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AN INVERSE EIGENVALUE FORMULATION FOR OPTIMIZING THE DYNAMIC BEHAVIOUR OF PIN-JOINTED STRUCTURES

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An efficient relationship between geometric and material properties of pin-jointed truss structures and their eigenvalues is established. The problem is formulated as an inverse eigenvalue problem. This formulation allows the determination of the required modifications on the structural members to achieve specified eigenfrequencies. In addition to the modification of the existing structural elements, the formulation also allows addition of new structural elements to obtain the desired frequencies. Using the proposed inverse method, two cases of plane as well as space truss structures are studied and the results are compared with those obtained using the conventional optimization techniques adopted by commercial finite element codes.

1. INTRODUCTION

Dynamic characteristics are one of the most fundamental considerations in the design of structural systems. The most basic feature in determining the vibration behaviour of a structure is its eigenfrequencies and the eigenmodes associated with each natural frequency. It is important for the designer to ensure that the natural frequencies of the structure do not coincide with the excitation frequencies. The common industrial practise for optimising the vibration behaviour of structures is to conduct a series of modifications on the simulated structure in order to achieve the required eigenfrequencies. Each modification requires the analysis of the modified structure, which is usually only slightly different from the structure previously analyzed. This approach, known as *forward variation approach or design load analysis cycle*, is extremely time consuming, expensive and rarely yields an optimum solution. The vibration minimization problem can be defined as an inverse engineering problem. Inverse engineering refers to problems, where the desired response of the system is known or decided but the physical system is unknown.

In the conventional approach in deciding which parameters to change in achieving the required resonance frequencies, the designer usually considers making structural modification by adding or removing material from certain parts of the structure or adding mass to others. Modification of damping characteristics of the structure is also a means of changing the dynamic behaviour of the structure. In using all these techniques, the previous experience of the engineer is relied on to make the necessary modifications. This implies a number of usually expensive finite element analysis (FEA) design iterations. A number of commercially available FEA codes provide some optimization modules. The algorithm used in these modules is such that the user defines the areas within the model, where the optimization is to be conducted. These are converted by the program to constraint equations. The user also defines the optimization parameters. These are used as state variables. Finally, the user defines the objective function, e.g., minimum eigenvalue. The program then goes through many forward iterations in order to obtain a solution.

To eliminate the need to re-analyze the whole structure, research efforts were conducted towards developing new concepts with sufficient information to find the exact modified parameters which yield the required natural frequencies. Early work in this direction [1–3] utilized the first order terms of a Taylor's series expansion and is based on Rayleigh's work. Chen and Garba [4] used the iterative method to modify structural systems. Later Baldwin and Hutton [5] presented a detailed review of structural modification techniques and classified them into three categories:

(1) techniques based on small modification;
(2) techniques based on localized modification;
(3) techniques based on modal approximation.

Further research on structural modification was carried out by Tsuei *et al.* [6–8], who presented a method of shifting the desired eigenfrequencies using the forced response of the system. The method is based on modification of either the mass or stiffness matrix by treating the modification of the system matrices as an external forced response. This external forced response is formulated in terms of the modification parameters, thus creating a modified eigenvalue problem. More recently, Zhang and Kim [9] investigated the use of mass matrix modification to achieve the desired natural frequencies. McMillan and Keane [10] investigated a method of shifting eigenfrequencies of a rectangular plate by adding concentrated mass elements.

Sivan and Ram [11–13] extended further the research on structural modification by studying the construction of mass and spring system with prescribed natural frequencies. They developed a new algorithm based on Joseph's work [14] to obtain a physically realizable solution. The resulting solutions were only approximate as they were based on an optimization approach rather than an exact solution.

In the last few years, the work on the inverse problem conducted by Gladwell [15] started to be taken seriously by engineers and researchers interested in this field of engineering. The work is applied to both discrete and continuous systems.

In this paper, a simple but efficient formulation between geometric or material properties of pin-jointed truss structures and their eigenvalues is established. The formulation allows the shifting of the frequency and solves for the required modification on chosen geometric and material parameters. The importance of the present formulation becomes more apparent when only a local modification is allowed due to practical constraints.

2. THEORETICAL CONSIDERATION

To construct a system with desired eigenfrequencies, it is necessary to find a relationship between the structural parameters of the system and its eigenfrequencies. For a discrete system such as a mass and spring system, and when only one or 2 degrees of freedom (d.o.f.) are involved, the formulation which accounts for such relationship is easily obtained and hence the change of stiffness or mass required for shifting the eigenvalues can easily be evaluated. However, for systems with a large number of d.o.fs and continuous systems special algorithms have to be developed.

