Journal of Sound and Vibration (2002) **253**(4), 909–916
doi:10.1006/jsvi.2001.4056, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

A UNIVERSAL SOLUTION FOR EVOLUTIONARY RANDOM RESPONSE PROBLEMS

T. Fang

Vibration Research Centre, Northwestern Polytechnical University, Xian 710072, *People's Republic of China. E-mail*: *tfang8@pub.xaonline.com*

J. Q. Li

Xian University of Technology, Xian 710048, *People's Republic of China*

and

M. N. Sun

Nanjing University of Science and Technology, Nanjing 210094, *People's Republic of China*

(*Received* 10 *January* 2001, *and in final form* 6 *September* 2001)

1. INTRODUCTION

It is well known that random processes can be classified into stationary ones and non-stationary ones. The most important feature of a stationary random process lies in that its statistical property in frequency domain does not depend on time. So the power spectral density function of a stationary random process is the best representative of its frequency characteristics. On the contrary, if the statistical properties of a random process vary with time, then the process must be a non-stationary one. In many cases, random excitations may be reasonably regarded as stationary ones, so large amounts of random vibration problems are stationary ones. However, in some other cases, random excitations must be treated as non-stationary ones, such as earthquakes, gusts, raging tides, blast waves, etc. Nevertheless, it is too complex to model a non-stationary random excitation in its general form, and the related response problem is too difficult to solve. Fortunately, there is a sub-class of non-stationary random processes, called evolutionary random processes. An evolutionary random process is a special kind of non-stationary random process, which results from a stationary one by deterministic modulation. It is Priestley [1] who first suggested the concept of an evolutionary power spectrum to describe the modulated characteristics of an evolutionary random process. Short after then, based on that concept, Hammond [2] and Shinozoka [3] developed a modal summation method for evolutionary random responses of classically damped Md.o.f. linear systems. Usually, by filtering a stationary random process through a linear time-dependent filter, the output of the filter would be an evolutionary one. However, there is another type of evolutionary random process, which results from non-linear transformation of the argument of a stationary random process, such as the excitation of road undulation to a vehicle travelling with variable speed. Though these two types of evolutionary random excitations result from stationary random process by two utterly different ways, the corresponding response problems have much in common. A unified approach to these two types of response problems was first suggested by Fang and Sun [4]. Another new contribution in reference [4] lies in

0022-460X/02/$35.00

a clear and definite conclusion that all these evolutionary random response problems can be reduced into transient response problems of the original system subject to some deterministic excitations. The statement in reference [4] was based on time-invariant dynamic systems. In this paper, we will extend these results to time-variant dynamic systems.

2. TWO TYPES OF EVOLUTIONARY RANDOM EXCITATIONS

The above-mentioned two types of evolutionary random excitations can be modelled as follows: one may be obtained by filtering a stationary random process through a linear time-dependent system, while another may result from a non-linear transformation of the argument of a stationary random process.

Suppose that $F(t)$ is a stationary random process with zero mean, and let its correlation functions be expressed through power spectral density $S(\omega)$ as follows:

$$E[F(t_1)F(t_2)] = \int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{j}\omega(t_2-t_1)}\,\mathrm{d}\omega. \tag{1}$$

Let the impulse response function of a time-variant linear filter be $a(t, \tau)$, which is the output at instant t of the filter due to a unit impulse input at instant $t-\tau$. Then, due to a stationary random input $F(t)$, the output $f(t)$ of the filter can be obtained as

$$f(t) = \int_{-\infty}^{\infty} a(t, \tau)F(t-\tau)\,\mathrm{d}\tau. \tag{2}$$

It is seen that if $F(t)$ has a zero mean value, so has $f(t)$. So the correlation functions of $f(t)$ can be obtained as

$$E[f(t_1)f(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1, \tau_1)a(t_2, \tau_2)E[F(t_1-\tau_1)F(t_2-\tau_2)]\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{3}$$

$$= \int_{-\infty}^{\infty} \alpha(\omega, t_1)\bar{\alpha}(\omega, t_2)S(\omega)\mathrm{e}^{\mathrm{j}\omega(t_2-t_1)}\,\mathrm{d}\omega,$$

where

$$\alpha(\omega, t) = \int_{-\infty}^{\infty} a(t, \tau)e^{\mathrm{j}\omega\tau}\,\mathrm{d}\tau. \tag{4}$$

