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SOME GENERAL EFFECTS OF STRONG HIGH-FREQUENCY EXCITATION: STIFFENING, BIASING AND SMOOTHENING

J. J. THOMSEN

Department of Mechanical Engineering, Solid Mechanics, Technical University of Denmark, Building 404, DK-2800 Lyngby, Denmark, E-mail: jjt@mek.dtu.dk

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Mechanical high-frequency (HF) excitation provides a working principle behind many industrial and natural applications and phenomena. This paper concerns three particular effects of HF excitation, that may change the apparent characteristics of mechanical systems: (1) *stiffening*, by which the apparent linear stiffness associated with an equilibrium is changed, along with derived quantities such as stability and natural frequencies; (2) *biasing* by which the system is biased towards a particular state, static or dynamic, which does not exist or is unstable in the absence of the HF excitation; and (3) *smoothening*, referring to a tendency for discontinuities to be effectively "smeared out" by HF excitation. Illustrating first these effects for a few specific systems, analytical results are provided that quantify them for a quite general class of mechanical systems. This class covers systems that can be modelled by a finite number of second order ordinary differential equations, generally non-linear, with periodically oscillating excitation terms of high frequency and small amplitude. The results should be useful for understanding the effects in question in a broader perspective than is possible with specific systems, for calculating effects for specific systems using well-defined formulas, and for possibly designing systems that display prescribed characteristics in the presence of HF excitation.

1. INTRODUCTION

High-frequency (HF) excitation may effectively change certain characteristics of mechanical structures and systems, such as their equilibrium states, linear stiffness, damping, and natural frequencies—and non-linear features such as the type of restoring forces and energy dissipation, frequency response and bifurcation behavior. This work focuses on three such effects, that occur across a wide range of systems, and from which some of the other effects may be derived: (1) *stiffening*, by which the apparent linear stiffness associated with an equilibrium is changed; (2) *biasing*, by which a system subjected to HF excitation is biased towards a particular state, static or dynamic, which does not exist or is unstable in the absence of the HF excitation; and (3) *smoothening*, referring to a tendency for discontinuities to be effectively smeared out by HF excitation.

Illustrating first these effects for a few specific and very simple systems, it is the purpose of the work to provide general analytical results that quantify the effects for a general class of mechanical systems. This class incorporates systems that can be adequately modelled by a finite number of second order ordinary differential equations, generally non-linear, with periodically oscillating excitation terms of high frequency and small amplitude. The HF excitation should be "strong", in the sense that displacements associated with the HF sources should be very small, while the energy should be significant.

A great many studies of specific effects of HF excitation for specific systems have been published, most of them by Blekhman, who has also devised convenient mathematical tools for predicting them [1]. Some early works in this field by Stephenson [2, 3], Hirsch [4], and Kapitza [5] deal with the now well-known effect of the upright stabilization of a pendulum with a rapidly oscillating support, and Chelomei [6] proposed to use similar effects for stabilizing elastic systems. More recent investigations include using resonant HF excitation to transport fluid or strings through pipes [7, 8], damp resonant vibrations of strings and beams [9–12], induce linear motion in systems with asymmetrical friction [13, 14], change the equilibriums, stability, natural frequencies, and non-linear frequency response for a two-bar link [15], increase the buckling load and natural frequencies for columns [16–18], quench friction-induced periodic or chaotic oscillations for non-linear oscillators [19–21], stabilize or change the non-linear behavior of follower-loaded double pendulums and articulated fluid-carrying pipes [22–24], explain the "floating" of the disk on the inverted so-called "Chelomei's pendulum" [16, 25, 26], and form other properties of non-linear mechanical systems [27]. Effects of HF excitation have been investigated for continuous flexible structures such as strings, beams and rotating disks [28–31], and with consideration to change in basic material properties such as creep, relaxation, plasticity [1, 32] (plus a number of works published only in Russian). Also, in the control sciences the use of HF excitation in the form of so-called "dither"-signals has been studied for some time, e.g., see references [33–37].

Mechanical HF excitation provides the basic working principle behind a range of different industrial applications, e.g., transport of granular material or solid bodies on vibration feeders, auto-focusing of camera lenses, submersion of piles, or separation of solid materials according to size or density. Nature occasionally utilizes slow effects of HF excitation, e.g., in making primitive living organisms able to drift and swim. And there are some obviously harmful effects, e.g., self-unscrewing bolts and nuts on vibrating machinery, and needle pointer instruments displaying bias error when operating in strongly vibrating environments.

There are several studies providing valuable general results regarding different effects of HF excitation, for more or less restricted classes of systems and excitations, e.g., references [1, 33–44]. Since, there seems to be no published results in the form of analytical predictions of the effects in question here, for generally non-linear finite-degree-of-freedom mechanical systems with general periodic HF excitation, it is the subject of the present work. The level of generality should be sufficient to cover a broad class of systems occurring in applications, but not so general that it becomes impossible to extract meaningful physical interpretation of the results. Results were aimed at those that would be useful for (1) understanding the effects in question in a broader perspective than it is possible with specific systems, (2) calculating effects for specific systems using well-defined formulas, and (3) possibly designing systems with prescribed characteristics in the presence of HF excitation.

Section 2 of the paper presents the concerned effects in terms of three very simple physical systems, and also presents in short form a number of other typical examples. Section 3 presents the general mathematical model to be studied, representing the example systems of the previous section as well as many others—and shows how this system can be transformed to provide only the essential averaged or "slow" motions that is of interest here. Section 4 then defines and analyses the stiffening, the biasing, and the smoothening effect based on these averaged equations.

2. EXAMPLES

Three specific examples are presented in some detail, followed by brief mention of some other systems in tabular form. The purpose is to illustrate the effects of concern in a simple physical setting, and to provide specific reference for the subsequent general analysis.

2.1. EXAMPLE 1: PENDULUM ON A VIBRATING SUPPORT (STIFFENING AND BIASING)

Studies of the pendulum with a vibrating support (Figure 1) have a long history [2–5, 45]. Here, it serves to illustrate the stiffening and biasing effects of HF excitation, which in turn influence the effective natural frequency and the existence and stability of equilibriums. The equation of motion is

$$\ddot{\theta} + 2\beta\dot{\theta}(1 - q\Omega^2 \sin(\Omega t))\sin\theta = 0, \tag{1}$$

where θ is the swing angle, $t = \omega_0 \tilde{t}$ is the non-dimensional time, $\tilde{t}$ the physical time, $\omega_0 = \sqrt{g/L}$ the linear natural frequency for oscillations near $\theta = 0$, $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$, β the damping ratio, q the relative amplitude of prescribed support oscillations, and $\Omega = \tilde{\Omega}/\omega_0$ the non-dimensional frequency of this excitation. We consider the case of small but rapid vibrations of the support, i.e., $q \ll 1$ and $\Omega \gg 1$. Approximate solutions can be obtained by a number of different perturbation methods, considering $\varepsilon = \Omega^{-1}$ as a small parameter. By the method to be described subsequently, the motions $\theta(t)$ are split into slow and fast components as follows:

$$\theta(t) = z(t) + \Omega^{-1}\varphi(t, \Omega t), \tag{2}$$

where z describes the *slow motions* (at the time-scale of free pendulum oscillations), and $\Omega^{-1}\varphi$ is a small overlay of *fast motions* (at the rapid rate of support vibrations). We perceive t as the *slow time scale* and Ωt as a *fast time scale*. The slow motions z are those of primary concern, whereas the details of the fast overlay φ are interesting only by their effect on z. By the method to be described, one finds the following approximation for φ:

$$\varphi(t, \Omega t) = -q\Omega \sin(z)\sin(\Omega t) \tag{3}$$

while the slow motions z are approximately governed by

$$\ddot{z} + 2\beta\dot{z} + (1 + \tfrac{1}{2}(q\Omega)^2\cos(z))\sin z = 0, \tag{4}$$

where in both equations small terms of the order Ω^{-1} have been neglected. Hence, by equations (2) and (3) the pendulum motions are given by

$$\theta(t) = z(t) - q\sin(z)\sin(\Omega t). \tag{5}$$

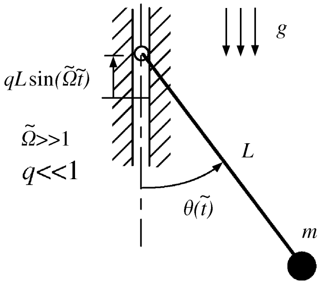


Figure 1. Pendulum with a vibrating support.

It appears, since $q \ll 1$, that the rapidly oscillating overlay is very small; it constitutes the *trivial effect* of the HF excitation. *Non-trivial* effects appear in the dynamics of the slow motions z. As appears from equation (4) z is governed by a differential equation quite similar to the original one (1), though, with the non-autonomous excitation term $q\Omega^2 \sin(\Omega t)$ replaced by the autonomous term $\frac{1}{2}(q\Omega)^2 \cos(z)$, describing an average effect of the HF excitation. The effective restoring force in the presence of HF frequency thus changes from $f_r(z) = \sin(z)$ to

$$f_r(z) = (1 + \tfrac{1}{2}(q\Omega)^2 \cos z)\sin z, \tag{6}$$

which is depicted in Figure 2(a) for three values of the excitation intensity $q\Omega$. Of concern here is the *effective stiffness*, which we define as the slope of the restoring force curve at the static equilibriums. As appears from Figure 2(b), at the two equilibriums corresponding to the straight downward or upward pointing pendulum ($z = 0, \pm\pi$), this stiffness increases in the presence of HF excitation (i.e. when $q\Omega \neq 0$). Specifically, using equation (6) we find that the stiffness for both equilibriums increases by the same quantity $\frac{1}{2}(q\Omega)^2$:

$$\frac{\mathrm{d}f_r}{\mathrm{d}z} = \begin{cases} 1 + \frac{1}{2}(q\Omega)^2 & \text{for } z = 0, \\ -1 + \frac{1}{2}(q\Omega)^2 & \text{for } z = \pm\pi. \end{cases} \tag{7}$$

For the equilibrium at $z = 0$ the change in effective stiffness corresponds to a change in effective natural frequency, from unity to ω_1,

$$\omega_1 = \sqrt{1 + \tfrac{1}{2}(q\Omega)^2}. \tag{8}$$

Thus, near the down-pointing equilibrium, the free pendulum oscillations occur at a higher frequency when the support is vibrating rapidly up and down. For example, with a pendulum clock mounted on a table vibrating at intensity of $q\Omega = 1$, the clock will run $\sqrt{1 + \frac{1}{2}} - 1 \approx 22\%$ faster than if the table is at rest.

For the up-pointing equilibrium at $z = \pm\pi$ the situation is different, because in the absence of HF excitation this equilibrium is unstable—as reflected by negative stiffness (cf., equation (7) and Figure 2(b)). Thus an increase in effective stiffness, if sufficiently large, may stabilize this equilibrium. As appears from equation (7) this occurs when $(q\Omega)^2 > 2$, which is a well-known result (e.g., references [1, 5, 45, 46]). With this condition fulfilled, small disturbances to the upright equilibrium of the pendulum decays at an effective natural frequency ω_2 that increases with the level $q\Omega$ of excitation:

$$\omega_2 = \sqrt{\tfrac{1}{2}q^2\Omega^2 - 1}. \tag{9}$$

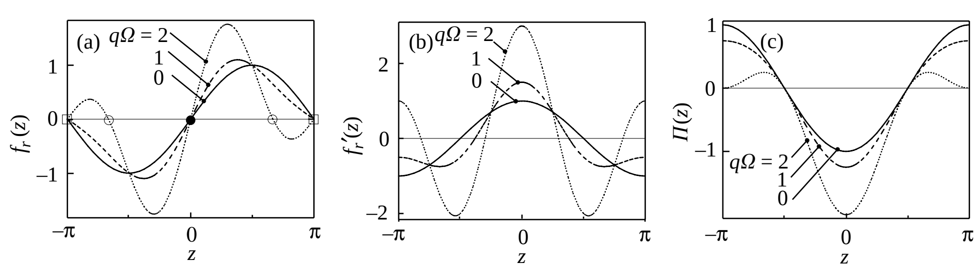


Figure 2. (a) Effective restoring force $f_r(z)$ in the presence of different levels of intensity $q\Omega$ of rapid support vibrations. Equilibrium points: ●, stable; ○, unstable; □, stable if $\mathrm{d}f_r/\mathrm{d}z > 0$ (i.e., only for the dotted curve). (b) Effective linear stiffness $f_r'(z) = \mathrm{d}f_r/\mathrm{d}z$; (c) potential energy $\Pi(z) = \int f_r\,\mathrm{d}z$.