A contribution in this direction was made by Esat and Akbar [16]. They presented the stiffness of a discrete system as a function of the desired eigenvalues and showed that the stiffness varies linearly with the eigenvalues. The formulation is very simple; however, the resulting stiffness of the modified system may not be physically implemented.

For the new system to be constructed, the modification carried out on the structural properties of the system must have a physical meaning (realizable). In the case of truss structures, both the elastic modulus and the cross-sectional area of the bars can be modified to shift the eigenfrequencies. Any modification on the elastic modulus would cause only stiffness modification of the structures. However, a modification in the area parameter would result in both stiffness as well as mass modification of the structure. In the following section, a formulation giving the cross-sectional area modification as a function of the required eigenfrequency is first developed. This formulation can then be used to obtain the elastic modulus variation as a function of the desired eigenfrequency.

3. MODIFICATION OF THE CROSS-SECTIONAL AREA PARAMETER

For a pin-jointed truss structure, both the stiffness and mass modifications can be given as functions of the area modification of any member in the structure:

$$[\Delta\mathbf{K}] = \Delta A[\mathbf{K}'], \qquad [\Delta\mathbf{M}] = \Delta A[\mathbf{M}'], \tag{1}$$

where $[\Delta\mathbf{K}]$ and $[\Delta\mathbf{M}]$ are the variations or modifications of the system stiffness and mass matrices, respectively, ΔA, which is a scalar, is the change in the area of an individual modified member and $[\mathbf{K}']$ and $[\mathbf{M}']$ are the matrices containing the coefficients of the stiffness and mass participation, respectively, of the modified member, i.e., they are the stiffness and mass matrices of the modified member, where the area A is taken as unity.

The equation of motion for free vibration of a dynamic system is given by

$$[\mathbf{K} - \lambda_o\mathbf{M}]\{\boldsymbol{\delta}\} = 0, \tag{2}$$

where $[\mathbf{K}]$ and $[\mathbf{M}]$ are the stiffness and mass matrices of the system, $\{\boldsymbol{\delta}\}$ is the displacement vector and λ_o is the eigenvalue of the original system.

If a modification ΔA is carried out on any member of the structure, this would result in modifications in both stiffness and mass matrices of the whole structure and hence the equation of motion becomes

$$[\mathbf{K} + \Delta\mathbf{K} - \lambda_d\mathbf{M} - \lambda_d\Delta\mathbf{M}]\{\boldsymbol{\delta}\} = 0, \tag{3}$$

where λ_d is the new eigenvalue of the modified structure. It should be noted that the dimensions of $[\Delta\mathbf{K}]$ and $[\Delta\mathbf{M}]$ are adjusted to make their addition to $[\mathbf{K}]$ and $[\mathbf{M}]$ possible.

Transformation to modal co-ordinates can be obtained by putting

$$\{\boldsymbol{\delta}\} = [\boldsymbol{\phi}]\{\mathbf{u}\},$$

where $[\boldsymbol{\phi}]$ is the full (and square) mass normalized modal matrix for the original system. Hence,

$$[\mathbf{K} + \Delta\mathbf{K} - \lambda_d\mathbf{M} - \lambda_d\Delta\mathbf{M}][\boldsymbol{\phi}]\{\mathbf{u}\} = \mathbf{0}, \tag{4}$$

$$[\mathbf{K}\boldsymbol{\phi} + \Delta\mathbf{K}\boldsymbol{\phi} - \lambda_d\mathbf{M}\boldsymbol{\phi} - \lambda_d\Delta\mathbf{M}\boldsymbol{\phi}]\{\mathbf{u}\} = \mathbf{0}. \tag{5}$$

If we pre-multiply the above equation by $[\boldsymbol{\phi}]^{\mathrm{T}}$ and use the orthogonality characteristic of $[\boldsymbol{\phi}]$ with respect to $[\mathbf{K}]$ and $[\mathbf{M}]$ one obtains the following equation:

$$[\boldsymbol{\Omega} + \boldsymbol{\phi}^{\mathrm{T}}\Delta\mathbf{K}\boldsymbol{\phi} - \lambda_d\mathbf{I} - \lambda_d\boldsymbol{\phi}^{\mathrm{T}}\Delta\mathbf{M}\boldsymbol{\phi}]\{\mathbf{u}\} = \mathbf{0}, \tag{6}$$

where $[\boldsymbol{\Omega}]$ is the diagonal eigenvalue matrix and $\mathbf{I}$ is the identity matrix.

