2\pi$, $K = 0.01$ and $\tau = 4\pi$, $K = 0.03$ when $A = 11.4$. In Figure 16, the delay time is T , which converts chaos to period-1 oscillation. In Figure 17, the delay time is $2T$, which converts chaos to period-2 oscillation. The difference between the delayed output signal and the output signal itself is used as a control signal. Only a simple delay time is required for this feedback control. To achieve the periodic motion of the system, two parameters, the time of delay τ and the weight K of the feedback, should be adjusted.

8.4. CONTROLLING OF CHAOS BY ADAPTIVE CONTROL

An adaptive control algorithm extends to a multi-parameter and higher-dimensional non-linear system [21–23]. This control law is remarkably effective in returning a system to

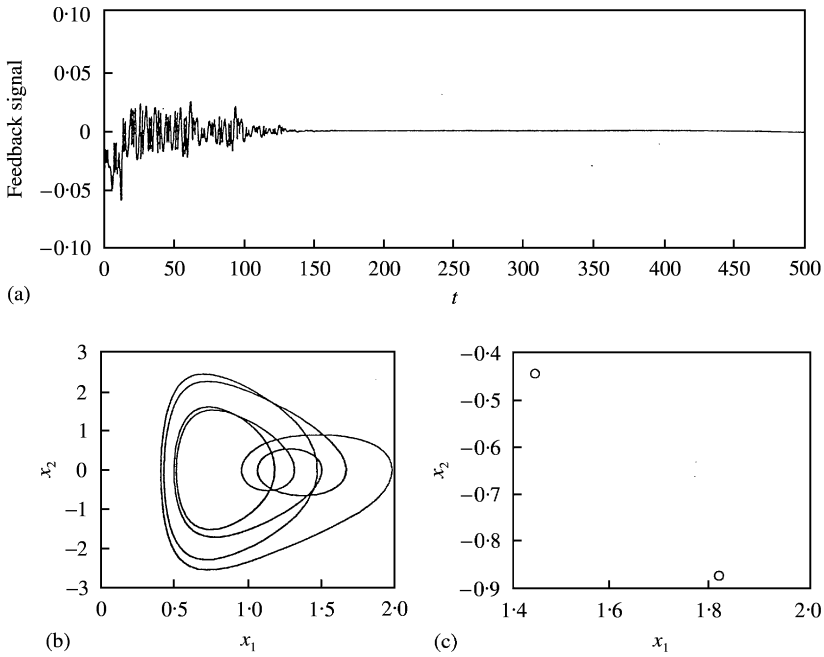


Figure 17. (a)-(c) Feedback control for $K = 0.03$: (a) feedback signal; (b) phase portrait; (c) Poincaré map.

its original dynamics after a sudden perturbation in the system parameter changes of the dynamical behavior. A simple and effective adaptive control algorithm has been suggested, which utilizes an error signal proportional to the difference between the desired output and the actual output of the system. This error signal governs the change of the parameter of the system, which readjusts so as to reduce the error to zero. For a general N -dimensional dynamical system

$$\dot{X} \equiv \frac{dX}{d\tau} = F(X, \tau, \mu), \tag{8.3}$$

where $X \equiv (x_1, x_2, \dots, x_N)$ are variables and $\mu \equiv (\mu_1, \mu_2, \dots, \mu_M)$ are parameters, τ is the dimensionless time, which determines the nature of the dynamics, the prescription for effecting adaptive control is through the additional dynamics,

$$\dot{\mu} = \varepsilon G(X - X_s), \tag{8.4}$$

where X_s is the desired steady state value, ε indicates the stiffness of control and $G(X - X_s)$ is some suitable function with $G(0) = 0$. In order to convert the dynamics of system (2.3) from chaotic motion to the desired periodic motion X_s , the chosen parameter A is perturbed as

$$\dot{A} = \varepsilon(X - X_s).$$

The results, Figures 18(a-c) and 19(a-c) show error function $X - X_s$, phase portrait and Poincaré maps for $\varepsilon = -0.2$ and -0.06 . In Figure 18, X_s is a period-1 motion, so the chaos converts to the period-1 motion. In Figure 19, X_s is a period-2 motion, so the chaos converts to the period-2 motion.