Also, under such conditions two new equilibriums are created, given by $z = \arccos(-2(q\Omega)^{-2})$, for which equation (6) has zeroes only when $\omega_2^2 > 0$. These equilibriums are always unstable, and thus act as "potential barriers" that has to be overcome if the pendulum is to be moved between the two stable equilibriums at $z = 0$ and $\pm\pi$ (cf., the curve for $q\Omega = 2$ in Figure 2(a) and 2(c)).

It appears that the change in effective stiffness, here considered the primary effect, has derived effects in the form of stabilization and creation of new equilibriums. Another derived effect is a change in the effective non-linearity of the system, which can be seen by Taylor-expanding (4), including leading order non-linearities. Then one finds the following approximate equations of motion, valid near the vertical equilibriums:

$$\begin{aligned} &\text{for } z \approx 0: \quad \ddot{z} + 2\beta\dot{z} + (1 + \tfrac{1}{2}q^2\Omega^2)z - \tfrac{1}{6}(1 + 2q^2\Omega^2)z^3 + \mathrm{O}(z^5) = 0, \\ &\text{for } z \approx \pm\pi: \quad \ddot{z} + 2\beta\dot{z} - (1 - \tfrac{1}{2}q^2\Omega^2)(z - \pi) + \tfrac{1}{6}(1 - 2q^2\Omega^2)(z - \pi)^3 + \mathrm{O}((z - \pi)^5) = 0. \end{aligned} \tag{10}$$

Considering the leading non-linear cubical term, it appears that for $z \approx 0$ its softening character becomes more pronounced with increased level $q\Omega$ of excitation, whereas for $z \approx \pi$ the hardening character of the non-linearity becomes softening for sufficiently large $q\Omega$.

The present system may also serve to illustrate an example of the *biasing* effect of HF excitation. For this we imagine that the pendulum is excited and swings in the horizontal plane, so that the gravity term (the number one) is replaced by zero in equations (1) and (4). Then, in the absence of the HF excitation, all positions θ or z are equilibriums for the pendulum, i.e., the pendulum has no preference or bias for pointing in any particular direction. However, as appears from equation (4) with the gravity term zeroed, when $q\Omega \neq 0$ there are four equilibriums: $z = 0, \pi, \pm\pi/2$, and one can easily show that only $z = 0, \pi$ are stable. Hence, in the presence of rapid support excitation the pendulum is biased to line up with the direction of excitation.

The effects mentioned above are not just mathematical artifacts; they can quite easily be demonstrated in the laboratory by using a small pendulum driven, e.g., by an electric jigsaw [47] or loudspeaker [48]. Also, the pendulum equation (1), in particular, if Taylor-expanded to order three, is representative of a great many other systems and structures. For example, a single-mode approximation for a beam with pulsating axial excitation has this form, and thus the effects described above is pertinent for this case as well. Then the results for the upright equilibrium apply if the average axial beam load exceeds the buckling load, whereas the results for the down pointing equilibrium apply for sub-critical loads. The situation with a horizontal pendulum—having no restoring force in the absence of support excitation—corresponds to a beam that has no transverse stiffness at all, i.e., to an untaught string.

2.2. EXAMPLE 2: MASS ON A VIBRATING PLANE (SMOOTHENING AND BIAS)

Figure 3(a) shows the system: a mass m of extension L, attached by a spring of stiffness K to a horizontal plane that vibrates at a small amplitude qL and high frequency $\tilde{\Omega}$. The friction between mass and plane is given by a simple asymmetric Coulomb form, i.e., the friction force is $mg\mu(\dot{X})$, where the coefficient μ depends on the direction of motion, as shown in Figure 3(b) (many materials possess this property, e.g., the skin or fur of animals,

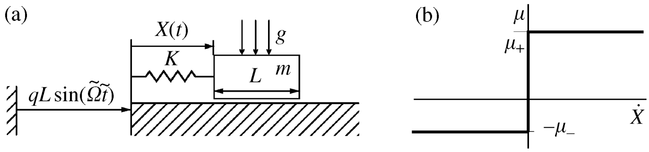


Figure 3. (a) Mass on a vibrating plane; (b) coefficients of dry friction.

as do certain asymmetrical processes such as vibrational piling and penetration):

$$\mu(\dot{X}) = \begin{cases} \mu_+ & \text{for } \dot{X} > 0, \\ -\mu_- & \text{for } \dot{X} < 0, \end{cases} \quad \mu(\dot{X}) \in [-\mu_-; \mu_+] \text{ for } \dot{X} = 0. \tag{11}$$

The non-dimensional equation of motion is

$$\ddot{x} + \omega^2 x + \bar{\mu}\,\mathrm{sgn}(\dot{x}) + \mu_\Delta = q\Omega^2 \sin(\Omega t) \quad (\mathrm{sgn}(0) \in [-1; 1]), \tag{12}$$

where $\bar{\mu}$ denotes the *average* coefficient of friction, and μ_Δ the *asymmetry* in friction:

$$\bar{\mu} = \tfrac{1}{2}(\mu_+ + \mu_-), \quad \mu_\Delta = \tfrac{1}{2}(\mu_+ - \mu_-) \tag{13}$$

and where $x = X/L$ denotes the position, $t = \omega_0 \tilde{t}$ the non-dimensional time ($\tilde{t}$ physical time), $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\omega_0 = \sqrt{g/L}$ a characteristic frequency, $\omega = \sqrt{K/M}/\omega_0$ the natural frequency of small oscillations when there is no friction, q the amplitude of the excitation, and $\Omega = \tilde{\Omega}/\omega_0$ its non-dimensional frequency. By using ω_0 rather than ω as the characteristic frequency, we allow for setting $\omega = 0$ to study what happens when there is no spring. The vibrations from the plane are assumed to be sufficiently strong to cause the mass to slide in both directions, i.e., $q\Omega^2 > \max(\mu_+, \mu_-)$.

Again we determine approximate solutions for the case $\Omega \gg 1$, $q \ll 1$, by splitting motions into slow and fast components:

$$x(t) = z(t) + \Omega^{-1}\varphi(t, \Omega t), \tag{14}$$

where z holds the slow motions, and $\Omega^{-1}\varphi$ is a small overlay of zero average fast motions. Proceeding as for Example 1 above, it is found that the fast motions are approximately

$$\varphi(t, \Omega t) = -q\Omega \sin(\Omega t) \tag{15}$$

while the slow motions are approximately

$$\ddot{z} + \omega^2 z + \bar{\mu}\langle \mathrm{sgn}(\dot{z} - q\Omega\cos(\Omega t))\rangle + \mu_\Delta = 0, \tag{16}$$

where small terms of the order Ω^{-1} have been neglected, and $\langle\ \rangle$ denotes the "fast time" averaging operator, defined for integrable functions $f(t, \Omega t)$ that are 2π-periodic in Ωt:

$$\langle f(t, \tau)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(t, \tau)\,\mathrm{d}\tau; \quad \tau = \Omega t, \quad t \text{ fixed.} \tag{17}$$

Hence, by equations (14) and (15), motions of the mass are given by

$$x(t) = z(t) + q\sin(\Omega t). \tag{18}$$

The average in equation (16) can be calculated by noting that the argument of the signum function is positive for $\Omega t \in [\Omega t_1; \Omega t_2]$, where $\Omega t_1 = \arccos(\dot{z}/q\Omega)$ and $\Omega t_2 =$

$2\pi - \Omega t_1$, and otherwise negative. As a result the equation of slow motions becomes

$$\ddot{z} + \omega^2 z + \bar{f}(\dot{z}) = 0,$$
$$\bar{f}(\dot{z}) = \begin{cases} \bar{\mu}\,\mathrm{sgn}(\dot{z}) + \mu_\Delta & \text{for } |\dot{z}| \geqslant q\Omega, \\ \bar{\mu}\left(1 - \dfrac{2}{\pi}\arccos(\dot{z}/q\Omega)\right) + \mu_\Delta & \text{for } |\dot{z}| < q\Omega. \end{cases} \tag{19}$$

Comparing with the original equation (12) for x, it appears that the non-autonomous excitation term has disappeared, and that the term representing friction force has changed for velocities $|\dot{z}| < q\Omega$. Figure 4 depicts the effective friction force $\bar{f}$. As compared with the unforced case (dashed line), it is seen that the HF excitation effectively *smooths* the discontinuity at $\dot{z} = 0$. In fact, it makes sense to linearize the otherwise essentially non-linear friction force: Taylor-expanding the arccos-term in equation (19) for $|\dot{z}| \ll q\Omega$, one finds that for small velocities the slow motions are governed by

$$\ddot{z} + \bar{\beta}\dot{z} + \mu_\Delta = 0, \quad \bar{\beta} = \frac{2}{\pi}\frac{\bar{\mu}}{q\Omega} \tag{20}$$

from which a well-known result appears (e.g., reference [1]): HF excitation can make dry friction appear as linear viscous damping, with the equivalent damping coefficient gradually vanishing as the frequency of excitation is increased. (Indeed this is an everyday experience, e.g., when placing small objects on a rapidly vibrating surface, they may seem to 'float' with little resistance to motions along the surface.)

Also, as a second effect the HF excitation may introduce *bias*—a preference to certain states over others. First, we note that in the absence of HF excitation the mass is in static equilibrium as long as the spring force is too small to overcome friction, i.e., as long as $-\mu_+ < \omega^2 x < \mu_-$ or, in terms of the average and the asymmetry in friction: $|\omega^2 x + \mu_\Delta| < \bar{\mu}$. To find static equilibriums in the presence of HF excitation we let $\dot{z} = \ddot{z} = 0$ in (19) solve for z and find the solution $z = \tilde{z}$,

$$\tilde{z} = -\frac{\mu_\Delta}{\omega^2}. \tag{21}$$

Thus, the mass is biased to occupy a particular position, rather than a range of positions, an effect that increases with asymmetry in friction and with spring flexibility ($\propto \omega^{-2}$). This kind of bias may explain misreading from scale instruments in strongly vibrating environments.

Another example of bias can occur when there is no restoring spring, i.e., $\omega^2 = 0$. Then it is found, letting $\ddot{z} = 0$ in equation (19) and solving for $\dot{z}$, that a steady equilibrium solution exists, corresponding to the mass moving at the constant speed $\dot{z} = \tilde{\dot{z}}$,

$$\tilde{\dot{z}} = -q\Omega \sin\left(\frac{\pi}{2}\mu_\Delta/\bar{\mu}\right) \approx -\frac{\pi}{2}q\Omega\mu_\Delta/\bar{\mu} \quad \text{for } \mu_\Delta/\bar{\mu} \ll 1. \tag{22}$$

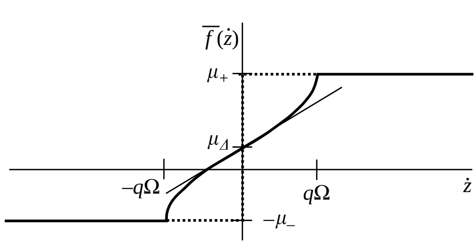


Figure 4. Effective friction force $\bar{f}(\dot{z})$ in the presence of high-frequency excitation of amplitude $q \ll 1$ and frequency $\Omega \gg 1$, as given by equation (19). ——, $q\Omega \neq 0$; - - - - -, $q\Omega = 0$.

This "drift" velocity vanishes as the two coefficients of friction become identical, i.e., as $\mu_\Delta \to 0$. Indeed, the asymmetry need not be in the friction coefficients. A quite similar effect is induced if the vibrating plane is excited along a skew direction (different from vertical and horizontal), or if the mass has an internal degree of freedom oriented at a skew angle with respect to gravity. This kind of bias accounts for many phenomena commonly observed in strongly vibrating environments, e.g., self-loosening of screws and nuts, and vibrational transportation of objects.