Letting $t_1 = t_2 = t$ in equation (3), we have

$$E[f^2(t)] = \int_{-\infty}^{\infty} \alpha(\omega, t)\bar{\alpha}(\omega, t)S(\omega)\,\mathrm{d}\omega. \tag{5}$$

Thus, the evolutionary power spectral density of $f(t)$ can be defined as

$$S_f(\omega, t) \equiv \alpha(\omega, t)\bar{\alpha}(\omega, t)S(\omega), \tag{6}$$

where the two-dimensional deterministic function, $\alpha(\omega, t)$, of t and ω is usually called a modulating function. It is the modulating function, which regulates a stationary random process into an evolutionary one. The non-stationary stochastic model of seismic motions during earthquake often takes this type of evolutionary random process.

Now let us turn to the random excitation experienced by a vehicle travelling with variable speed due to the road undulation. It is reasonable to assume the undulation of a highway surface as a homogeneous random process with respect to distance x. For analysis in time domain, it is necessary to transform the argument of the random process, $F(x)$, from the distance domain x to the time domain t. Suppose that the transformation is a non-linear one, i.e.,

$$x = \varphi(t). \tag{7}$$

Letting $F(x)$ correspond to $f(t)$, and x_i correspond to t_i, we have

$$x_i = \varphi(t_i), \quad i = 1, 2.$$

Then, the correlation functions of $f(t)$ can be expressed as

$$\begin{aligned} E[f(t_1)f(t_2)] &= E[F(x_1)F(x_2)] \\ &= \int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{j}\omega(x_2 - x_1)}\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{j}\omega[\varphi(t_2) - \varphi(t_1)]}\,\mathrm{d}\omega. \end{aligned} \tag{8}$$

3. MEAN SQUARE EVOLUTIONARY RANDOM RESPONSES OF A TIME-VARIANT LINEAR SYSTEM

Suppose an n-d.o.f. time-variant linear system is subject to an evolutionary random excitation with a common source $f(t)$. Its differential equation for response x can be expressed as

$$m(t)\ddot{x} + c(t)\dot{x} + k(t)x = bf(t), \tag{9}$$

where $m(t)$, $c(t)$, and $k(t)$ are time-variant matrices of mass, damping and stiffness, respectively, among which $m(t)$ being a positive-definite matrix; b is a real constant n-vector, and $f(t)$ is a scalar evolutionary random process with a zero mean value and with correlation functions expressed by equation (3) or equation (8). By introducing the state variables

$$y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

equation (9) can be reduced to a system of first order equations of state variables

$$\dot{y} = A(t)y + B(t)f(t), \tag{10}$$

where

$$A(t) = \begin{bmatrix} 0 & I \\ -m^{-1}(t)k(t) & -m^{-1}(t)c(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 \\ m^{-1}(t)b \end{bmatrix}.$$

The homogeneous equation corresponding to equation (10) is

$$\dot{y} = A(t)y, \quad y(t_0) = y_0. \tag{11}$$

Its general solution can be expressed as

$$y(t) = \Phi(t, t_0)y_0, \tag{12}$$

where $\Phi(t, t_0)$ is called a transition matrix with the following properties:

(1) $\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I,$

(2) $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0), \quad t_2 > t_1 > t_0,$

(3) $\Phi^{-1}(t, t_0) = \Phi(t_0, t).$

Then, for $t_0 = 0$, $y_0 = 0$, the transient solution for a sample excitation of equation (10) can be obtained as

$$y(t) = \int_0^t \Phi(t, u)B(u)f(u)\,\mathrm{d}u. \tag{13}$$

And the corresponding mean square response matrix can be expressed as

$$E[y(t)y^{\mathrm{T}}(t)] = \int_0^t \int_0^t \Phi(t, u)B(u) \cdot B^{\mathrm{T}}(v)\Phi^{\mathrm{T}}(t, v)E[f(u)f(v)]\,\mathrm{d}u\,\mathrm{d}v.$$