Or:

$$[\boldsymbol{\Omega} - \lambda_d \mathbf{I}]\{\mathbf{u}\} = -[\boldsymbol{\phi}^{\mathrm{T}} \Delta \mathbf{K} \boldsymbol{\phi} - \lambda_d \boldsymbol{\phi}^{\mathrm{T}} \Delta \mathbf{M} \boldsymbol{\phi}]\{\mathbf{u}\}. \tag{7}$$

Hence,

$$\{\mathbf{u}\} = -([\boldsymbol{\Omega} - \lambda_d \mathbf{I}]^{-1}[\boldsymbol{\phi}^{\mathrm{T}} \Delta \mathbf{K} \boldsymbol{\phi} - \lambda_d \boldsymbol{\phi}^{\mathrm{T}} \Delta \mathbf{M} \boldsymbol{\phi}])\{\mathbf{u}\}. \tag{8}$$

By pre-multiplying both sides by $[\boldsymbol{\phi}]$ and substituting $[\Delta \mathbf{K}]$, $[\Delta \mathbf{M}]$ and $[\boldsymbol{\phi}]\{\mathbf{u}\}$ by $\Delta A[\mathbf{K}']$, $\Delta A[\mathbf{M}']$ and $\{\boldsymbol{\delta}\}$, respectively, gives

$$\{\boldsymbol{\delta}\} = -\Delta A([\boldsymbol{\phi}][\boldsymbol{\Omega} - \lambda_d \mathbf{I}]^{-1}[\boldsymbol{\phi}^{\mathrm{T}}][\mathbf{K}' - \lambda_d \mathbf{M}'])\{\boldsymbol{\delta}\}. \tag{9}$$

This can be written as

$$\{\boldsymbol{\delta}\} = -\Delta A[\mathbf{F}][\mathbf{G}]\{\boldsymbol{\delta}\}, \tag{10}$$

where

$$[\mathbf{F}] = [\boldsymbol{\phi}][\boldsymbol{\Omega} - \lambda_d \mathbf{I}]^{-1}[\boldsymbol{\phi}]^{\mathrm{T}} \tag{11}$$

and

$$[\mathbf{G}] = [\mathbf{K}' - \lambda_d \mathbf{M}']. \tag{12}$$

Equation (10) can be simplified in a matrix form as

$$\begin{bmatrix} \Delta A^{-1} + (\mathbf{FG})_{1,1} & (\mathbf{FG})_{1,2} & \cdots & (\mathbf{FG})_{1,n} \\ (\mathbf{FG})_{2,1} & \Delta A^{-1} + (\mathbf{FG})_{2,2} & \cdots & (\mathbf{FG})_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ (\mathbf{FG})_{n,1} & (\mathbf{FG})_{n,2} & \cdots & \Delta A^{-1} + (\mathbf{FG})_{n,n} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdots \\ \delta_n \end{Bmatrix} = \mathbf{0}, \tag{13}$$

where the terms $(\mathbf{FG})_{i,j}$ are elements of the matrix $[\mathbf{FG}]$ which is a function of the eigenvalue $\lambda_d$and the suffix n denotes the nth term in the matrix. It should be noted that the solutions to equation (13) are not necessarily real and positive. This is because, although the matrices $[\mathbf{F}]$ and $[\mathbf{G}]$ are symmetric, their product is not. However, a necessary and sufficient condition for solvability of this equation is set in the solution algorithm by requiring that $(\Delta A/A) \leqslant 1$. By imposing this condition and noting that ΔA represents the variation in the cross-sectional area (which can be negative) only real solutions to equation (13) are retained.

The characteristic equation of the modified system for the eigenvalue λ_d is given by

$$\begin{vmatrix} \Delta A^{-1} + (\mathbf{FG})_{1,1} & (\mathbf{FG})_{1,2} & \cdots & (\mathbf{FG})_{1,n} \\ (\mathbf{FG})_{2,1} & \Delta A^{-1} + (\mathbf{FG})_{2,2} & \cdots & (\mathbf{FG})_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ (\mathbf{FG})_{n,1} & (\mathbf{FG})_{n,2} & \cdots & \Delta A^{-1} + (\mathbf{FG})_{n,n} \end{vmatrix} = 0. \tag{14}$$

Equations (13) are for global modification, where all the bars are to be modified at the same time and with the same ΔA. In this case, n is equal to the total number of unconstrained d.o.fs. However, if this is not the case, then only the terms corresponding to the nodes associated with the modified bars are retained. For example, if only one bar is to be modified then equations (13) would be reduced to four equations for a plane truss and to six equations for a space truss, where the associated nodes are not constrained. It is more usual to conduct a sensitivity analysis on the structure first by considering the variation of ΔA with respect to change of eigenfrequency for each structural member. The modifications are then carried out on the members which are most sensitive to this change. A solution ΔA for the above problem can be obtained by solving the characteristic equation (14) once the desired eigenvalue λ_d is specified.