2.3. EXAMPLE 3: BRUMBERG'S PIPE (SMOOTHENING AND BIASING)

Figure 5 shows Brumberg's pipe [49], which is shaken horizontally and vertically at high frequencies $\tilde{\Omega}$ and $2\tilde{\Omega}$, respectively, in a gravity field g. Inside the pipe slides a solid mass of characteristic length L and coefficient of dry friction μ. Under proper relative phasing of the two excitations, the longitudinal inertia forces acting on the mass are directed upwards at just those intervals of time where the transverse normal forces, and thus the friction forces, are weakest—whereas at time intervals where the longitudinal inertia is directed downwards, the friction forces are strongest. As a result the mass may travel up the pipe, against gravity, at a constant average speed with a small overlay of rapid oscillations. For the present purpose we consider harmonic excitations with a relative phasing that maximizes this effect (see reference [1] for results with arbitrary phasing):

$$Q_x = -a_x L \sin(2\tilde{\Omega}\tilde{t}), \quad Q_y = a_y L \sin(\tilde{\Omega}\tilde{t}) \tag{23}$$

by which the non-dimensional equation of motion becomes:

$$\ddot{x} + 1 + \mu a_y \Omega^2 |\sin(\Omega t)| \operatorname{sgn}(\dot{x}) + 4 a_x \Omega^2 \sin(2\Omega t) = 0 \quad (\operatorname{sgn}(0) \in [-1; 1]), \tag{24}$$

where $x = x/L$ denotes the position, $t = \omega_0 \tilde{t}$ is the non-dimensional time, $\omega_0 = \sqrt{g/L}$ a characteristic frequency, $\dot{x} = \mathrm{d}x/\mathrm{d}t$, a_x and a_y are the horizontal and vertical excitation amplitudes respectively, and $\Omega = \tilde{\Omega}/\omega_0$ denotes the fundamental excitation frequency. For the magnitudes of parameters we assume the excitation is high in frequency and small in amplitude, and that friction is weak, i.e. $\Omega = \mathrm{O}(\varepsilon^{-1})$, $a_{x,y} = \mathrm{O}(\varepsilon)$, $\mu = \mathrm{O}(\varepsilon)$, $\varepsilon \ll 1$, which implies $a_{x,y}\Omega = \mathrm{O}(1)$ and $\mu\Omega = \mathrm{O}(1)$.

Brumberg obtained an exact but complicated solution to equation (24). Here, we follow reference [1] and split the motions into slow and fast components:

$$x(t) = z(t) + \Omega^{-1} \varphi(t, \Omega t), \tag{25}$$

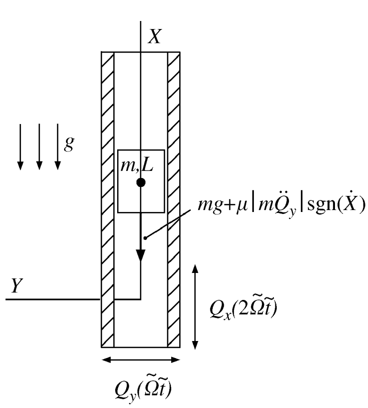


Figure 5. Brumberg's pipe.

where φ is a small overlay of fast oscillations having zero average:

$$\varphi(t, \Omega t) = a_x \Omega \sin(2\Omega t) \tag{26}$$

and z describes the slow motions, approximately governed by

$$\ddot{z} + 1 + \mu a_y \Omega^2 \langle \mathrm{sgn}(\dot{z} + 2a_x \Omega \cos(2\Omega t)) |\sin(\Omega t)| \rangle = 0, \tag{27}$$

where $\langle\ \rangle$ is the averaging operator defined by equation (17). Calculating the average in question, the final equation for the slow motions becomes

$$\ddot{z} + 1 + \frac{2}{\pi} \mu a_y \Omega^2 \bar{h}(\dot{z}) = 0, \quad \bar{h}(\dot{z}) = \begin{cases} \mathrm{sgn}(\dot{z}) & \text{for } |\dot{z}| \geqslant 2a_x\Omega, \\ 1 - \sqrt{2(1 - \dot{z}/(2a_x\Omega))} & \text{for } |\dot{z}| < 2a_x\Omega. \end{cases} \tag{28}$$

Seeking stationary solutions corresponding to the mass moving at the constant average speed $\tilde{\dot{z}}$ up the pipe, we let $(\dot{z}, \ddot{z}) = (\tilde{\dot{z}}, 0)$ and find that

$$\tilde{\dot{z}} = a_x \Omega \left(2 - \left(1 + \frac{1}{(2/\pi)\mu a_y \Omega^2} \right)^2 \right), \quad \frac{2(\sqrt{2} - 1)}{\pi} \mu a_y \Omega^2 \geqslant 1, \tag{29}$$

where the latter condition ensures existence and stability of the solution, and the main assumptions behind the analysis should be recalled: $a_{x,y}\Omega = \mathrm{O}(1)$, $\mu\Omega = \mathrm{O}(1)$, and $\Omega \gg 1$. Hence, in the presence of HF excitation of proper strength and phasing, the mass is biased to move at constant speed against gravity.

Brumberg's pipe shares features with the mass on a vibrating plane (Example 2 above), as is revealed when Taylor-expanding (28) for small $\dot{z}$, rearranging, and neglecting small terms of order $\dot{z}^2$ and higher:

$$\ddot{z} + \frac{1}{\sqrt{2\pi}} \mu \Omega \frac{a_y}{a_x} \dot{z} = \frac{2(\sqrt{2} - 1)}{\pi} \mu a_y \Omega^2 - 1. \tag{30}$$

Two features are apparent: (1) A smoothening effect, where the essentially non-linear dissipation term in the original equation (24) (representing dry friction) is replaced by a linearizable dissipative term (representing equivalent viscous damping); and (2) A bias effect, represented by the first term of the right-hand side of equation (30), which may change the original bias of the system mass from downwards to upwards in gravity.

From a mathematical point of view, the distinctive feature of this example is the occurrence of a velocity variable in a rapidly oscillating term (here $\dot{x}$ in equation (24)). This may prevent using perturbation methods unless certain restrictions are imposed on the order magnitudes of parameters, since with HF excitation velocities can be large even if fluctuations in positions are small. Hence terms of this type should be included in the general model to be discussed.

2.4. OTHER EXAMPLES

Figure 6 shows a range of recently examined systems displaying stiffening, biasing, smoothening, or combined effects to HF excitation—with brief descriptions and literature references given in the legends. All of them are covered by the general mathematical model to be described next. Blekhman's book [1] should be consulted for earlier work on a wide variety of similar systems and phenomena, including granular materials, fluid flow, synchronization, vibrorheology, vibrotransportation, industrial processes, celestial mechanics, and "dynamic" materials.

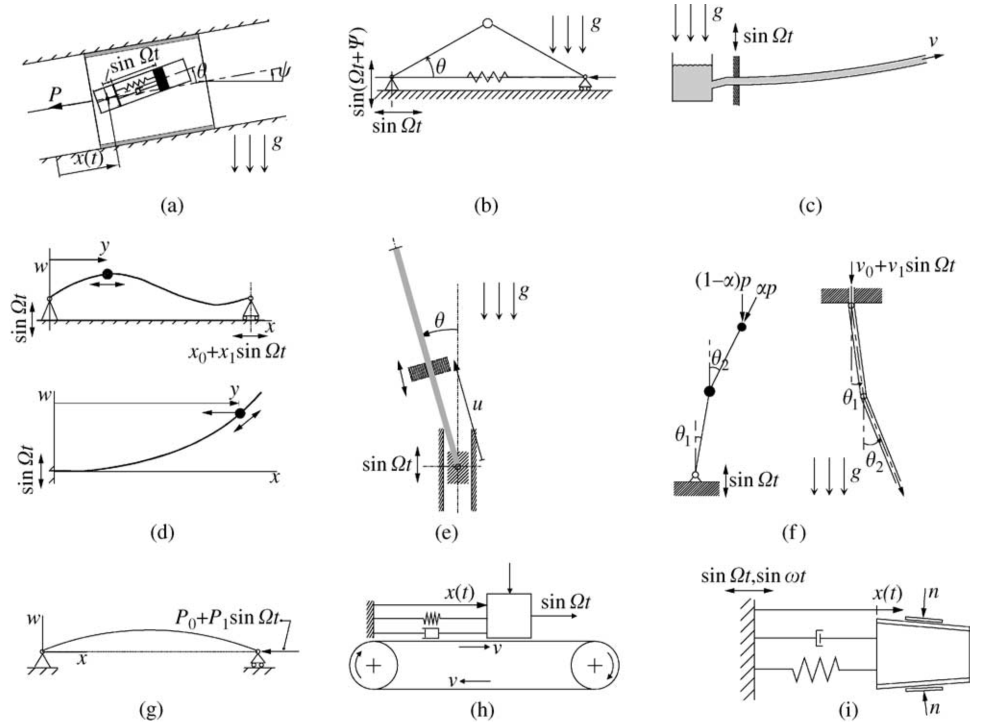


Figure 6. High-frequency excited systems, representative of the effects of stiffening (b,e,f,g), biasing (a,b,c,d,e,h), and smoothening (a,h,i). (a) Vibration-induced movement using friction layers and a HF-resonator [13, 14]. (b) Change of equilibriums, stability, natural frequencies, and non-linear response for a HF-excited two-bar link [15]. (c) Using resonant HF excitation to pump fluid [7] or continuous material [8] through pipes. (d) Using vibration-induced sliding to damp resonant vibrations of strings and beams [9–12, 55]. (e) Chelomei's pendulum: has a freely sliding disk. With HF excitation both the disk and the pendulum stabilizes against gravity [16, 25, 26]. (f) Stabilization and change of non-linear behavior for follower loaded double pendulums with HF support excitation (left) [22, 23], and for articulated pipes carrying HF pulsating fluid (right) [24]. (g) Increasing the buckling load and natural frequencies for a column using HF excitation [16–18]. (h) Quenching friction-induced stick–slip vibration by using HF excitation [20]. (i) Quenching chaotic oscillations for a stick–slip friction oscillator using HF excitation [21].

3. THE GENERAL SYSTEM, AND ITS AVERAGED FORM

3.1. SCOPE AND APPROACH

We consider a general class of systems with HF excitation, broad enough to cover a large variety of systems of scientific and industrial interest, but also sufficiently specific to provide results that are physically interpretable. The purpose of the analysis is to set up expressions quantifying the stiffening, the biasing, and the smoothening effects for this general system. As should be clear from the above examples, the effects are understood to characterize the *averaged* behavior of the system, i.e., what can be observed when ignoring the details of the small overlay of HF oscillations. This corresponds to lowpass filtering the observed response—either digitally, if the response consists of computer simulated or real sampled data, or analogously, as with response perceived by band limited devices such as physical instruments and human senses.

However, we here want to make predictions regarding the filtered response, without actually knowing the unfiltered response: Since the general system to be studied is non-linear, setting up general solutions is not possible. Instead we transform the original problem into another one whose solution is the filtered response, i.e., its equations govern the *slow* or average motions of the system. For this one can consider Ω^{-1} as a small parameter, Ω being a fundamental frequency of the excitation, and then employ a number of different perturbation techniques, e.g., standard averaging or multiple time-scaling methods (see references [15, 29, 30, 38, 50] for example using multiple time scaling for systems with HF excitation, or reference [50] on using generalized KBM averaging).

This work employs a method which is especially convenient for dealing with non-resonantly HF-excited systems, the so-called *Method of Direct Separation of Motions* [1]. By this, a set of generally non-linear differential equations is transformed into two exact subsets, that may each be approximately solved: one describing the fast components of motion, and the other describing slow components, the latter typically being those of primary interest. Originating from Kapitza's heuristic approach for a specific problem [5], this method was generalized and applied to a wide variety of physical systems and phenomena by Blekhman (e.g., references [1, 51], see references [14, 20, 22, 26, 39] for recent applications).

3.2. THE SYSTEM

We consider dynamical systems that can be modelled by a finite number of second order ordinary differential equations, generally non-linear, and with time-explicit HF excitation:

$$\mathbf{M}(\mathbf{u}, t)\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} + \mathbf{s}\left(\mathbf{u}, \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}, t\right) + \sum_{j=1}^{m}\left(\mathbf{h}_j(\mathbf{u}, \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}, t) + \Omega\mathbf{f}_j(\mathbf{u}, t)\right)\frac{\partial^2}{\partial(\Omega t)^2}\xi_j(t, \Omega t) = \mathbf{0},$$

$$\mathbf{u} = \mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0, \quad (\mathrm{d}\mathbf{u}/\mathrm{d}t)_{t=0} = \dot{\mathbf{u}}_0, \quad \Omega \gg 1, \tag{31}$$

where $\mathbf{u}$ is an n-vector describing the positional state of the system at time t, $\mathbf{M}$ is a positive definite $(n \times n)$-matrix describing inertia forces, $\mathbf{s}$ is an n-vector of "slow" forces, Ω is a large number representing a fundamental excitation frequency, $\mathbf{h}_j$ and $\mathbf{f}_j$ are n-vectors that, jointly with the scalar time functions ξ_j, describe "fast" or rapidly oscillating excitations, and the functions ξ_j and their first and second derivatives with respect to Ωt are 2π-periodic in Ωt with zero average (i.e., possible non-zero averages should be extracted and moved to the $\mathbf{s}$ functions). Thus, the three main terms of the matrix equation describe a balance between inertia forces ($\mathbf{M}$), slowly changing internal and external forces ($\mathbf{s}$), and rapidly oscillating forces ($\mathbf{h}_j, \mathbf{f}_j, \xi_j$). The class of systems described by equation (31) is sufficiently broad to cover many applications involving finite-degree-of-freedom or discretized continuous mechanical systems, including those described in section 2 and Figure 6.

