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LETTERS TO THE EDITOR

STABILITY OF A SHORT UNIFORM CANTILEVER COLUMN SUBJECTED TO AN INTERMEDIATE FOLLOWER FORCE

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1. INTRODUCTION

An excellent survey of the stability of columns subjected to follower forces can be seen in references [1, 2]. During the 1970s, finite element formulations were developed to study this problem based on energy method [3, 4] and Galerkin method [5, 6]. The stability boundaries of columns subjected to an intermediate follower force have been presented in reference [7]. In this study, the column is assumed as a slender one and the effects of shear deformation and rotary inertia are not considered. However, Euler buckling studies of shear flexible columns with intermediate load and with overhang are presented in references [8, 9]. Consideration of shear deformation and rotary inertia is necessary in the case of columns subjected to intermediate follower forces as the load comes closer to the fixed end where the immovability condition of axial displacement is imposed. This aspect is well established in the study of one of the authors while dealing with the Euler-type intermediate load [8].

The aim of the present paper is to study the effect of shear deformation and rotary inertia on the stability of cantilever columns subjected to an intermediate concentrated follower force. This aspect of intermediate follower force is not studied till now, to the best of the author's knowledge; even though the stability behaviour of short cantilever columns with a tip concentrated follower force is available in reference [10].

The formulation of the present problem is briefly discussed in the next section followed by numerical results, discussion and conclusions.

2. FINITE ELEMENT FORMULATION

A uniform cantilever column of length L subjected to a concentrated intermediate follower force P is shown in Figure 1. The column is idealized into a number of finite elements, the length of a typical finite element being l. The various energies/work done, considering shear deformation and rotary inertia, for the problem considered here, are

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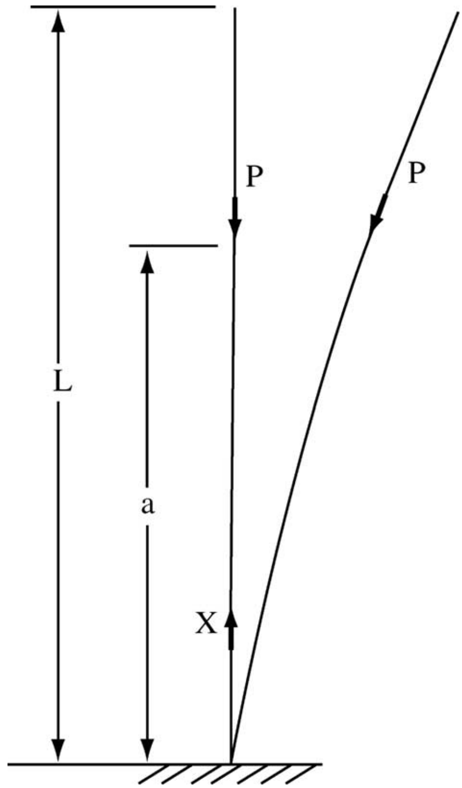

Figure 1. A uniform cantilever column subjected to an intermediate follower force.

given by

$$U = \frac{EI}{2}\int_0^l (\theta)^2\,\mathrm{d}x + \frac{kGA}{2}\int_0^l (w' + \theta)^2\,\mathrm{d}x, \tag{1}$$

$$T = \frac{\rho A\omega^2}{2}\int_0^l (w)^2\,\mathrm{d}x + \frac{\rho I\omega^2}{2}\int_0^l (\theta)^2\,\mathrm{d}x, \tag{2}$$

$$W_c = \frac{P}{2}\int_0^l (w')^2\,\mathrm{d}x, \quad W_{nc} = -\,Pw'(a)w(a), \tag{3, 4}$$

where, U is the strain energy, T is the kinetic energy, W_c is the conservative part of the work done by P, W_{nc} is the non-conservative part of the work done by P, E is the Young's modulus, I is the area moment of inertia, k is the shear correction factor (taken as 5/6), G is the shear modulus, A is the area of cross-section, ρ is the mass density, w is the lateral displacement, θ is the section rotation ($= \gamma - w'$, γ being the shear rotation), a is the global axial co-ordinate X where the concentrated follower force is applied and $(\,)'$ denotes the differentiation with respect to the element axial co-ordinate x.

Assuming cubic displacement distribution for w and θ over the element as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3, \tag{5}$$

$$\theta = \alpha_5 + \alpha_6 x + \alpha_7 x^2 + \alpha_8 x^3 \tag{6}$$

and applying the standard procedure [11], we obtain the element matrices with respect to the degrees of freedom w_1, w'_1, θ_1, θ'_1, w_2, w'_2, θ_2, θ'_2 (subscripts 1 and 2 representing the two ends of the beam element) as