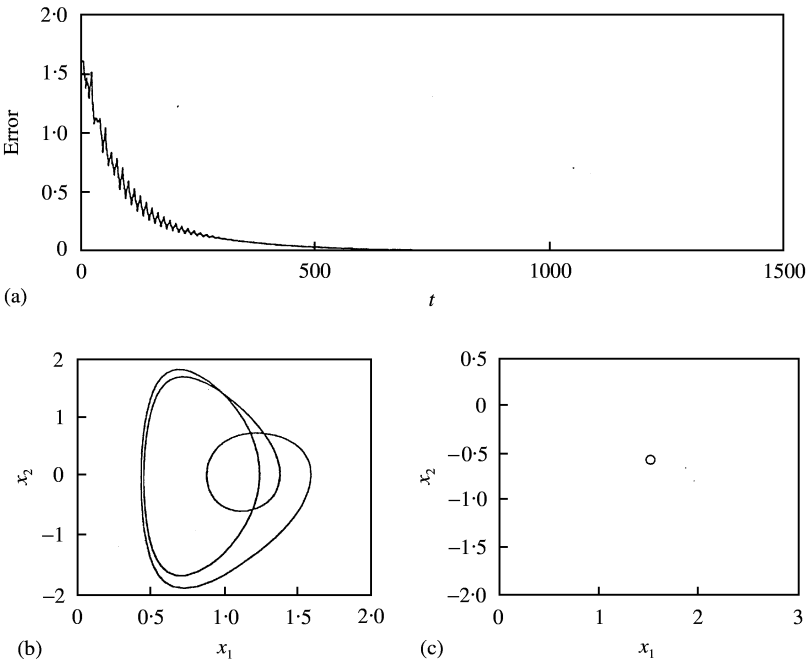


Figure 18. (a)–(c) adaptive control for period-1: (a) error function; (b) phase portrait; (c) Poincaré map.

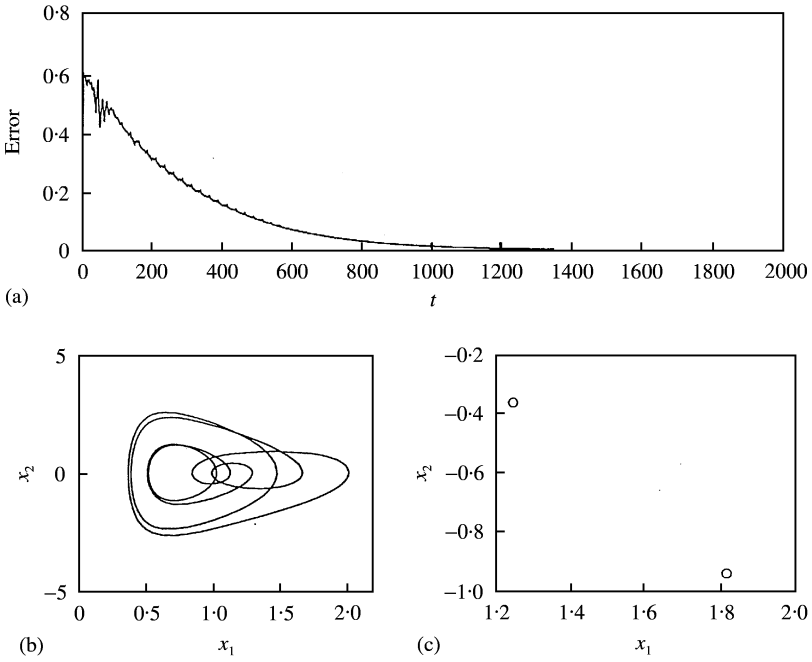


Figure 19. (a)–(c) adaptive control for period-2: (a) error function; (b) phase portrait; (c) Poincaré map.

8.5. CONTROLLING OF CHAOS BY BANG-BANG CONTROL

Define the error function as follows:

$$e(t) = X(t) - X(t - T), \tag{8.5}$$

where T is the external torque period. Define $V(t) = e(t)^2$ which is always positive or zero. Then we can obtain the derivative of V :

$$\dot{V} = 2e(t)\dot{e}(t). \tag{8.6}$$

If $\dot{V} \leq 0$ then $V(t) \rightarrow 0$, i.e., $e(t) \rightarrow 0$, it means that $X(t) \rightarrow X(t - T)$ and the periodic behavior is achieved. By considering equation (8.6), the feedback signal of the control law can be determined as follows:

$$u(t) = -K(F_1(x_1, x_2, x_3, t) - \dot{x}_1(t - T)) \quad \text{when } e(t) > 0,$$

$$u(t) = K(F_1(x_1, x_2, x_3, t) - \dot{x}_1(t - T)) \quad \text{when } e(t) < 0,$$

where K is a positive adjustable constant. In our case, $u(t)$ is added instead of the constant torque M_1 in equation (8.1).

When the trajectory is close to our periodic orbit, the control signal approaches to zero. That is why the method is called ‘‘Bang-Bang’’ control. Figure 20 shows the phase portrait and Poincaré map for $K = 0.03$.