The above notions of "slow" and "fast" refer to two distinct characteristic time scales or frequencies characterizing motions of the system: There is a time scale t and a characteristic frequency ω describing motions of the system when all $\xi_j = 0$, for example, one can take as ω the largest natural frequency of the linearized unloaded system, i.e., the largest real root of the characteristic polynomial $|\mathbf{J}(\mathbf{0},\mathbf{0},t) - \omega^2\mathbf{M}(\mathbf{0},t)| = 0$, where $\mathbf{J} \equiv \partial\mathbf{s}/\partial\mathbf{u}$. Then there is another time scale Ωt, much faster than t, which describes fluctuations imposed by external excitations of characteristic frequency $\Omega \gg \omega$, with time variations specified by ξ_j. It is further assumed that all functions in equation (31) are generally non-linear functions of their arguments, that they are of magnitude order unity or lower (i.e., $\mathrm{O}(\|\mathbf{M}\|) = \mathrm{O}(\|\mathbf{s}\|) = \mathrm{O}(\|\mathbf{h}_j\|) = \mathrm{O}(\|\mathbf{f}_j\|) = \mathrm{O}(\|\mathbf{M}\|) = \mathrm{O}(\xi_j) \leqslant 1$, where O is the order symbol), that ξ_j are bounded on $[0; 2\pi]$, and that $\mathbf{f}_j$ and $\mathbf{M}$ are bounded with continuous first derivatives with respect to $\mathbf{u}$ (whereas, $\mathbf{h}_j$ and $\mathbf{s}$ are not necessarily continuous). The assumptions on magnitude order and boundedness are not required to hold, in general (this would preclude even linear functions such as $\mathbf{s}=\mathbf{u}$), but only for the type of solutions under consideration; typically any kind of unstable or strongly resonant solution causes the assumptions to be violated.

Next, we mention example functions of the above type, and then attempt to approximate equation (31) by a simpler system, in which time variations of $\mathbf{u}$ on the fast scale Ωt is accounted for only by their integrated influence on the slow—or average—behavior of the system.

3.3. EXAMPLE FUNCTIONS

For the example systems of section 2 one has, for the pendulum with a vibrating support (section 2.1): $\mathbf{u}=\theta$, $\mathbf{M}=1$, $\mathbf{s}=2\beta\dot{\theta}+\sin\theta$, $\mathbf{h}=0$, $\mathbf{f}=-q\Omega\sin\theta$, $\xi=-\sin\Omega t$; for the mass on a vibrating plane (section 2.2): $\mathbf{u}=u$, $\mathbf{M}=1$, $\mathbf{s}=\omega^2 u+\bar{\mu}\,\mathrm{sgn}(\dot{u})+\mu_\Delta$, $\mathbf{h}=0$, $\mathbf{f}=-q\Omega$, $\xi=-\sin\Omega t$; and for Brumberg's pipe (section 2.3): $\mathbf{u}=x$, $\mathbf{M}=1$, $\mathbf{s}=1$, $\mathbf{h}_1=0$, $\mathbf{f}_1=-a_x\Omega$, $\xi_1=\sin(2\Omega t)$, $\mathbf{h}_2=-\mu a_y\Omega^2\,\mathrm{sgn}(\dot{x})$, $\mathbf{f}_2=0$, $\xi_2=|\sin(\Omega t)|-2/\pi$.

3.4. THE AVERAGED SYSTEM GOVERNING "SLOW" MOTIONS

To analyze the general system (31) a fast time is introduced as a new independent variable:

$$\tau=\Omega t \tag{32}$$

and it is assumed that solutions can be separated into slow and fast components as follows:

$$\mathbf{u}=\mathbf{u}(t,\tau)=\mathbf{z}(t)+\Omega^{-1}\boldsymbol{\varphi}(t,\tau), \tag{33}$$

where $\mathbf{z}(t)$ holds the "slow" or average motions, and $\Omega^{-1}\boldsymbol{\varphi}$ is a rapidly oscillating overlay which has small amplitude, is 2π-periodic in the fast time τ, and has zero fast-time average:

$$\langle\boldsymbol{\varphi}(t,\tau)\rangle\equiv\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{\varphi}(t,\tau)\,\mathrm{d}\tau=0 \tag{34}$$

where $\langle\ \rangle$ defines the fast-time averaging operator. Note that $\langle\ \rangle$ is a linear operator, and that for any function $h(t)$ or $h(t,\tau)$ it holds that: $\langle h(t)\rangle=h(t)$, $\langle\partial h(t,\tau)/\partial t\rangle=\partial\langle h(t,\tau)\rangle/\partial t$, and, if h is 2π–periodic in τ, $\langle\partial h(t,\tau)/\partial\tau\rangle=\langle\partial^2 h(t,\tau)/\partial\tau^2\rangle=0$. By these definitions and formulas one has

$$\langle\mathbf{u}\rangle=\mathbf{z}(t) \tag{35}$$

so that $\mathbf{z}(t)$ describes the fast-time average of the total motion $\mathbf{u}$. Also, time derivatives of $\mathbf{u}$ transform into partial derivatives with respect to the two times scales t and τ as follows:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\dot{\mathbf{z}}+\boldsymbol{\varphi}'+\Omega^{-1}\dot{\boldsymbol{\varphi}},\quad \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2}=\Omega\boldsymbol{\varphi}''+\ddot{\mathbf{z}}+2\dot{\boldsymbol{\varphi}}'+\Omega^{-1}\ddot{\boldsymbol{\varphi}}, \tag{36}$$

where $(\dot{\cdot})\equiv\partial/\partial t$ and $()'\equiv\partial/\partial\tau$. Inserting this and equation (33) into equation (31) one obtains

$$\mathbf{M}(\mathbf{z}+\Omega^{-1}\boldsymbol{\varphi},t)(\Omega\boldsymbol{\varphi}''+\ddot{\mathbf{z}}+2\dot{\boldsymbol{\varphi}}'+\Omega^{-1}\ddot{\boldsymbol{\varphi}})+\mathbf{s}(\mathbf{z}+\Omega^{-1}\boldsymbol{\varphi},\dot{\mathbf{z}}+\boldsymbol{\varphi}'+\Omega^{-1}\dot{\boldsymbol{\varphi}},t)$$

$$+\sum_j(\mathbf{h}_j(\mathbf{z}+\Omega^{-1}\boldsymbol{\varphi},\dot{\mathbf{z}}+\boldsymbol{\varphi}'+\Omega^{-1}\dot{\boldsymbol{\varphi}},t)+\Omega\mathbf{f}_j(\mathbf{z}+\Omega^{-1}\boldsymbol{\varphi},t))\xi_j''(t,\tau)=\mathbf{0} \tag{37}$$

or Taylor-expanding for small Ω^{-1} and rearranging

$$\begin{aligned}\varphi'' = & -\mathbf{M}^{-1}(\mathbf{z},t)\sum_j \mathbf{f}_j(\mathbf{z},t)\xi_j''(t,\tau) - \Omega^{-1}(\ddot{\mathbf{z}} + 2\dot{\boldsymbol{\varphi}}') - \Omega^{-1}\mathbf{M}^{-1}(\mathbf{z},t)\mathbf{s}(\mathbf{z},\dot{\mathbf{z}}+\boldsymbol{\varphi}',t) \\ & - \Omega^{-1}\mathbf{M}^{-1}(\mathbf{z},t)\left\{\sum_j (\mathbf{h}_j(\mathbf{z},\dot{\mathbf{z}}+\boldsymbol{\varphi}',t) + \nabla\mathbf{f}_j(\mathbf{z},t)\boldsymbol{\varphi})\xi_j''(t,\tau) + \nabla\mathbf{m}(\mathbf{z},t,\boldsymbol{\varphi})\right\} + \mathrm{O}(\Omega^{-2}),\end{aligned} \tag{38}$$

where ∇ is used to indicate derivatives with respect to position variables as follows:

$$\begin{aligned}\nabla\mathbf{f}_j(\mathbf{u},t) &\equiv \frac{\partial \mathbf{f}_j}{\partial \mathbf{u}} = \left[\frac{\partial \mathbf{f}_j}{\partial \mathbf{u}_{(1)}} \quad \cdots \quad \frac{\partial \mathbf{f}_j}{\partial \mathbf{u}_{(n)}}\right], \\ \nabla\mathbf{m}_{(i)}(\mathbf{u},t,\boldsymbol{\varphi}) &\equiv (\boldsymbol{\varphi}'')^{\mathrm{T}}\nabla\mathbf{M}_{(i)}\boldsymbol{\varphi}, \\ \nabla\mathbf{M}_{(i)}(\mathbf{u},t) &\equiv \frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}} = \left[\frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}_{(1)}} \quad \cdots \quad \frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}_{(n)}}\right],\end{aligned} \tag{39}$$

where a subscript in parenthesis denotes a particular vector element or matrix column, and T denotes vector or matrix transpose. Solving equation (38) for $\boldsymbol{\varphi}$ one finds that

$$\boldsymbol{\varphi}(t,\tau) = \hat{\boldsymbol{\varphi}}(t,\tau) + \mathrm{O}(\Omega^{-1}), \tag{40}$$

where $\hat{\boldsymbol{\varphi}}$ is the zero order approximate solution that will be used in place of $\boldsymbol{\varphi}$, satisfying $\langle\hat{\boldsymbol{\varphi}}\rangle = 0$ and $|\hat{\boldsymbol{\varphi}}| = \mathrm{O}(1)$,

$$\hat{\boldsymbol{\varphi}}(t,\tau) = -\mathbf{M}^{-1}(\mathbf{z},t)\sum_j \mathbf{f}_j(\mathbf{z},t)\xi_j(t,\tau). \tag{41}$$

To obtain an equation governing the slow motions $z(t)$ we employ the averaging operator $\langle\ \rangle$ to equation (38), recalling that $\langle\xi_j\rangle = \langle\xi_j'\rangle = \langle\xi_j''\rangle = 0$ by assumption, and then insert solution (40) for $\boldsymbol{\varphi}$, multiply by $\Omega\mathbf{M}$, neglect terms of order Ω^{-1} and lower, rearrange, and obtain

$$\mathbf{M}(\mathbf{z},t)\ddot{\mathbf{z}} + \mathbf{s}(\mathbf{z},\dot{\mathbf{z}},t) + \mathbf{v}(\mathbf{z},\dot{\mathbf{z}},t) = \mathbf{0}, \tag{42}$$

where $\mathbf{v}(\mathbf{z}, \dot{\mathbf{z}}, t)$ defines an example of what reference [1] calls *vibrational force*:

$$\begin{aligned}\mathbf{v}(\mathbf{z},\dot{\mathbf{z}},t) = & \langle\mathbf{s}(\mathbf{z},\dot{\mathbf{z}}+\hat{\boldsymbol{\varphi}}',t) - \mathbf{s}(\mathbf{z},\dot{\mathbf{z}},t)\rangle \\ & + \sum_j \langle\mathbf{h}_j(\mathbf{z},\dot{\mathbf{z}}+\hat{\boldsymbol{\varphi}}',t)\xi_j''(t,\tau)\rangle \\ & + \sum_j \nabla\mathbf{f}_j(\mathbf{z},t)\langle\hat{\boldsymbol{\varphi}}\xi_j''(t,\tau)\rangle + \langle\nabla\mathbf{m}(\mathbf{z},t,\hat{\boldsymbol{\varphi}})\rangle\end{aligned} \tag{43}$$

with $\hat{\boldsymbol{\varphi}}$ given by equation (41).

It appears that equation (42) for the slow motions $\mathbf{z}$ is similar in form to the original equation of motion (31) for the full motions $\mathbf{u}$, though, with the integrated influence of *fast forces* $(\mathbf{h}_j + \Omega\mathbf{f}_j)\xi_j''$ accounted for by the vibrational forces $\mathbf{v}$, which depends only on the slowly changing variables $\mathbf{z}$ and t. Any explicit dependence on the fast time variable $\tau = \Omega t$ has been averaged out, which makes the equation for $\mathbf{z}$ easier to solve than the original equation for $\mathbf{u}$. This holds for analytical solutions, not least because there are powerful mathematical tools that applies only to autonomous systems. And it holds for numerical solutions, where much larger time steps can be used because there is no need to keep track of the rapid oscillations at frequency Ω, and the numerical ill-posedness or "stiffness" is reduced that is associated with solving systems of differential equations on vastly different time scales.