By using equation (3) or equation (8), the above equation can be rewritten as

$$E[y(t)y^{\mathrm{T}}(t)] = \int_0^\infty H(\omega, t)\bar{H}^{\mathrm{T}}(\omega, t)S(\omega)\,\mathrm{d}\omega, \tag{14}$$

where for equation (3),

$$H(\omega, t) = \int_0^t \Phi(t, u)B(u)\alpha(\omega, u)\mathrm{e}^{-\mathrm{j}\omega u}\,\mathrm{d}u \tag{15}$$

and for equation (8),

$$H(\omega, t) = \int_0^t \Phi(t, u)B(u)\mathrm{e}^{-\mathrm{j}\omega\varphi(u)}\,\mathrm{d}u. \tag{16}$$

It is easy to see from equation (14) that $H(\omega, t)$ is just the modulating function matrix of the wanted evolutionary random response. By comparing equation (15) or equation (16) with equation (13), one can also see that as long as ω is taken as a fixed parameter, $H(\omega, t)$ is just the transient response of the original system subject to a deterministic excitation, $B(t)\alpha(\omega, t)\mathrm{e}^{-\mathrm{j}\omega t}$ [for equation (15)] or $B(t)\mathrm{e}^{-\mathrm{j}\omega\varphi(t)}$ [for equation (16)], under a zero initial condition. In this sense, the mean square evolutionary random response problem is finally reduced to a transient response problem of the original system subject to some deterministic excitation.

In general, the transition matrix of a time-variant linear system is too difficult to obtain in analytical form. Fortunately, what we are looking for is only a particular solution of the system under a zero initial condition. This kind of particular solution is not difficult to find through any effective numerical methods, such as Runge–Kutta method, etc.

In case the random excitation is a stationary one, we have $\alpha(\omega, t) \equiv 1$, or $\varphi(t) \equiv t$. Then, both equations (15) and (16) degenerate to

$$H(\omega, t) = \int_0^t \Phi(t, u)B(u)\mathrm{e}^{-\mathrm{j}\omega u}\,\mathrm{d}u. \tag{17}$$

In case the original system is time-invariant, we have $A(t) \equiv A$. $B(t) \equiv B$, and $\Phi(t, u) \equiv \mathrm{e}^{A(t-u)}$. Then equations (15) and (16) convert separately into

$$H(\omega, t) = \int_0^t \mathrm{e}^{A(t-u)}B\alpha(\omega, u)\mathrm{e}^{-\mathrm{j}\omega u}\,\mathrm{d}u$$

and

$$H(\omega, t) = \int_0^t \mathrm{e}^{A(t-u)}B\mathrm{e}^{-\mathrm{j}\omega\varphi(u)}\,\mathrm{d}u.$$

The above results coincide exactly with those in reference [4].

4. A NUMERICAL EXAMPLE [5]

The response problem of a vehicle passing across a bridge is taken as an example. For simplifying analysis, a vehicle is taken as a 2-d.o.f. mass–spring system, and a bridge as a simply supported uniform beam. Though the two separated sub-systems are time-invariant ones, while the vehicle travelling along the bridge, the coupled system should be a time-variant one, as shown in Figure 1. Suppose that the vehicle is travelling with a constant speed v, then the distance travelled can be expressed as $x = vt$. The bridge undulation, $\eta(x)$, is supposed to be a homogeneous random process along the distance x, with a zero mean and with correlation functions as

$$E[\eta(x_1)\eta(x_2)] = \int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{j}\omega(x_2 - x_1)}\,\mathrm{d}\omega. \tag{18}$$

The vertical responses $y(x, t)$, y_1, and y_2 of the coupled system under the influence of moving vehicle weight and the bridge undulation are considered here. To derive the

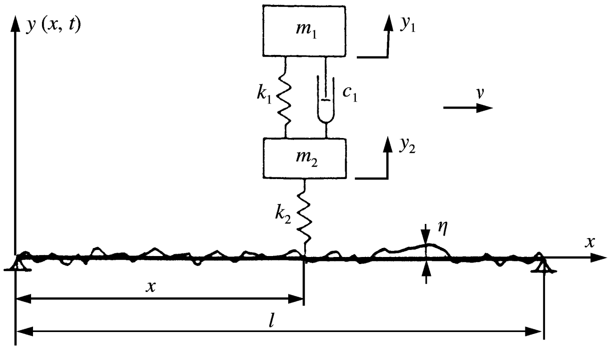

Figure 1. Mathematical model of a coupled and vehicle and bridge system.