8.6. CONTROLLING OF CHAOS BY OPTIMAL CONTROL

In this section, we use optimal control method to control chaotic behavior of the system. Optimal control is a well-established engineering control strategy, and is useful for both linear and non-linear system with linear or non-linear controller. Also, the system under control can be either continuous-time or discrete-time.

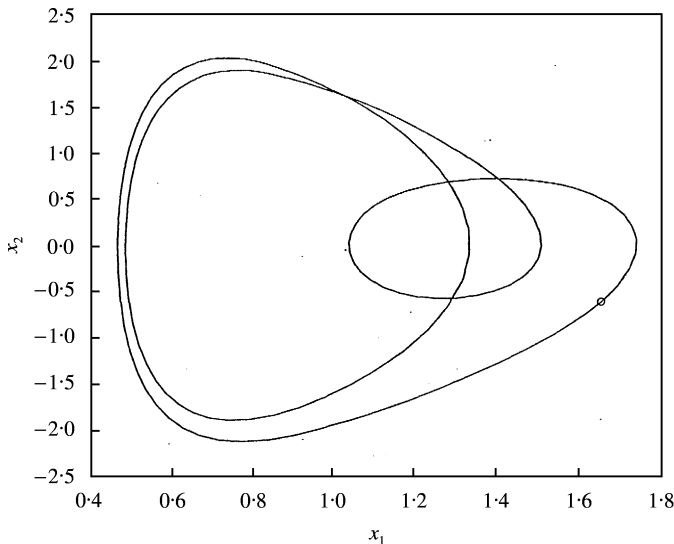


Figure 20. Poincaré maps and phase portraits with $K = 0.03$ for Bang-Bang control.

We consider system (2.3) with controller u

$$\begin{aligned} \dot{x}_1 &= f_1(x) = x_2, \\ \dot{x}_2 &= f_2(x) = \frac{1}{2m_1 + 4m_2 \sin^2 x_1} \left(\frac{-2m_2 g \sin x_1}{\omega^2 l} - \frac{2m_2 A \omega^2 \sin \tau \sin x_1}{\omega^2 l} \right. \\ &\quad \left. - 4m_2 \sin x_1 \cos x_1 x_2^2 + 2m_1 \sin x_1 \cos x_1 x_3^2 - \frac{k_1}{\omega^2 l^2} x_1 - \frac{k_2}{\omega l^2} x_2 \right), \\ \dot{x}_3 &= f_3(x) = \frac{-2 \cos x_1 x_2 x_3 + u}{\sin x_1}. \end{aligned} \tag{8.7}$$

By defining the Hamilton function $H = p_1 f_1 + p_2 f_2 + p_3 f_3$, where the Lagrange multiplier $p(t) = [p_1, p_2, p_3]$, then the optimal controller, u , can be obtained via the Hamilton principle:

$$\frac{\partial H}{\partial u} = \frac{p_3}{\sin x_1} = 0$$

and, consequently

$$\frac{-4m_1 \sin x_1 \cos x_1 x_3}{2m_1 + 4m_2 \sin^2 x_1} p_2 = 0, \tag{8.8}$$

$$\begin{aligned} p_1 x_2 + p_2 \frac{1}{2m_1 + 4m_2 \sin^2 x_1} \left(\frac{-2m_2 g \sin x_1}{\omega^2 l} - \frac{2m_2 A \omega^2 \sin \tau \sin x_1}{\omega^2 l} \right. \\ \left. - 4m_2 \sin x_1 \cos x_1 x_2^2 + 2m_1 \sin x_1 \cos x_1 x_3^2 - \frac{k_1}{\omega^2 l^2} x_1 - \frac{k_2}{\omega l^2} x_2 \right) = 0. \end{aligned} \tag{8.9}$$

This yields a non-zero solution for (p_1, p_2) if and only if

$$\frac{4m_1 \sin x_1 \cos x_1 x_2 x_3}{2m_1 + 4m_2 \sin^2 x_1} = 0,$$

the feedback controller can be obtained

$$u = -k_b \operatorname{sgn} \left[\frac{4m_1 \sin x_1 \cos x_1 x_2 x_3}{2m_1 + 4m_2 \sin^2 x_1} \right],$$

where the signum function

$$\operatorname{sgn}[v] = \begin{cases} +1 & \text{if } v > 0, \\ 0 & \text{if } v = 0, \\ -1 & \text{if } v < 0. \end{cases}$$

Figure 21 shows the periodic motion of the system after control for $k_b = 0.1$.