The initial conditions needed to solve equation (42) for $\mathbf{z}$ is obtained from the original initial conditions in equation (31), which transforms under equations (33), (36), and (40)–(41) into

$$\mathbf{z}(0) = \mathbf{u}_0 - \Omega^{-1}\hat{\boldsymbol{\varphi}}(0,0), \quad \dot{\mathbf{z}}(0) = \dot{u}_0 - \hat{\boldsymbol{\varphi}}'(0,0) - \Omega^{-1}\dot{\hat{\boldsymbol{\varphi}}}(0,0). \tag{44}$$

This completes the separation of the full motions $\mathbf{u}$ into slow and fast components $\mathbf{z}$ and $\boldsymbol{\varphi}$, as given by equations (33) and (41), and equations (42)–(43).

3.5. INTERPRETATION OF AVERAGED FORCING TERMS

In equation (42) for the slow motions $\mathbf{z}$, the vibrational forces $\mathbf{v}$ describe average effects. They correspond to real physical forces having the same effect as the HF excitation, on the average. To an observer or measuring instrument that is filtering out small vibrations at high frequencies, the response $\mathbf{u}$ from equation (31) is similar to the response $\mathbf{z}$ from equation (42). It should be recalled that the average of the HF excitation itself is assumed to be zero, $\langle \xi_j \rangle = 0$. In many cases its average *effect* is also zero, $\mathbf{v} = \mathbf{0}$. So, it is instructive to consider each of the terms making up $\mathbf{v}$, to see when non-zero average effects can appear:

The first term in equation (43) expresses the average of the non-linear velocity-dependent terms of the slow forces $\mathbf{s}$. This term vanishes if $\mathbf{s}$ is independent of velocity, or is linear in velocity. However, it may be non-zero when $\mathbf{s}$ is non-linear in velocity terms, such as for systems with dry friction (cf., the examples in sections 2.2, 2.3, and Figure 6(h,i)) or non-linear damping.

The second term expresses the average effect of the velocity-dependent parts $\mathbf{h}_j$ of the fast forces. This term disappears when the fast forces are independent of velocity, but generally does not disappear even if the velocity dependence is linear. The example systems of sections 2.3 and Figure 6(a,f(right)) display non-zero average terms of this type.

The third and fourth terms describe the average effect of strong, velocity-independent parametrical excitation terms. These terms disappear if $\mathbf{f}_j$ and $\mathbf{M}$ are independent on $\mathbf{u}$. Non-zero averages of this kind of terms are quite common where non-trivial effects of HF excitation appear, e.g., they occur with the example systems of section 2.1 and Figure 6(a–g).

It is important to note that the vibrational force is generally a non-linear function of ξ_j, since $\hat{\boldsymbol{\varphi}}$ depends on ξ_j (cf., equation (41)). Consequently, if $\mathbf{v}_1$ is the vibrational force corresponding to a specific time variation of the excitation $\xi_1(t, \Omega t)$, and $\mathbf{v}_2$ is the force corresponding to another excitation $\xi_2(t, \Omega t)$, then the force $\mathbf{v}$ corresponding to the simultaneous action of both excitations does *not*, in general, equal $\mathbf{v}_1 + \mathbf{v}_2$. Thus one cannot readily predict the effects of combined excitations based on the knowledge of isolated effects. However, consideration to equation (43) shows that when $\mathbf{M}$ and $\mathbf{f}_j$ are independent of $\mathbf{u}$, and $\mathbf{h}_j$ is independent of $\mathrm{d}\mathbf{u}/\mathrm{d}t$, and $\mathbf{s}$ is linear or independent of $\mathrm{d}\mathbf{u}/\mathrm{d}t$—then one can indeed calculate the vibrational force for a sum of excitations by summing the vibrational forces for each excitation. None of the examples in section 2 or Figure 6 fulfills this necessary condition for the validity of superposition.

4. THE EFFECTS

Next, we make general predictions for the stiffening, biasing, and smoothening effects, based on the averaged equations (42) with vibrational forces as given by equation (43). It should be noted that the result (42) is valid on the quite general assumptions stated below the original system (31), whereas various additional assumptions may be stated during the following analysis. However, it will not be explicitly stated whether in each case the system functions $\mathbf{M}$, $\mathbf{s}$, $\mathbf{h}$, and $\mathbf{f}$ are allowed to depend on the slow time t, since it will be obvious from the context; thus the t in e.g., $\mathbf{f}(\mathbf{u}, t)$ will be kept even when $\mathbf{f}$ is actually not allowed to vary with t.

We shall base several results on a linearization of the averaged general system (42)–(43) near a possible static equilibrium $(\mathbf{z}, \dot{\mathbf{z}}) = (\tilde{\mathbf{z}}, \mathbf{0})$, where $\tilde{\mathbf{z}}$ is defined by

$$\mathbf{s}(\tilde{\mathbf{z}}, \mathbf{0}, t) + \mathbf{v}(\tilde{\mathbf{z}}, \mathbf{0}, t) = \mathbf{0}. \tag{45}$$

The linearization is

$$\mathbf{M}(\tilde{\mathbf{z}}, t)\ddot{\mathbf{z}} + (\mathbf{C}(\tilde{\mathbf{z}}) + \Delta\mathbf{C}(\tilde{\mathbf{z}}))\dot{\mathbf{z}} + (\mathbf{K}(\tilde{\mathbf{z}}) + \Delta\mathbf{K}(\tilde{\mathbf{z}}))(\mathbf{z} - \tilde{\mathbf{z}}) = \mathbf{0}, \tag{46}$$

where, with derivative operators $\nabla\mathbf{g}(\mathbf{u}, \dot{\mathbf{u}}) \equiv \partial\mathbf{g}/\partial\mathbf{u}$ and $\dot{\nabla}\mathbf{g}(\mathbf{u}, \dot{\mathbf{u}}) \equiv \partial\mathbf{g}/\partial\dot{\mathbf{u}}$,

$$\begin{aligned} \mathbf{K}(\tilde{\mathbf{z}}) &= \nabla\mathbf{s}(\tilde{\mathbf{z}}, \mathbf{0}, t), \quad \Delta\mathbf{K}(\tilde{\mathbf{z}}) = \nabla\mathbf{v}(\tilde{\mathbf{z}}, \mathbf{0}, t), \\ \mathbf{C}(\tilde{\mathbf{z}}) &= \dot{\nabla}\mathbf{s}(\tilde{\mathbf{z}}, \mathbf{0}, t), \quad \Delta\mathbf{C}(\tilde{\mathbf{z}}) = \dot{\nabla}\mathbf{v}(\tilde{\mathbf{z}}, \mathbf{0}, t). \end{aligned} \tag{47}$$

4.1. STIFFENING

By stiffness we refer to the change in resistive static force per unit deformation near a static equilibrium. Examples of stiffening appear in section 2.1 and Figure 6(b,e,f,g). For convenience it is assumed that in the absence of HF excitation the general system (31) has a static equilibrium at $\mathbf{u} = \mathbf{0}$, so that $\mathbf{s}(\mathbf{0}, \mathbf{0}, t) = \mathbf{0}$ (other equilibriums $\mathbf{u} = \tilde{\mathbf{u}} \neq \mathbf{0}$ are treated by a transform of co-ordinates, $\mathbf{u} \to \mathbf{u} - \tilde{\mathbf{u}}$), and that $\mathbf{s}$ has continuous derivatives with respect to $\mathbf{u}$ at that equilibrium. It is also assumed that this equilibrium remains an equilibrium in the presence of HF excitation, i.e., $\mathbf{v}(\mathbf{0}, \mathbf{0}, 0) = \mathbf{0}$ (the case $\mathbf{v}(\mathbf{0}, \mathbf{0}, 0) \neq \mathbf{0}$ is treated in section 4.2 on biasing). We then define the stiffening effect of HF excitation as the additional infinitesimal change in effective (generalized) force, be it positive or negative, which appears in response to an infinitesimal change in position near the equilibrium. Using equations (46)–(47) with $\tilde{\mathbf{z}} = 0$ we find that the stiffness changes from $\mathbf{K}(\mathbf{0})$ to $\mathbf{K}(\mathbf{0}) + \Delta\mathbf{K}(\mathbf{0})$,

$$\mathbf{K}(\mathbf{0}) = \nabla\mathbf{s}(\mathbf{0}, \mathbf{0}, t), \quad \Delta\mathbf{K}(\mathbf{0}) = \nabla\mathbf{v}(\mathbf{0}, \mathbf{0}, t) \tag{48}$$

Only in rare cases will the slow forces $\mathbf{s}$ contribute to the stiffening effect $\Delta\mathbf{K}(\mathbf{0})$. First, conferring with equation (43) one finds that only those components of $\mathbf{s}$ that are non-linear in velocity *and* linear in position can contribute to $\Delta\mathbf{K}(\mathbf{0})$. For the examples in this paper, only the one in Figure 6(i) has such a component, and in that case the relevant average vanishes at the equilibrium. Similarly, the velocity-dependent part $\mathbf{h}_j$ of the fast forces may theoretically contribute to the stiffening effect, but it does not occur for the examples in this paper.

So, the main responsible terms for the stiffening effects are the two last ones of equation (43); we denote their contribution by $\Delta\mathbf{K}^{\mathbf{f},\mathbf{m}}(\mathbf{0})$, so that by equations (48) and (43):

$$\Delta\mathbf{K}^{\mathbf{f},\mathbf{m}}(\mathbf{0}) = \frac{\partial}{\partial\mathbf{z}}\bigg|_{\mathbf{z}=\mathbf{0}} \left(\sum_j \nabla\mathbf{f}_j(\mathbf{z}, t)\langle\hat{\boldsymbol{\varphi}}\xi_j''(t, \tau)\rangle + \langle\nabla\mathbf{m}(\mathbf{z}, t, \hat{\boldsymbol{\varphi}})\rangle \right). \tag{49}$$

Inserting equation (41) for $\hat{\boldsymbol{\varphi}}$, we find that the first averaging term becomes

$$\langle\hat{\varphi}\xi_j''(t, \tau)\rangle = -\mathbf{M}^{-1}(\mathbf{z}, t)\sum_k \langle\xi_k\xi_j''\rangle\mathbf{f}_k(\mathbf{z}, t) = \mathbf{M}^{-1}(\mathbf{z}, t)\sum_k \langle\xi_k'\xi_j'\rangle\mathbf{f}_k(\mathbf{z}, t), \tag{50}$$

where the last equality follows from the fact that $\langle\xi_k\xi_j''\rangle = -\langle\xi_k'\xi_j'\rangle$ when ξ_k is 2π-periodic in τ and has zero average (this can be verified by representing ξ_k by its Fourier series and calculating the average of $\xi_k\xi_j''$). In particular $-\langle\xi_k\xi_k''\rangle = (\xi_k')^2$, which is the squared r.m.s. value of the velocity of the excitation.

Similarly, for the ith component of the second averaging term in equation (49) one finds

$$\begin{aligned}\langle \nabla \mathbf{m}_{(i)}(\mathbf{z}, t, \hat{\boldsymbol{\varphi}})\rangle &= \langle (\hat{\boldsymbol{\varphi}}'')^{\mathrm{T}} \nabla \mathbf{M}_{(i)} \hat{\boldsymbol{\varphi}}\rangle \\ &= \langle (\mathbf{M}^{-1}(\mathbf{z}, t) \sum_j \mathbf{f}_j(\mathbf{z}, t)\xi_j'')^{\mathrm{T}} \nabla \mathbf{M}_{(i)}(\mathbf{z}, t)\mathbf{M}(\mathbf{z}, t)^{-1} \sum_k \mathbf{f}_k(\mathbf{z}, t)\xi_k\rangle \\ &= \sum_{j,k} \langle \xi_j' \xi_k' \rangle (\mathbf{M}^{-1}(\mathbf{z}, t)\mathbf{f}_j(\mathbf{z}, t))^{\mathrm{T}} \nabla \mathbf{M}_{(i)}(\mathbf{z}, t)\mathbf{M}^{-1}(\mathbf{z}, t)\mathbf{f}_k(\mathbf{z}, t). \end{aligned} \tag{51}$$

Inserting these results into equation (49) and rearranging, the result becomes

$$\Delta \mathbf{K}^{\mathbf{f},\mathbf{m}}(\mathbf{0}) = \sum_{j,k} \langle \xi_j' \xi_k' \rangle \frac{\partial}{\partial \mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{z}=\mathbf{0}} \left(\nabla \mathbf{f}_j \mathbf{M}^{-1} \mathbf{f}_k + \begin{Bmatrix} (\mathbf{M}^{-1}\mathbf{f}_j)^{\mathrm{T}} \nabla \mathbf{M}_{(1)} \mathbf{M}^{-1} \mathbf{f}_k \\ \vdots \\ (\mathbf{M}^{-1}\mathbf{f}_j)^{\mathrm{T}} \nabla \mathbf{M}_{(n)} \mathbf{M}^{-1} \mathbf{f}_k \end{Bmatrix} \right). \tag{52}$$