4. ALGORITHM

For a given truss structure:

(1) Obtain the stiffness and mass matrices $[\mathbf{K}]$ and $[\mathbf{M}]$,
(2) run a modal analysis to obtain the natural eigenvalues $[\mathbf{\Omega}]$ and the corresponding eigenvectors,
(3) compute the mass normalized modal matrix $[\boldsymbol{\phi}]$,
(4) obtain the desired eigenvalue λ_d from the desired frequency f_d,
(5) compute the matrix $\mathbf{F} = [\boldsymbol{\phi}][\mathbf{\Omega} - \lambda_d \mathbf{I}]^{-1}[\boldsymbol{\phi}]^{\mathrm{T}}$,
(6) specify the member to be modified,
    (a) compute the stiffness and mass matrices $[\mathbf{K}']$ and $[\mathbf{M}']$ by taking the cross-sectional area of the member as unity,
    (b) obtain the matrix $[\mathbf{G}] = [\mathbf{K}' - \lambda_d \mathbf{M}']$,
    (c) carry out the matrix multiplication $[\mathbf{FG}] = [\mathbf{F}][\mathbf{G}]$,
    (d) compute ΔA^{-1} from the characteristic equation (14), $|\Delta A^{-1}[\mathbf{I}] + [\mathbf{F}][\mathbf{G}]| = 0$,
    (e) determine ΔA which represents the necessary variation of the cross-sectional area of the member considered for the desired frequency.
(7) Repeat step 6 to consider another member.

The algorithm allows for the addition of new members to the structure to obtain the required frequency. In this case, if the new bar is to be added between two existing nodes then only the other geometric and material properties and the nodes numbers are to be entered in step 6 for $[\mathbf{K}']$ and $[\mathbf{M}']$ to be computed. However, if the new bar is to be connected to a new external node, then of course that node has to be considered in step 1 with any corresponding boundary conditions. If more than one member is to be altered simultaneously, as in the case of the second example in the next section, then in step 6 above all the numbers of the members to be modified must be entered, and in step 6(a), $[\mathbf{K}']$ and $[\mathbf{M}']$ refer to the assembled stiffness and mass matrices for the considered members.

5. NUMERICAL APPLICATION

5.1. THREE-BAR TRUSS

A three-bar truss structure is shown in Figure 1 and has the following dimension and material properties:

$$E = 2 \times 10^{11}\ \mathrm{kN/m^2},$$

$$\rho = 7860\ \mathrm{kg/m^3}.$$

Cross-sectional area for all members is

$$A = 0{\cdot}3871 \times 10^{-4}\ \mathrm{m^2}.$$

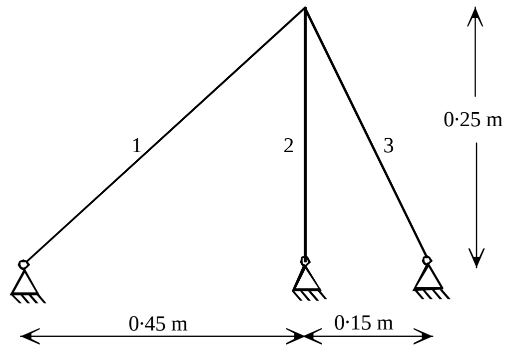

Figure 1. Three-bar truss.

The stiffness and mass matrices for the structure are

$$K_o = \begin{bmatrix} 0{\cdot}185216 \times 10^8 & -0{\cdot}533072 \times 10^7 \\ -0{\cdot}533072 \times 10^7 & 0{\cdot}540406 \times 10^8 \end{bmatrix}$$

$$M_o = \begin{bmatrix} 0{\cdot}107133 & 0 \\ 0 & 0{\cdot}107133 \end{bmatrix}.$$

The resulting eigenvalues and mass normalized modal matrices are obtained as

$$\Omega_o = \begin{bmatrix} 0{\cdot}165577 \times 10^9 & 0 \\ 0 & 0{\cdot}511732 \times 10^9 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} 3{\cdot}02277 & 0{\cdot}443877 \\ 0{\cdot}443877 & -3{\cdot}02277 \end{bmatrix}.$$

The lowest frequency $f_1 = \sqrt{\lambda_1}/2\pi = 2047{\cdot}95$ Hz, if this is to be increased by 5% the new (desired) frequency would be $f_{1d} = 2150{\cdot}35$ Hz and the corresponding eigenvalue $\lambda_d = 0{\cdot}182549 \times 10^9$.