differential equations of the coupled system, the first n natural modes of a simply supported uniform beam

$$\psi_i(x) = \sqrt{2}\sin(i\pi x/l), \quad i = 1, 2, \ldots, n$$

are taken as the assumed modes of the bridge in the coupled system. Then, $y(x, t)$ can be expressed as

$$y(x, t) = \sum_{i=1}^{n} \psi_i(x) q_i(t), \tag{19}$$

where $q_i(t)$ are the modal response corresponding to $\psi_i(x)$. By using the assumed modal method, the differential equations of the coupled system can finally be obtained as follows:

$$\ddot{q}_i(t) + \frac{m_1}{m_b}\psi_i(x)\ddot{y}_i(t) + \frac{m_2}{m_b}\psi_i(x)\ddot{y}_2(t) + \frac{c}{\rho A}\dot{q}_i(t) + \omega_i^2 q_i(t)$$
$$= -\frac{m_1 + m_2}{m_b} g\psi_i(x) \equiv f_i(t), \quad i = 1, 2, \ldots, n, \tag{20}$$

$$m_1\ddot{y}_1(t) + c_1\dot{y}_1(t) - c_1\dot{y}_2(t) + k_1 y_1(t) - k_1 y_2(t) = 0, \tag{21}$$

$$m_2 y_2(t) - c_1\dot{y}_1(t) + c_1\dot{y}_2(t) - k_2 \sum_{j=1}^{n} \psi_j(x) q_j(t) - k_1 y_1(t) + (k_1 + k_2) y_2(t) = k_2\eta(x) \equiv f(t), \tag{22}$$

where $\omega_i^2 = (i\pi/l)^4 EI/(\rho A)$, $m_b = \rho Al$; $f_i(t)$ is a deterministic excitation resulting from the moving vehicle weight, and $f(t)$ is a random excitation resulting from the bridge undulation. Therefore, the total response of the coupled system consists of two components, a deterministic one due to $f_i(t)$'s, and a random one due to $f(t)$. Since $f_i(t)$'s and $f(t)$ are mutually independent, the corresponding two component responses can be obtained separately. The former can be obtained by the conventional method and the latter can be obtained by the above-stated method.

In the following calculations, data are taken as follows: $EI = 2\,658\,069\,\text{kN m}^2$, $\rho A = 6067\,\text{kg/m}$, $m_1 = 12\,000\,\text{kg}$, $m_2 = 500\,\text{kg}$, $k_1 = 280\,000\,\text{N/m}$, $k_2 = 156\,000\,\text{N/m}$,

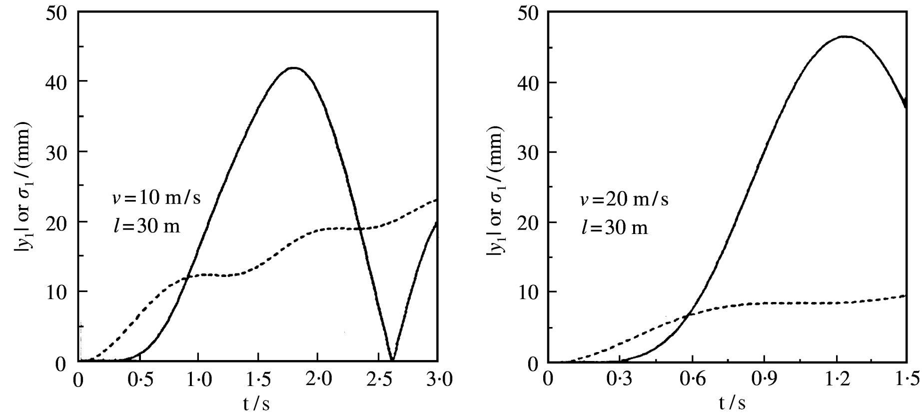

Figure 2. Response of m_1 for $l = 30$ m, and $v = 10$ m/s (left) and $v = 20$ m/s (right): ——, $|y_1|$; - - - -, σ_1.