To examine the meaning of this, we first note that the second term vanishes if $\mathbf{M}$ is constant, since then $\nabla \mathbf{M}_{(i)} = \mathbf{0}$. But this is just what can always be achieved by multiplying equation (31) by $\mathbf{M}^{-1}$ (since $|\mathbf{M}| > 0$ is assumed), whereby an equivalent system is obtained with a new mass matrix $\mathbf{M} = \mathbf{I}$, the unit diagonal matrix. Hence this second term carries no essential information which cannot be inferred from the first term. We term the contribution to stiffness from this important first term (with $\mathbf{M} = \mathbf{I}$) *parametrically induced stiness*, and denote it by $\Delta \mathbf{K}^{\mathbf{f}}(\mathbf{0})$:

$$\Delta \mathbf{K}^{\mathbf{f}}(\mathbf{0}) = \sum_{j,k} \langle \xi_j' \xi_k' \rangle \frac{\partial((\nabla \mathbf{f}_j)\mathbf{f}_k)}{\partial \mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{z}=\mathbf{0}} = \sum_{j,k} \langle \xi_j' \xi_k' \rangle (\nabla \mathbf{f}_j \nabla \mathbf{f}_k + \mathbf{f}_k^{\mathrm{T}} \otimes \nabla^2 \mathbf{f}_j)\big|_{\mathbf{u}=\mathbf{z}=\mathbf{0}}, \tag{53}$$

where the last term is an $(n \times n)$ matrix whose ith column is given by

$$(\mathbf{f}_k^{\mathrm{T}} \otimes \nabla^2 \mathbf{f}_j)_{(i)} \equiv \mathbf{f}_k^{\mathrm{T}} (\nabla^2 \mathbf{f}_j)_{(i)} \tag{54}$$

and $(\nabla^2 \mathbf{f}_j)_{(i)}$ is the Hessian matrix corresponding to $\mathbf{f}_{j(i)}$:

$$(\nabla^2 \mathbf{f}_j)_{(i)} \equiv \begin{bmatrix} \dfrac{\partial^2 \mathbf{f}_{j(i)}}{\partial \mathbf{u}_{(1)}^2} & \cdots & \dfrac{\partial^2 \mathbf{f}_{j(i)}}{\partial \mathbf{u}_{(1)} \partial \mathbf{u}_{(n)}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \mathbf{f}_{j(i)}}{\partial \mathbf{u}_{(n)} \partial \mathbf{u}_{(1)}} & \cdots & \dfrac{\partial^2 \mathbf{f}_{j(i)}}{\partial \mathbf{u}_{(n)}^2} \end{bmatrix}. \tag{55}$$

Expression (53) allows some general statements to be made about the parametrically induced stiffening effects induced by $\mathbf{f}_k$ (or $\mathbf{M}^{-1}\mathbf{f}_k$) and ξ:

(1) At least one of the functions $\mathbf{f}_k$ (or $\mathbf{M}$) must depend on $\mathbf{u}$ in order for the effective stiffness to change. This is equivalent to saying that the equations of motion written in standard second-order form (i.e., multiplying equation (31) by $\mathbf{M}^{-1}$) must have HF excitation terms that are *parametrical* in character, whereas pure *external* excitation will not produce such effects.

(2) The first contributive term in equation (53) exists only if $\mathbf{f}_k$ is linearizable with a non-zero gradient at zero. It disappears for even functions, $\mathbf{f}_k(-\mathbf{u}, t) = \mathbf{f}_k(\mathbf{u}, t)$, and for functions that are essentially non-linear. We may thus term its contribution *linearly induced parametrical stiffness*. If $\langle \xi_j' \xi_k' \rangle = 0$ for $j \neq k$, as is often the case in

applications, then the contribution will always be positive definite, since $|\nabla \mathbf{f}_j \nabla \mathbf{f}_k| = |\nabla \mathbf{f}_k||\nabla \mathbf{f}_k| > 0$ for $j = k$.

(3) The second term in equation (53) exists only when $\mathbf{f}_k$, has both a constant and a quadratic part when Taylor-expanded near the equilibrium; we may term it *non-linearly induced parametrical stiffness*, since only non-linear functions $\mathbf{f}_k$ can contribute to it. The change in stiffness can be positive or negative, and it disappears for odd functions, $\mathbf{f}_k(-\mathbf{u}, t) = -\mathbf{f}_k(\mathbf{u}, t)$.

(4) Terms of order three and higher of the Taylor-expansion of $\mathbf{f}_k$, do not contribute to stiffness (to the level of approximation employed), since their gradients vanish at the equilibrium.

(5) The stiffening effect is linear, by the definition employed above. But this does not imply that approximately correct results are always obtained by using linearized equations of motions. For example, if $n = 1$ and $\mathbf{f} = f(u) = \sin u$, then the sum of the two terms in the parenthesis of equation (53) becomes $(f')^2 + ff'' \equiv p(u) = \cos^2 u - \sin^2 u$, so that $p(0) = 1$, which is the same as when starting with the linearization $f(u) \approx u$. However, if instead $f(u) = \cos u - 2$, then one again finds $p(0) = 1$, while using the linearization $f(u) \approx -1$ one finds $p(0) = 0$, so that, by an otherwise appropriate linearization, the linear stiffening effect is in this case totally overlooked. In general, referring to items (2)–(4) above, if $\mathbf{f}$ contain constant terms, then quadratic non-linearities in $\mathbf{u}$ (when Taylor-expanded) should be retained, or stiffening effects could be inadvertently overlooked.

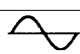
(6) The magnitude or "strength" of the stiffening effect increases linearly with the squared r.m.s. velocity of the excitation velocity, i.e., with the input level of energy. Hence the particular details of the excitation time signals are unimportant for the effect, as long as the signals are periodic and have high frequency and small amplitude. Table 1 provides cross-averages $\langle \xi'_j \xi'_k \rangle$ for some typical input signal forms for reference.

Changes in effective linear stiffness may cause changes in derived linear quantities, such as natural frequencies and stability. Using equations (46)–(47) with $\tilde{\mathbf{z}} = \mathbf{0}$, the linearized dynamics of the system near the equilibrium is seen to be governed by

$$\mathbf{M}(\mathbf{0}, t)\ddot{\mathbf{z}} + (\mathbf{C}(\mathbf{0}) + \varDelta\mathbf{C}(\mathbf{0}))\dot{\mathbf{z}} + (\mathbf{K}(\mathbf{0}) + \varDelta\mathbf{K}(\mathbf{0}))\mathbf{z} = \mathbf{0}. \tag{56}$$

TABLE 1

Example input signals ξ_j and their corresponding accelerations ξ''_j and cross-averages $\langle \xi'_j \xi'_k \rangle$. Here, δ_{jk} is the Kronecker delta, $\delta(\tau)$ is Dirac's delta function, all ξ-functions have zero average and are 2π-periodic in $\tau = \Omega t$ (modulo 2π), j and k are integers, and siv(τ) *is a sawtooth function having the same zeroes and extremums as* $\sin(\tau)$

		ξ_j	ξ''_j	$\langle \xi'_j \xi'_k \rangle$
Sine		$\sin(j\tau)$	$-j^2\sin(j\tau)$	$\frac{1}{2} j^2 \delta_{jk}$
Sawtooth		$\mathrm{siv}(j\tau)$	$\frac{2j}{\pi}\sum_{p=1}^{2j}(-1)^p \delta\left(\tau - \frac{\pi}{2j}(2p-1)\right)$	$\frac{18}{2\pi^2} j^2 \delta_{jk}$
Rectified sine		$\lvert\sin(j\tau)\rvert - 2/\pi$	$-j^2\lvert\sin(j\tau)\rvert + 2j\sum_{p=1}^{2j} \delta\left(\tau - \frac{\pi}{j}p\right)$	$\frac{1}{2} j^2 \delta_{jk}$
Phase-shifted sine		$\sin(\tau + \psi_j)$	$-\sin(\tau + \psi_j)$	$\frac{1}{2}\cos(\psi_j - \psi_k)$

Thus, the natural frequencies and stability associated with the equilibrium are determined by the eigenvalues $\lambda = \lambda_r$, $r = 1, \ldots, 2n$, which are roots of the characteristic polynomial:

$$|\lambda^2 \mathbf{M}(\mathbf{0}, t) + \lambda(\mathbf{C}(\mathbf{0}) + \Delta\mathbf{C}(\mathbf{0})) + \mathbf{K}(\mathbf{0}) + \Delta\mathbf{K}(\mathbf{0})| = 0. \quad (57)$$

The natural frequencies are given by $|\mathrm{Im}(\lambda_r)|$, while the equilibrium in question is stable to small disturbances only if $\mathrm{Re}(\lambda_r) < 0$ for all r. Both quantities are likely to differ from the values obtained when there is no HF excitation, i.e., when $\Delta\mathbf{C}(\mathbf{0}) = \Delta\mathbf{K}(\mathbf{0}) = \mathbf{0}$ in equation (57). For assessing stability in specific cases, it might be convenient to use Ziegler's system of classification [52], by which statements on stability are inferred from properties such as symmetry and definiteness of the system matrices. Also, since effective structural properties (stiffness, damping, stability, etc.) can be controlled by changing the structure or the HF excitation or both, some interesting inverse problems could be posed on using HF excitation as a design variable in designing structures with specific or optimal low-frequency properties. Proper response sensitivity analysis would be essential for this; however, for systems like equation (56) one could rely on well-proven methods and results (e.g., reference [53]).

4.2. BIASING

We here define biasing as a change in (average) static equilibrium position as a consequence of HF excitation. Two important kinds will be considered: *positional bias* and *velocity bias*.

4.2.1. *Positional bias*

Positional bias refers to a fixed translation of a static equilibrium. Examples of positional bias appear in sections 2.1–2.2, and Figure 6(b,d,e,h). Assuming, as in the previous section, that the general system (31) has a static equilibrium at $\mathbf{u} = \mathbf{0}$ when unexcited, then HF excitation may change this state of affairs, so that the equilibrium for the slow components of motion changes from $(\mathbf{z}, \dot{\mathbf{z}}) = (\mathbf{0}, \mathbf{0})$ to $(\tilde{\mathbf{z}}, \mathbf{0})$, where $\tilde{\mathbf{z}} \neq \mathbf{0}$ is constant valued. We term $\tilde{\mathbf{z}}$ a *quasi*-equilibrium for the full motions $\mathbf{u}$, because these will actually oscillate at small amplitude and high frequency of about $\tilde{\mathbf{z}}$ (cf., equation (33)). To calculate $\tilde{\mathbf{z}}$ we use equation (42), inserting $(\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}) = (\tilde{\mathbf{z}}, \mathbf{0}, \mathbf{0})$, and find that $\tilde{\mathbf{z}}$ is the solution of the generally non-linear set of algebraic equations (45), i.e.,

$$\mathbf{s}(\tilde{\mathbf{z}}, \mathbf{0}, t) + \mathbf{v}(\tilde{\mathbf{z}}, \mathbf{0}, t) = \mathbf{0} \quad (58)$$

with $\mathbf{v}$ given by equation (43). The equilibrium $\tilde{\mathbf{z}}$ is stable when the real parts of the roots λ of the following characteristic polynomial are all negative:

$$|\lambda^2 \mathbf{M}(\tilde{\mathbf{z}}, t) + \lambda(\mathbf{C}(\tilde{\mathbf{z}}) + \Delta\mathbf{C}(\tilde{\mathbf{z}})) + \mathbf{K}(\tilde{\mathbf{z}}) + \Delta\mathbf{K}(\tilde{\mathbf{z}})| = 0, \quad (59)$$

where the matrices $\mathbf{C}$ and $\mathbf{K}$ are given by equation (47). More than one solution for $\tilde{\mathbf{z}}$ may exist, since $\mathbf{s}$ and $\mathbf{v}$ are generally non-linear functions of position, or there may be no solutions at all. Here, we consider the typical case of a relatively small bias, $|\tilde{\mathbf{z}}| \ll 1$, for which it makes sense to Taylor-expand (58) and solve for $\tilde{\mathbf{z}}$, which yields, recalling that $\mathbf{s}(\mathbf{0},\mathbf{0},t) = \mathbf{0}$:

$$\tilde{\mathbf{z}} = -(\mathbf{K}(\mathbf{0}) + \Delta\mathbf{K}(\mathbf{0}))^{-1}\mathbf{v}(\mathbf{0}, \mathbf{0}, t) + \mathrm{O}(|\tilde{\mathbf{z}}|^2), \quad (60)$$

where the last term denotes small higher order contributions. Thus, to first order the positional bias is proportional to the HF-induced vibrational force $\mathbf{v}$ at the original equilibrium, and inversely proportional to the effective linear stiffness. Using equations