If a modification is to be carried on the cross-sectional area of member 1 to achieve the new desired frequency, then both matrices [**K′**] and [**M′**] for member 1 have to be calculated. These are equal to

$$K' = \begin{bmatrix} 0{\cdot}296884 \times 10^{12} & 0{\cdot}164935 \times 10^{12} \\ 0{\cdot}164935 \times 10^{12} & 0{\cdot}916307 \times 10^{12} \end{bmatrix},$$

$$M' = \begin{bmatrix} 1348{\cdot}73 & 0 \\ 0 & 1348{\cdot}73 \end{bmatrix}.$$

Both matrices [**F**] and [**G**] as defined in equations (11) and (12), can now be computed and by applying equations (13) gives:

$$\begin{bmatrix} \Delta A^{-1} - 0{\cdot}409636 \times 10^5 & -0{\cdot}758481 \times 10^5 \\ -0{\cdot}154944 \times 10^4 & \Delta A^{-1} - 0{\cdot}162078 \times 10^5 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = 0.$$

By solving the above characteristic equation: one obtains $\Delta A = 0{\cdot}222 \times 10^{-4}$ m^2. Therefore, in order to shift the frequency f_1 to 2150·35 Hz the cross sectional area of bar 1 needs to be increased by 57·36%.

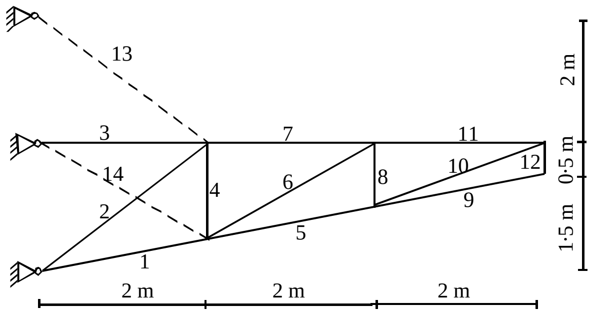

Figure 2. Plane truss structure.

TABLE 1

Five lowest natural frequencies (Hz) of the plane truss structure

Software	Five lowest natural frequencies				
Present results	39·2741	99·7494	165·864	187·710	362·533
ANSYS	39·274	99·749	165·86	187·71	362·53

5.2. PLANE TRUSS STRUCTURE

(1) The second case considered is a 12-bar truss cantilever as shown in Figure 2. Bars 13 and 14 are initially not included in this section. The material properties and the cross-sectional area of the bars are

$E = 2 \times 10^{11}$ N/m^2,
$\rho = 7860$ kg/m^3,
$A = 5 \times 10^{-4}$ m^2 for all bars.

As shown in the previous section the method consists of two main steps. These are:

(1) The modal analysis to calculate the eigenvalues and eigenvectors of the original system as well as the required data for subsequent use.
(2) The specification of the frequency (eigenvalue) and the determination of the required modification.

Therefore, the developed code has been validated by first applying it to modal analysis of the truss cantilever and the results obtained for the first five frequencies are checked against those obtained from ANSYS [17]. These are shown in Table 1, where it is seen that a very good comparison is obtained.

In the second step, the lowest natural frequency was increased by 5% through increments of 1% and for each increment, the required modification in the cross-sectional area of bar 1 was determined. The same problem was now presented to ANSYS [17] as an optimization problem, where the objective function was defined as minimization of the bar cross-sectional area subject to feasible upper and lower bounds. The state variable and design variables were, respectively, defined as the required eigenfrequency and the bar cross-sectional area. The ANSYS program uses the conventional forward iteration optimization algorithm. This involved 14 iterations to arrive at the optimum solution.

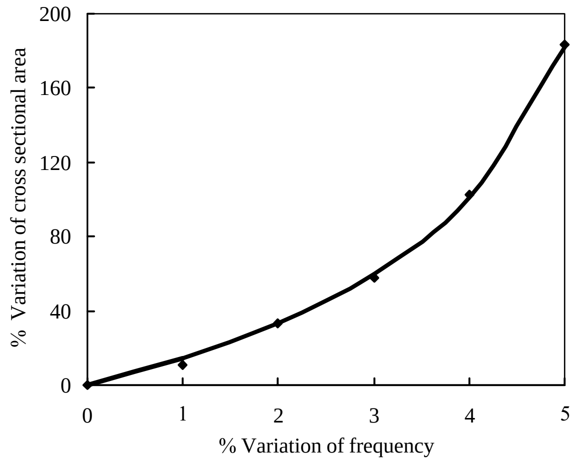

Figure 3. Variation of first frequency with the required modification in the cross-sectional area of bar 1 (——, present Sol.; ◆, ANSYS).