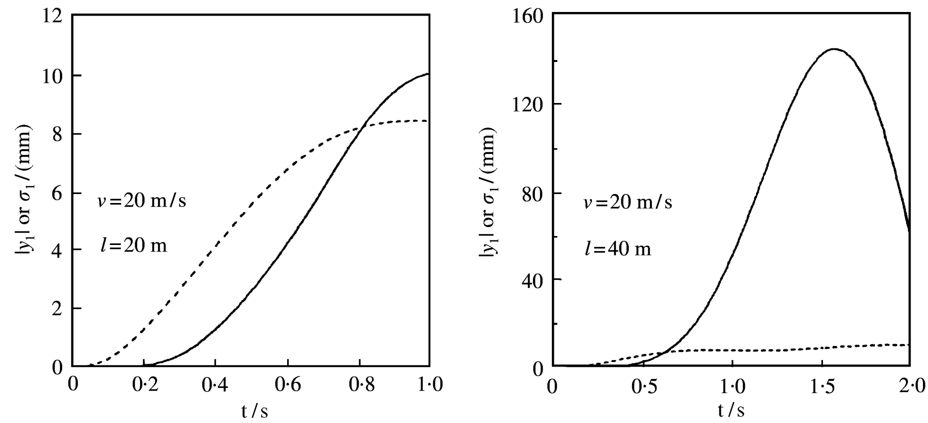

Figure 3. Response of m_1 for $v = 20$ m/s, and $l = 20$ m (left) and $l = 40$ m (right): ——, $|y_1|$; - - - -, σ_1.

$c_1 = 11{\cdot}59 \times 10^3$ N s/m. The power spectral density of the homogeneous random undulation is taken as

$$S(\omega) = \frac{1}{\pi} \frac{4\gamma\alpha\beta\omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2},$$

where $\omega_0^2 = \alpha^2 + \beta^2$, $\alpha = 0{\cdot}1$, $\beta = 0{\cdot}3$, $\gamma = 1\,\text{cm}^2$. The numerical results for $v = 10$ and 20 m/s, and $l = 30$ m are shown in Figure 2, and those for $v = 20$ m/s, and $l = 20$ and 40 m are shown in Figure 3. In these figures, solid curves represent the absolute value of the deterministic component responses, while the dotted curves represent the r.m.s. random component responses. It reveals in calculations that taking the number of the selected assumed modes $n = 4$ is good enough. All the results for $n \geqslant 4$ coincide with each other.

It is seen in Figure 2 that for a fixed bridge length, the faster the travelling speed, the smaller the r.m.s. value of the random response as compared with the deterministic response. And it is seen in Figure 3 that for a fixed travelling speed, the longer the bridge length, the smaller the r.m.s. value of the random response as compared with the deterministic response.

5. CONCLUSIONS

(1) A deep understanding of evolutionary random response problems is introduced here. The key for solving an evolutionary random response problem is to find the modulating function of the response, or equivalently to find a transient response of the original system due to some deterministic excitation. In this sense, an evolutionary random response problem can be finally reduced to a deterministic transient response problem.

(2) No matter what the original system is, time-invariant or time-variant, problems for modulating functions of evolutionary random responses can be formulated in a unique way and solved by any effective numerical methods, including the Runge–Kutta method.

REFERENCES

1. M. B. PRIESTLEY 1967 *Journal of Sound and Vibration* **6**, 86–97. Power spectral analysis of non-stationary random process.
2. J. K. HAMMOND 1968 *Journal of Sound and Vibration* **7**, 393–416. On the response of single and MDOF system to non-stationary random excitation.
3. M. SHINOZUKA 1970 *Journal Engineering Mechanics* **96**, 543–545. Random process with evolutionary power.
4. T. FANG and M. N. SUN 1997 *Archive of Applied Mechanics* **67**, 496–506. A unified approach to two types of evolutionary random response problems in engineering.
5. J. Q. LI 2000 *Doctoral Thesis* (in Chinese), *Northwestern Polytechnical University*, *Xian*, *China* A unified approach to evolutionary random response problems with applications.