(43) and (41) one finds

$$\mathbf{v}(\mathbf{0},\mathbf{0},t) = \langle \mathbf{s}(\mathbf{0},\hat{\boldsymbol{\varphi}}_0',t)\rangle + \sum_j \langle \mathbf{h}_j(\mathbf{0},\hat{\boldsymbol{\varphi}}_0',t)\xi_j''(t,\tau)\rangle + \sum_j \nabla\mathbf{f}_j(\mathbf{0},t)\langle\hat{\boldsymbol{\varphi}}_0\xi_j''(t,\tau)\rangle + \langle\nabla\mathbf{m}(\mathbf{0},t,\hat{\boldsymbol{\varphi}}_0)\rangle, \tag{61}$$

where

$$\hat{\boldsymbol{\varphi}}_0(t,\tau) = -\mathbf{M}^{-1}(\mathbf{0},t)\sum_j \mathbf{f}_j(\mathbf{0},t)\xi_j(t,\tau). \tag{62}$$

Thus, there are several sources to bring up positional bias, corresponding to each of the four terms in equation (61) having a non-zero average. The contribution of the $\mathbf{s}$ term, to be denoted by $\mathbf{v}^{\mathbf{s}}$, can be partly examined by splitting $\mathbf{s}$ into components:

$$\mathbf{s}(\mathbf{u},\dot{\mathbf{u}},t) = \mathbf{s}_0 + \mathbf{S}_{11}\mathbf{u} + \mathbf{S}_{12}\dot{\mathbf{u}} + \mathbf{r}(\mathbf{u},\dot{\mathbf{u}},t), \tag{63}$$

where $\mathbf{s}_0$, and $\mathbf{S}_{11,12}$ are constant-valued vectors and matrices, and $\mathbf{r}$ is the remainder part, which is essentially non-linear in $\mathbf{u}$ and $\dot{\mathbf{u}}$ and satisfies $\mathbf{r}(\mathbf{0},\mathbf{0},t)+\mathbf{s}_0=\mathbf{0}$. Then $\mathbf{v}^{\mathbf{s}}$ becomes, recalling that the fast motions $\hat{\boldsymbol{\varphi}}$ has zero average:

$$\mathbf{v}^{\mathbf{s}} = \langle \mathbf{s}(\mathbf{0},\hat{\boldsymbol{\varphi}}_0',t)\rangle = \mathbf{s}_0 + \langle \mathbf{r}(\mathbf{0},\hat{\boldsymbol{\varphi}}_0',t)\rangle \tag{64}$$

from which it appears that no bias occurs from this source if $\mathbf{s}$ is a linear function of $\mathbf{u}$ and $\dot{\mathbf{u}}$, since then $\mathbf{r}=\mathbf{0}$ which implies $\mathbf{s}_0=\mathbf{0}$. This also holds if $\mathbf{s}$ is (generally) non-linear in $\mathbf{u}$, but linear in $\dot{\mathbf{u}}$. Thus, $\mathbf{s}$ should be non-linear in $\dot{\mathbf{u}}$, e.g., as for systems with dry friction, for bias to occur from this source.

The second term of equation (61) may, or may not, contribute to positional bias effect, dependent on the details of the functions $\mathbf{h}_j$, though it does not occur for the examples in this paper.

Considering the two last terms of equation (61), denoting their contribution by $\mathbf{v}^{\mathbf{f},\mathbf{m}}$, we use equations (50) and (51) to substitute the average terms and find that

$$\mathbf{v}^{\mathbf{f},\mathbf{m}} = \sum_{k,j}\langle\xi_k'\xi_j'\rangle\left(\nabla\mathbf{f}_j\mathbf{M}^{-1}\mathbf{f}_k + \left\{\begin{array}{c}(\mathbf{M}^{-1}\mathbf{f}_k)^{\mathrm{T}}\nabla\mathbf{M}_{(1)}\mathbf{M}^{-1}\mathbf{f}_l\\ \vdots \\ (\mathbf{M}^{-1}\mathbf{f}_k)^{\mathrm{T}}\nabla\mathbf{M}_{(n)}\mathbf{M}^{-1}\mathbf{f}_l\end{array}\right\}\right)_{\mathbf{u}=\mathbf{z}=\mathbf{0}}. \tag{65}$$

Then, by the same arguments as used below equation (52), we note that the essential information is carried by the first term, which we denote $\mathbf{v}^{\mathbf{f}}$ when $\mathbf{M}=\mathbf{I}$, i.e.,

$$\mathbf{v}^{\mathbf{f}}(\mathbf{0},\mathbf{0},t) = \sum_{j,k}\langle\xi_j'\xi_k'\rangle\nabla\mathbf{f}_j(\mathbf{0},t)\mathbf{f}_k(\mathbf{0},t). \tag{66}$$

Inspecting this expression, it appears that $\mathbf{v}^{\mathbf{f}}\neq\mathbf{0}$ requires that at least one of the functions $\mathbf{f}_k$ depends on $\mathbf{u}$, i.e., for bias to occur from this source, the HF excitation should be parametric in character. Furthermore, at least one $\mathbf{f}_k$ should be linearizable with a non-zero gradient at zero, while essentially non-linear functions $\mathbf{f}_k$ does not contribute to $\mathbf{v}^{\mathbf{f}}$. Likewise, functions $\mathbf{f}_k$ that possesses symmetry—be it odd, $\mathbf{f}_k(-\mathbf{u},t) = -\mathbf{f}_k(\mathbf{u},t)$, or even, $\mathbf{f}_k(-\mathbf{u},t) = \mathbf{f}_k(\mathbf{u},t)$—does not contribute to $\mathbf{v}^{\mathbf{f}}$. The latter requirement on asymmetry is essential, and can be illustrated by the simple function $\mathbf{f} = f(u) = 1 + u$, where the first term is evenly and the other one is oddly symmetrical, while their sum is asymmetric. For this case the contribution to $\mathbf{v}^{\mathbf{f}}$ is proportional to $\nabla\mathbf{f}(\mathbf{0})\mathbf{f}(\mathbf{0}) = f'(0)f(0) = 1$, whereas this value is zero if calculated for each of the terms 1 or u separately. In physical terms, the function $1 + u$ is representative, e.g., of structures subjected to a combination of external and parametrical HF excitation (cf., Figure 6(b)). Then bias may occur, even though the time average of the excitation is zero, whereas it disappears if either of the two excitations are switched off.

As special cases of positional bias we mention those where, in the absence of HF excitation, the system has (1) no static equilibriums at all, i.e., $\mathbf{s}(\mathbf{u},\mathbf{0},t)=\mathbf{0}$ has no real-valued solutions, or (2) has a continuous range of equilibriums, i.e., $\mathbf{s}(\mathbf{u},\mathbf{0},t)=\mathbf{0}$ has infinitely many solutions. Even in these cases the HF excitation may create well-defined equilibriums, as given by possible solutions $\tilde{\mathbf{z}}$ to equation (45), though equation (60) only holds when these happens to be close to zero. Examples are here the horizontal pendulum (last part of sections 2.1), and the mass with no spring on a vibrating plane (last part of section 2.2).

4.2.2. *Velocity bias*

By velocity bias we refer to the possible emergence of states of steady drifting with constant velocity, i.e., with slow motions of the form $(\mathbf{z},\dot{\mathbf{z}})=(\tilde{\dot{\mathbf{z}}}t,\tilde{\dot{\mathbf{z}}})$, where $\tilde{\dot{\mathbf{z}}}$ is the drift velocity. Examples of velocity bias appear in the last part of section 2.2 (mass with no spring on vibrating plane), section 2.3, and Figure 6(a,c). Using equation (42), we find that $\tilde{\dot{\mathbf{z}}}$ should be a solution of

$$\mathbf{s}(\tilde{\dot{\mathbf{z}}}t,\tilde{\dot{\mathbf{z}}},t)+\mathbf{v}(\tilde{\dot{\mathbf{z}}}t,\tilde{\dot{\mathbf{z}}},t)=\mathbf{0}. \tag{67}$$

If such a solution exist, one can easily show, the condition for its stability is that the real parts of the roots λ of the following characteristic polynomial are all negative:

$$|\lambda\mathbf{M}(\tilde{\dot{\mathbf{z}}}t,t)+\dot{\nabla}\mathbf{s}(\tilde{\dot{\mathbf{z}}}t,\tilde{\dot{\mathbf{z}}},t)+\dot{\nabla}\mathbf{v}(\tilde{\dot{\mathbf{z}}}t,\tilde{\dot{\mathbf{z}}},t)|=0. \tag{68}$$

A necessary condition for velocity bias to exist is that $\mathbf{M}$ and the expression in equation (67) are independent of t, that is $\mathbf{s}$ and $\mathbf{v}$ or their sum, and $\mathbf{M}$, should be independent of $\mathbf{u}$ and of t.

For typical cases where the velocity bias is relatively small, we may solve equation (67) approximately by Taylor-expanding near $\tilde{\dot{\mathbf{z}}}=0$ and solving for the drift velocity $\tilde{\dot{\mathbf{z}}}$, the result becoming

$$\tilde{\dot{\mathbf{z}}}=-(\dot{\nabla}\mathbf{s}(\tilde{\dot{\mathbf{z}}}t,\mathbf{0},t)+\dot{\nabla}\mathbf{v}(\tilde{\dot{\mathbf{z}}}t,\mathbf{0},t))^{-1}(\mathbf{s}(\tilde{\dot{\mathbf{z}}}t,\mathbf{0},t)+\mathbf{v}(\tilde{\dot{\mathbf{z}}}t,\mathbf{0},t)) \tag{69}$$

which again is only a proper solution if independent of t (in typical cases the terms with $\tilde{\dot{\mathbf{z}}}t$, cancel each other, so that $\tilde{\dot{\mathbf{z}}}$ need not to be known in order to compute the right-hand side).

4.3. SMOOTHENING

Sometimes HF excitation has the effect of apparently removing discontinuities of a system; we refer to this as the *smoothening effect*. A classical example is systems modelled with Coulomb friction, where HF excitation may cause the discontinuity at zero velocity to seemingly disappear, and the damping appears viscous in character (cf., Sections 2.2–2.3, and Figure 6(a,h,i)). To study this effect we assume that the slow forces $\mathbf{s}$ of the system (31) have a single discontinuity at zero velocity when there is no HF excitation, i.e.,

$$\mathbf{s}_+(\mathbf{u},t)\neq\mathbf{s}_-(\mathbf{u},t),\quad \mathbf{s}_+(\mathbf{u},t)\equiv\lim_{\dot{\mathbf{u}}\to\mathbf{0}_+}\mathbf{s}(\mathbf{u},\dot{\mathbf{u}},t),\quad \mathbf{s}_-(\mathbf{u},t)\equiv\lim_{\dot{\mathbf{u}}\to\mathbf{0}_-}\mathbf{s}(\mathbf{u},\dot{\mathbf{u}},t). \tag{70}$$

As appears from equation (42), the slow forces for the averaged system is $\mathbf{s}+\mathbf{v}$, and by equation (43):

$$\mathbf{s}(\mathbf{z},\dot{\mathbf{z}},t)+\mathbf{v}(\mathbf{z},\dot{\mathbf{z}},t)=\langle\mathbf{s}(\mathbf{z},\dot{\mathbf{z}}+\hat{\boldsymbol{\varphi}}',t)\rangle+\sum_j\langle\mathbf{h}_j(\mathbf{z},\dot{\mathbf{z}}+\hat{\boldsymbol{\varphi}}',t)\xi_j''(t,\tau)\rangle+\bar{\mathbf{r}}(\mathbf{z},\dot{\mathbf{z}},t), \tag{71}$$

where the fast motions $\hat{\boldsymbol{\varphi}}$ are given by equation (41), while $\bar{\mathbf{r}}$ is a remainder term holding continuous functions that are unrelated to $\mathbf{s}$. It seems intuitively plausible that $\mathbf{s}+\mathbf{v}$ can be continuous at $\dot{\mathbf{z}}=\mathbf{0}$, even though $\mathbf{s}$ (and perhaps $\mathbf{h}_j$) is discontinuous: The right-hand side is calculated by averaging $\mathbf{s}$ (and $\mathbf{h}_j$) while the velocity argument $\dot{\mathbf{z}}+\hat{\boldsymbol{\varphi}}$ is racing back and forth across the discontinuity, and this average may not change much in the vicinity of $\dot{\mathbf{z}}=\mathbf{0}$.