8.7. CONTROLLING OF CHAOS BY THE ADDITION OF A PERIODIC IMPULSE

We can add a periodic impulse input to the system to control the chaotic behavior. Consider system (8.1) with a periodic impulse input I^* replacing the constant

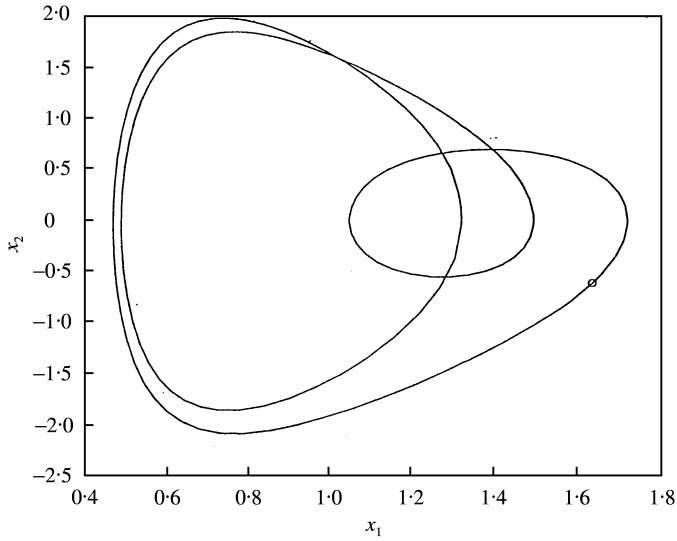


Figure 21. Poincaré maps and phase portraits with $K = 0.1$ for optimal control.

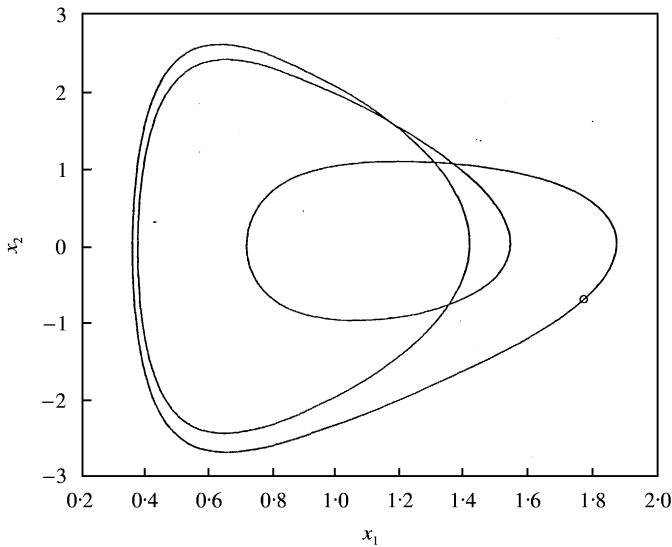


Figure 22. Poincaré maps and phase portraits with $a = -5$ for adding periodic impulse.

torque M_1 :

$$I^* = a \sum_{j=1}^{\infty} \delta(i - jk_p), \tag{8.10}$$

where a is a constant impulse intensity, k_p is the period between two consecutive impulses, and δ is the delta function.

With different values of a and k_p , the controlled system can be stabilized at different periodic solutions. Figure 22 shows the phase portrait and Poincaré maps for $a = -5$ when $k_p = 0.01$.

9. CONCLUSIONS

In the previous sections, both analytical and computational methods are used to obtain the characteristics of the tachometer with a vibrating support system. The analytical analyses for the non-linear dynamical system are obtained by Lyapunov direct method and center manifold theorem. Lyapunov direct method is applied to obtain the conditions of stability of the relative equilibrium position. The center manifold theorem determines the conditions of stability for the autonomous system in a critical case.

Further, various numerical analyses are used to study the dynamical behaviors of the tachometer system. All of those methods present a variety of periodic solutions and the phenomena of chaotic motion. The effect of the change of parameters on the system can be found in the bifurcation and parametric diagrams. The phase portrait is a collection of trajectories that represent the solutions of these equations of motion in the phase space. A more informative representation of the periodic solutions is Poincaré map method. Chaotic motion is the motion that has a sensitive dependence on the initial condition in deterministic physical systems. The chaotic motion has been detected by using Lyapunov exponents and Lyapunov dimensions.

The presence of chaotic behavior is generic for suitable non-linearities, ranges of parameters and external force, where one wishes to avoid or control so as to improve the performance of a dynamical system. We present seven methods to control chaos: the addition of a constant torque, the addition of a periodic torque, delayed feedback control, adaptive control, Bang–Bang control, optimal control and the addition of a periodic impulse.

By using a number of analytical or computational methods, the non-linear behaviors of the tachometer system such as different types of periodic solutions, the effects on the solutions caused by different parameters and initial conditions, the stability analysis of solutions have been studied here. In spite of that these method are different, the results obtained match each other. The conclusion is that there exist complicated phenomena in the dynamics of the non-linear system.

ACKNOWLEDGMENTS

This research was supported by the National Science Council, Republic of China, under Grant Number NSC 87-2212-E-009-019.

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