Results obtained from the present formulation as well as from ANSYS optimization analysis are shown in Figure 3. In comparing the two approaches it should be pointed out that:

- In the present formulation no iterations or convergence were involved and only the solution of a quadratic equation (equation (14)) was required to obtain an exact solution, i.e., if the calculated modifications were implemented and a modal analysis was carried out the exact desired frequency would be obtained.
- Using commercial software such as ANSYS requires the reanalysis of the whole structure with a large number of iterations. In addition, the results obtained for the modification are only approximate, i.e., if the calculated modifications were implemented, and a modal analysis was carried out, only an approximate solution to the desired frequency would be obtained.

Having validated the program, it is now used to further illustrate the efficiency of the developed formulation. To investigate the required modifications of the cross-sectional area of the bars for desired frequencies, the lowest natural frequency of the truss cantilever was increased by $\Delta f_1 = 5\%$ through steps of 0·5% and for each step the required change in the cross-sectional area of each bar is obtained. These are shown in Figure 4. It can be seen that while an increase in the cross-sectional area of some bars, for example, 1, 2, 3 and 7, is necessary to achieve the desired frequency, other bars require their areas to be decreased. This is due to the fact that the cross-sectional area affects both the mass and stiffness matrices of the structure. It is also noted that the fixed frequency may not be achieved by varying the areas of some bars, for example in this case, by shifting the frequency by 2% no solution is obtained by modifying bars 2, 7 or 8. Similarly, a shifting of the lowest frequency by 5% can be achieved by modifying the cross-sectional area of bars 1, 3 or 9 only. Therefore, if no restriction is made on which bar is to be modified to shift the frequency, the designer can compare the set of results and choose the structural member to be modified.

(2) As mentioned above, the method can also deal with the possibility of adding new bars to an existing structure to achieve a desired frequency. For example, Figure 2 shows the addition of new bars to the truss structure. In this example, bars 13 and 14 are added separately in order to shift the lowest frequency. Figure 5 shows that to shift the frequency

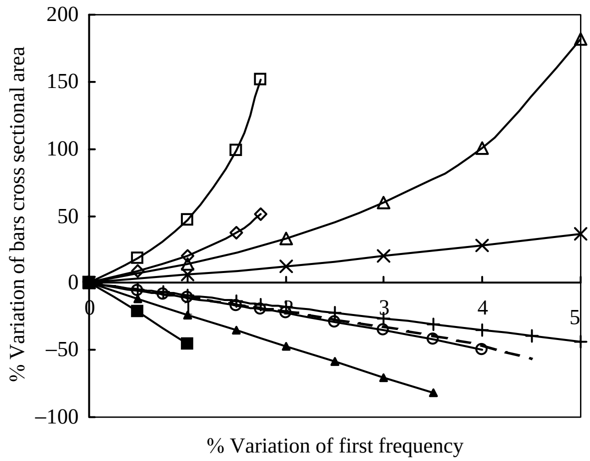

Figure 4. Variation of first frequency with the required modification in the cross-sectional area of bars. Only one bar at a time varied (—△—, bar 1; —□—, bar 2; —×—, bar 3; —◇—, bar 7; —■—, bar 8; —+—, bar 9; – – – , bar 10; —○—, bar 11; —▲—, bar 12).

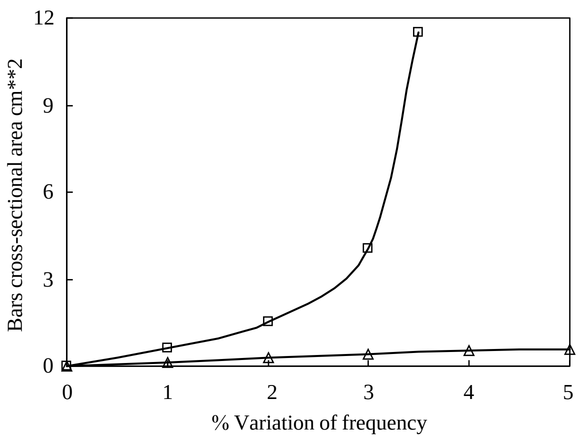

Figure 5. Variation of first frequency with the cross-sectional area of the added bars (—△—, bar 13; —□—, bar 14).

by 5% only a small sectional area for bar 13 is required and this cannot be achieved by adding the bar 14 alone no matter how large is its cross-sectional area. This last example demonstrates how efficiently the method can be used to modify the frequencies of the existing structures.