To set up conditions for smoothening to occur, we first note that the vector character of the functions and their dependence on $\mathbf{u}$ and t is inessential for this, since the averaging process is performed elementwise, and with $\mathbf{u}$ and t considered constant. It therefore suffices to consider a general scalar function $s(\dot{z})$, bounded on R, and having a single discontinuity at $\dot{z}=0$, and then consider the *smoothed image* $\tilde{s}$ of s:

$$\tilde{s}(\dot{z}) \equiv \langle s(\dot{z}+\varphi'(\tau))\rangle, \tag{72}$$

where the fast motions $\varphi(\tau)$ are 2π-periodic in τ with zero average. Using definition (34) of fast-time averaging one finds, splitting up the period of integration, that

$$\tilde{s}(\dot{z}) = \frac{1}{2\pi}\left[\int_{\tau_-} s(\dot{z}+\varphi'(\tau))\,\mathrm{d}\tau + \int_{\tau_0} s(\dot{z}+\varphi'(\tau))\,\mathrm{d}\tau + \int_{\tau_+} s(\dot{z}+\varphi'(\tau))\,\mathrm{d}\tau\right], \tag{73}$$

where τ_-, τ_0, and τ_+ are the intervals of time where $\dot{z}+\varphi'(\tau)<0, =0$, and >0, respectively, and $\tau_- \cup \tau_0 \cup \tau_+ = [0;2\pi]$. The limits of these intervals are solutions to the algebraic equation $\dot{z}+\varphi'(\tau)=0$; we assume there are $n_\tau = n_\tau(\dot{z})$ solutions and denote them $\tau_j = \tau_j(\dot{z})$, $j=1,2,\ldots,n_\tau$. By these definitions the first and the third integrands are assured to be continuous in $\dot{z}$, while continuity of the corresponding integrals requires continuity also of the limits of the integration intervals. The following observations regarding the continuity of $\tilde{s}(\dot{z})$ can then be made, noting that for every value of $\dot{z}$, the number of solutions n_τ is either zero, finite, or infinite:

(1) If $n_\tau=0$ (for some range of $\dot{z}$) then $\tilde{s}$ is continuous (in that range). Namely, when $\dot{z}+\varphi'(\tau)$ newer sweeps the discontinuity at zero, then $\tilde{s}$ is given by either the first or the third integral in equation (73), which is a continuous function of $\dot{z}$ because the integrand s is continuous during τ_- or τ_+, and the interval of integration $[0;2\pi]$ is independent of $\dot{z}$.

(2) If n_τ is infinite, the discontinuity of s at zero is still present in $\tilde{s}$. Namely, $n_\tau \to \infty$ means that $\dot{z}+\varphi'=0$ over finite sub-intervals of $[0;2\pi]$, i.e., τ_0 is not a point set. Then the second integral in equation (73) is to be performed over finite intervals of time where $\dot{z}+\varphi'(\tau)=0$, i.e., at values where s is discontinuous (undefined); this integral therefore is discontinuous, and so is $\tilde{s}$.

(3) If n_τ is finite, and the zeroes τ_j of $\dot{z}+\varphi'(\tau)$ are all simple, then $\tilde{s}$ is continuous for the corresponding values of $\dot{z}$. This is because a finite value of n_τ means that $\dot{z}+\varphi'(\tau)$ sweeps the discontinuity a finite number of times during a period, i.e., τ_0 is a point set. Then the second integral in equation (73) vanishes, because s is bounded and the integration intervals are infinitely short. Consequently, continuity is assured if the two other integrals are continuous. Their integrands are continuous, because they are to be evaluated for the intervals τ_- and τ_+, where $\dot{z}+\varphi'(\tau)$ has constant sign and thus only sweeps the continuous part of s. But the boundaries of the integration intervals should be continuous as well, that is, τ_j should be continuous functions of $\dot{z}$. By the implicit function theorem this is true when $\varphi''(\tau_j)\neq 0$, that is, if τ_j are simple zeroes of $\dot{z}+\varphi'(\tau)$.

(4) In particular, it follows from the above results that $\tilde{s}$ is continuous at $\dot{z}=0$ if $\varphi'(\tau)$ has a finite number of simple zeroes.

As described, the smoothening effect on the discontinuity at zero occurs only if the fast motions actually sweeps the discontinuity, and does not "rest" there. One example function that fails to satisfy these requirements is $\varphi' = -\sin(2\tau)$ for $\tau \in [0; \pi]$; $\varphi' = 0$ for $\tau \in [\pi; 2\pi]$, which has simple zeroes at $\tau = 0, \pi/2$, but infinitely many non-simple zeroes for $\tau \in [\pi; 2\pi]$; hence this function does not smoothen discontinuities.

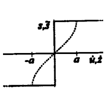
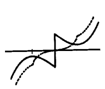
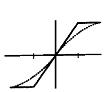
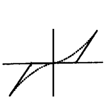
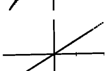
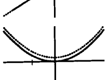
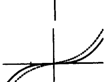
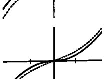
An example function that *does* satisfy the requirements is a simple harmonic, e.g., $\varphi' = -a\cos(\tau)$, which has simple zeroes at $\pi/2$ and $3\pi/2$ and is thus smoothening. Its effect on the discontinuous signum function appears in Table 2(a); it could represent the smoothening of the dry friction characteristic for a physical system subjected to HF excitation (see also Example 2.2 and Figure 4).

Table 2(b) shows how the same fast motions smoothen the discontinuity of a function that may more realistically model friction. Here, s has negative slope at the origin and thus may cause unstable oscillations of a corresponding physical system, whereas the smoothed image has positive slope everywhere (see reference [20] for an application of this effect).

Discontinuities in derivatives may also be smoothened, as is easily realized when changing s and $\tilde{s}$ with $\mathrm{d}s/\mathrm{d}\dot{z}$ and $\mathrm{d}\tilde{s}/\mathrm{d}\dot{z}$ (or higher order derivatives) in the above expressions and arguments. Table 2(c,d) shows two such examples, which might represent

TABLE 2

Examples of slow functions $s(\dot{u})$ and their corresponding smoothed images $\tilde{s}(\dot{z})$ for $\varphi(\tau) = -a\sin(\tau)$. Lengthy expressions for $\tilde{s}_{c1}$ and $\tilde{s}_{c2}$ in (c) and (d) are omitted; they are both smooth functions with smooth transitions at the boundaries of their intervals of definition

$s(\dot{u})$	$\tilde{s}(\dot{z}) \equiv \langle s(\dot{z} + \varphi'(\tau))\rangle$, $\varphi' = -a\cos\tau$	s (——), $\tilde{s}$(- - -)
(a) $\mathrm{sgn}(\dot{u})$	$\begin{cases} 1 - \frac{2}{\pi}\arccos(\dot{z}/a), & \lvert\dot{z}\rvert \leqslant a \\ \mathrm{sgn}(\dot{z}), & \lvert\dot{z}\rvert > a \end{cases}$	
(b) $\mathrm{sgn}(\dot{u}) - \dot{u} + \frac{1}{3}\dot{u}^3$	$\begin{cases} 1 - \frac{2}{\pi}\arccos(\dot{z}/a) + (\frac{1}{2}a^2 - 1)\dot{z} + \frac{1}{3}\dot{z}^3 & \text{for } \lvert\dot{z}\rvert \leqslant a \\ \mathrm{sgn}(\dot{z}) + (\frac{1}{2}a^2 - 1)\dot{z} + \frac{1}{3}\dot{z}^3 & \text{for } \lvert\dot{z}\rvert > a \end{cases}$	
(c) $\begin{cases} \dot{u}, & \lvert\dot{u}\rvert \leqslant 1 \\ \mathrm{sgn}(\dot{u}), & \lvert\dot{u}\rvert > 1 \end{cases}$	$\begin{cases} \dot{z}, & \lvert\dot{z}\rvert \leqslant 1 - a \\ \tilde{s}_{c1}(\dot{z}), & \lvert\dot{z}\rvert \in [1 - a; 1 + a] \\ \mathrm{sgn}(\dot{z}), & \lvert\dot{z}\rvert > 1 + a \end{cases}$	
(d) $\begin{cases} 0, & \lvert\dot{u}\rvert \leqslant 1 \\ \dot{u} - \mathrm{sgn}(\dot{u}), & \lvert\dot{u}\rvert > 1 \end{cases}$	$\begin{cases} 0, & \lvert\dot{z}\rvert \leqslant 1 - a \\ \tilde{s}_{c2}(\dot{z}), & \lvert\dot{z}\rvert \in [1 - a; 1 + a] \\ \dot{z} - \mathrm{sgn}(\dot{z}), & \lvert\dot{z}\rvert > 1 + a \end{cases}$	
(e) $\dot{u}$	$\dot{z}$	
(f) $\dot{u}^2$	$\dot{z}^2 + \frac{1}{2}a^2$	
(g) $\dot{u}^3$	$\dot{z}^3 + \frac{3}{2}a^2\dot{z}$	
(h) $\lvert\dot{u}\rvert\dot{u}$	$\begin{cases} (1 - \frac{2}{\pi}\arccos(\dot{z}/a))(\dot{z}^2 + \frac{1}{2}a^2) + \frac{3}{\pi}a\dot{z}\sqrt{1 - (\dot{z}/a)^2}, & \lvert\dot{z}\rvert \leqslant a \\ \lvert\dot{z}\rvert\dot{z} + \frac{1}{2}a^2\mathrm{sgn}(\dot{z}), & \lvert\dot{z}\rvert > a \end{cases}$	

idealized physical processes involving saturation and "barrier" behavior respectively. For both cases the discontinuous changes in slope are smoothed by the HF excitation.

Even if s has no discontinuity, the smoothed image $\tilde{s}$ may still differ from s. Table 2(e–h) shows smoothed images of functions that are typically encountered in applications. The essential properties of such functions are captured by a third order polynomial with zero constant term and coefficients $s_{1,2,3}$:

$$s(\dot{z}) = s_1\dot{z} + s_2\dot{z}^2 + s_3\dot{z}^3 \tag{74}$$

whose smoothed image is, by equation (72), and recalling that $\langle \varphi' \rangle = 0$:

$$\tilde{s}(\dot{z}) = (s_2\langle(\varphi')^2\rangle + s_3\langle(\varphi')^3\rangle) + (s_1 + 3s_3\langle(\varphi')^2\rangle)\dot{z} + s_2\dot{z}^2 + s_3\dot{z}^3. \tag{75}$$

Interestingly, the non-linear parts of s change the linear properties of $\tilde{s}$, whereas the non-linear properties themselves are unaffected. For example, the slope of $\tilde{s}$ at $\dot{z} = 0$ is seen to increase, as compared with s, if the cubic non-linearity of s is progressive ($s_3 > 0$), and decrease if it is recessive ($s_3 < 0$). Thus, the effective linear damping of a physical system might be partially controlled by using HF excitation (or be created out of nothing, as when $s_1 = 0$, $s_3 \neq 0$). Also, a constant drifting term is seen to appear, so that $\tilde{s}(0) \neq 0$ when there are non-linearities in s.

Finally, we recall that the above results for s and φ hold as well for the corresponding vector functions $\mathbf{s}$ and $\hat{\varphi}$ of the general system (31).

5. SUMMARY AND CONCLUSIONS

Three common effects of HF excitation (stiffening, biasing, and smoothening) have been analyzed for a general class of mechanical systems that can be modelled by equation (31), i.e., by a finite set of second order ordinary differential equations, generally non-linear, with periodically oscillating excitation terms of high frequency and small amplitude. The analysis was accomplished by analytically splitting the unknown solutions to these equations into slowly changing and rapidly oscillating components, respectively, and then extracting equations (42) that describes the slow components. The latter equations are similar to the original ones, except that the influence of the rapidly oscillating terms occurs only by their average effect.

This enables predictions to be made on the effects of concern, without actually solving the equations of motion, and without recoursing to numerical simulation. Such results are given in sections 4.1, 4.2 and 4.3, respectively, for the stiffening, the biasing, and the smoothening effect. They can be used for understanding the effects in question in a broader perspective than is possible with specific systems, and for calculating effects for specific systems using well-defined formulas.

Possible future work might include using the general expressions as an aid to design specific systems that have prescribed characteristics in the presence of HF excitation, deriving optimal locations and wave forms of HF excitations, or calculating how the strength of the effects changes when the HF excitation is quasi-periodic, chaotic, or random (results of the present study indicate that the details of the excitation waveform is unimportant, what counts is the average kinetic input energy). Also, there are other effects of HF excitation that occurs across a range of systems, and might deserve a more general treatment, e.g., apparent changes in bifurcation type and non-linear frequency response (see e.g., references [7, 17, 22, 54] for examples).

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