5.3. SPACE TRUSS STRUCTURE

The second example corresponds to the tower shown in Figure 6. The cross-sectional areas of the bars are:

$A = 3 \times 10^{-4}$ m^2 for C_1 and C_2 bars (corner columns in bottom and top levels respectively),
$A = 1{\cdot}5 \times 10^{-4}$ m^2 for B_1 bars (horizontal members in bottom level),
$A = 0{\cdot}8 \times 10^{-4}$ m^2 for B_2 bars (horizontal members in top level),

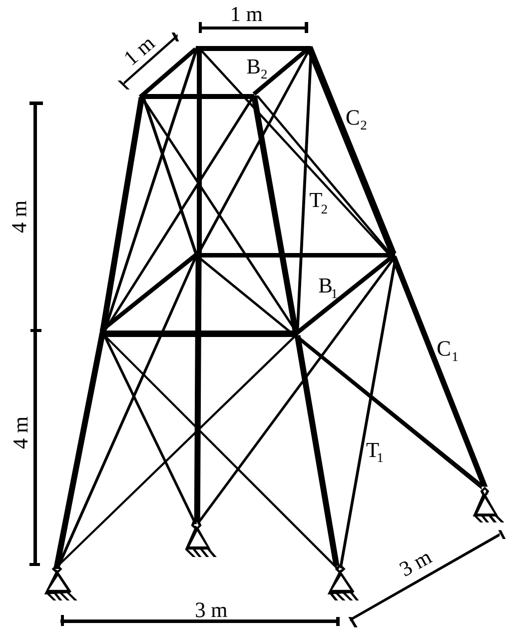

Figure 6. Space truss structure.

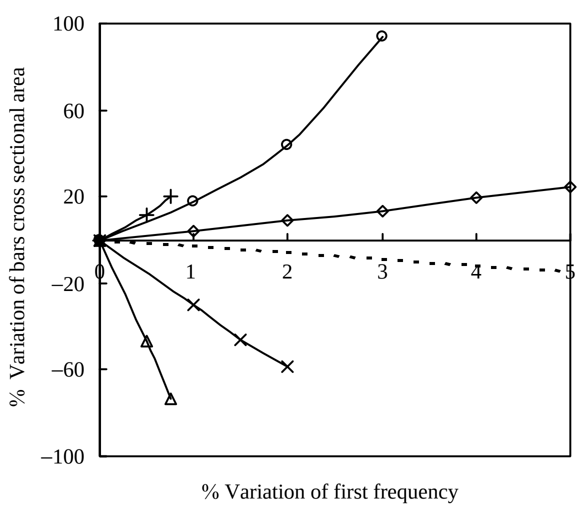

Figure 7. Variation of first frequency with the required modification in the cross-sectional area of bars (—◇—, C_1 bars; —⊟—, C_2 bars; -△—, B_1 bars; —×—, B_2 bars; —○—, T_1 bars; —+—, T_2 bars).

$A = 0{\cdot}8 \times 10^{-4}\ \mathrm{m}^2$ for T_1 bars (diagonal members in bottom level),
$A = 0{\cdot}4 \times 10^{-4}\ \mathrm{m}^2$ for T_2 bars (diagonal members in top level).

The cross-sectional areas given above are for each bar, the elasticity modulus is equal to $2{\cdot}1 \times 10^{11}\ \mathrm{N/m}^2$ and the material density is $\rho = 7860\ \mathrm{kg/m}^3$.

The sensitivity of the lowest natural frequency to any modification on the cross-sectional area of the different bars is first investigated. Figure 7 shows the percentage variation of the

TABLE 2

The five lowest frequencies (Hz) of the original and modified structures

	Modified structure	
Original structure	Cross-sectional area of bars C_1 increased	Cross-sectional area of bars C_2 decreased
26·96	30·00	30·21
31·21	32·91	35·00
40·89	39·65	46·61
66·59	60·28	68·86
67·97	62·10	71·07

first natural frequency with the required percentage variation on the cross-sectional area of the bars. It is seen that the first frequency is most sensitive to bars C_1 and C_2.

To further illustrate the effectiveness of the developed formulation, two practical applications are carried out on the tower structure. In the first case, it was desired to shift the first frequency of the original structure from 26·70 to 30·00 Hz by restricting alteration to the cross-sectional area of C_1 bars. The results obtained from the developed method showed that an increase in the cross-sectional area of bars C_1 from $3{\cdot}00 \times 10^{-4}$ to $5{\cdot}60 \times 10^{-4}\,\mathrm{m}^2$ is required. The results showing the lowest five natural frequencies for both the original and modified structure are given in Table 2.

In the second case, it was desired to shift the second frequency of the original structure from 31·21 to 35·00 Hz by restricting alteration to the cross-sectional area of C_2 bars. The solution obtained showed that a reduction in the cross-sectional area of C_2 bars from $3{\cdot}00 \times 10^{-4}$ to $1{\cdot}95 \times 10^{-4}\mathrm{m}^2$ is required. The results showing the lowest five natural frequencies for both original and modified structures for this case are also given in Table 2.

6. CONCLUSIONS

In this paper, a method for determining the required structural property modification to achieve desired frequencies for pin-jointed structures is developed. The formulation allows the determination of the necessary modifications on the bars cross-sectional area to shift any of the frequencies to desired positions. The approach can be used to increase or decrease the frequencies, and the structural modifications can also include the addition of new structural members. This approach provides the structural designers with an efficient algorithm, which is formulated in such a way that no iterations or convergence are involved in the process and only few calculations are required to obtain the necessary modifications. Further research work is currently being undertaken to develop the formulation and algorithm for two-dimensional and higher order elements.

REFERENCES

1. J. H. WILKINSON 1963 *The Algebraic Eigenvalue Problem*. Oxford: Oxford University Press, 62–109.
2. H. VAN BELLE 1982 *American Institute of Aeronautics and Astronautics Journal* **20**, 286–288. Higher order sensitivities in structural design.

3. P. Vanhonacker 1980 *American Institute of Aeronautics and Astronautics Journal* **18**, 1511–1514. Differential and difference sensitivities of natural frequencies and mode shapes of natural structures.
4. J. A. Chen and J. A. Garba 1980 *American Institute of Aeronautics and Astronautics Journal* **18**, 684–690. Analytical model improvement using modal test results.
5. J. Baldwin and S. Hutton 1985 *American Institute of Aeronautics and Astronautics Journal* **23**, 1737–1743. Natural modes of modified structures.
6. Y. G. Tsuei and E. Yee 1989 *Journal of Dynamic Systems* **111**, 403–408. Measurement and control. A method for modifying dynamic properties of undamped mechanical systems.
7. E. Yee and Y. G. Tsuei 1991, in: *Proc. ASME Conf. on Modal Analysis*, *Modelling*, *Diagnostics and Control—Analytical and Experimental* 1991, DE-Vol. **38**, 101–106. Modification of stiffness for shifting natural frequencies of damped mechanical systems.
8. E. Yee and Y. G. Tsuei 1991 *American Institute of Aeronautics and Astronautics Journal* **29**, 1973-1977. Method for shifting natural frequencies of damped mechanical systems.
9. Ki-ook Kim 1993 *Journal of Computers and Structures* **46**, 1041–1048. A review of mass matrices for eigenproblems.
10. A. J. McMillan and A. J. Keane 1996 *Journal of Sound and Vibration* **192**, 549–562. Shifting resonances from a frequency band by applying concentrated masses to a thin rectangular plate.
11. Y. M. Ram 1994 *Journal of Sound and Vibration* **176**, 225–234. Enlarging a spectral gap by structural modification.
12. D. Sivan and Y. M. Ram 1996 *Communication in Numerical Methods in Engineering* **12**, 531–542. Mass and stiffness modification to achieve desired natural frequencies.
13. D. Sivan and Y. M. Ram 1997 *Journal of Sound and Vibration* **201**, 323–334. Optimal construction of mass-spring system with prescribed model and spectral data.
14. K. T. Joseph 1992 *American Institute of Aeronautics and Astronautics Journal* **30**, 2890–2896. Inverse eigenvalue problem in structural design.
15. G. M. L. Gladwell 1997 *Inverse Problems* **29**, 421–434. Inverse vibration problems for finite element models.
16. I. I. Esat and S. Akbar 1997 *American Society of Mechanical Engineers ASIA*' 97, Congress and Exhibition, Singapore. Synthesis of multi-body systems for desired eigenfrequencies.
17. *ANSYS User's Manual*: *Procedures*, Vol. I, Swanson Analysis Systems, Inc. Version 5.8, 1999.