$$[k] =$$

$$\begin{bmatrix}
\frac{-6}{5l}kGA & \frac{-1}{10}kGA & \frac{1}{2}kGA & \frac{l}{10}kGA & \frac{6}{5l}kGA & \frac{-1}{10}kGA & \frac{1}{2}kGA & \frac{-1}{10}kGA \\
\frac{-1}{10}kGA & \frac{-2l}{15}kGA & \frac{-l}{10}kGA & 0 & \frac{1}{10}kGA & \frac{l}{30}kGA & \frac{l}{10}kGA & \frac{-l^2}{60}kGA \\
\frac{1}{2}kGA & \frac{-l}{10}kGA & \frac{-6}{5l}EI - \frac{13l}{35}kGA & \frac{-1}{10}EI - \frac{11l^2}{210}kGA & \frac{-1}{2}kGA & \frac{l}{10}kGA & \frac{6}{5l}EI - \frac{9l}{70}kGA & \frac{-1}{10}EI + \frac{13l^2}{420}kGA \\
\frac{l}{10}kGA & 0 & \frac{-1}{10}EI - \frac{11l^2}{210}kGA & \frac{-2l}{15}EI - \frac{l^3}{105}kGA & \frac{-l}{10}kGA & \frac{l^2}{60}kGA & \frac{1}{10}EI - \frac{13l^2}{420}kGA & \frac{l}{30}EI + \frac{l^3}{140}kGA \\
\frac{6}{5l}kGA & \frac{1}{10}kGA & \frac{-1}{2}kGA & \frac{-l}{10}kGA & \frac{-6}{5l}kGA & \frac{1}{10}kGA & \frac{-1}{2}kGA & \frac{l}{10}kGA \\
\frac{-1}{10}kGA & \frac{l}{30}kGA & \frac{l}{10}kGA & \frac{l^2}{60}kGA & \frac{1}{10}kGA & \frac{-2l}{15}kGA & \frac{-l}{10}kGA & 0 \\
\frac{1}{2}kGA & \frac{l}{10}kGA & \frac{6}{5l}EI - \frac{9l}{70}kGA & \frac{1}{10}EI - \frac{13l^2}{420}kGA & \frac{-1}{2}kGA & \frac{-l}{10}kGA & \frac{-6}{5l}EI - \frac{13l}{35}kGA & \frac{1}{10}EI + \frac{11l^2}{210}kGA \\
\frac{-1}{10}kGA & \frac{-l^2}{60}kGA & \frac{-1}{10}EI + \frac{13l^2}{420}kGA & \frac{l}{30}EI + \frac{l^3}{140}kGA & \frac{l}{10}kGA & 0 & \frac{1}{10}EI + \frac{11l^2}{210}kGA & \frac{-2l}{15}EI - \frac{l^3}{105}kGA
\end{bmatrix} \tag{7}$$

$$[m] = \rho \begin{bmatrix}
\frac{13}{35}Al & \frac{11}{210}Al^2 & 0 & 0 & \frac{9}{70}Al & \frac{-13}{420}Al^2 & 0 & 0 \\
\frac{11}{210}Al^2 & \frac{1}{105}Al^3 & 0 & 0 & \frac{13}{420}Al^2 & \frac{-1}{140}Al^3 & 0 & 0 \\
0 & 0 & \frac{13}{35}Il & \frac{11}{210}Il^2 & 0 & 0 & \frac{9}{70}Il & \frac{-13}{420}Il^2 \\
0 & 0 & \frac{11}{210}Il^2 & \frac{1}{105}Il^3 & 0 & 0 & \frac{13}{420}Il^2 & \frac{-1}{140}Il^3 \\
\frac{9}{70}Al & \frac{13}{420}Al^2 & 0 & 0 & \frac{13}{35}Al & \frac{-11}{210}Al^2 & 0 & 0 \\
\frac{-13}{420}Al^2 & \frac{-1}{140}Al^3 & 0 & 0 & \frac{-11}{210}Al^2 & \frac{1}{105}Al^3 & 0 & 0 \\
0 & 0 & \frac{9}{70}Il & \frac{13}{420}Il^2 & 0 & 0 & \frac{13}{35}Il & \frac{-11}{210}Il^2 \\
0 & 0 & \frac{-13}{420}Il^2 & \frac{-1}{140}Il^3 & 0 & 0 & \frac{-11}{210}Il^2 & \frac{1}{105}Il^3
\end{bmatrix}, \tag{8}$$

$$[g_c] = \begin{bmatrix}
\frac{6}{5l}P & \frac{1}{10}P & 0 & 0 & \frac{-6}{5l}P & \frac{1}{10}P & 0 & 0 \\
\frac{1}{10}P & \frac{2l}{15}P & 0 & 0 & \frac{-1}{10}P & \frac{-l}{30}P & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-6}{5l}P & \frac{-1}{10}P & 0 & 0 & \frac{6}{5l}P & \frac{-1}{10}P & 0 & 0 \\
\frac{1}{10}P & \frac{-l}{30}P & 0 & 0 & \frac{-1}{10}P & \frac{2l}{15}P & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{9}$$

and

$$[g_{nc}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

where $[k]$ is the element elastic stiffness matrix, $[m]$ is the element mass matrix, $[g_c]$ is the conservative part of the element geometric matrix and $[g_{nc}]$ is the non-conservative part of the element geometric stiffness matrix. It is to be noted here that in the case of an intermediate load, $[g_c]$ is a null matrix for unloaded elements and $[g_{nc}]$ is the only non-zero matrix where an element is first loaded with an intermediate follower force.

After the usual assembly procedure, the governing equation of motion for the stability of the column subjected to a follower force is written as

$$[K]\{\delta\} - [G_c]\{\delta\} - [G_{nc}]\{\delta\} - \omega^2[M]\{\delta\} = 0, \tag{11}$$

where $[K]$, $[G_c]$, $[G_{nc}]$, $[M]$ and $\{\delta\}$ are the assembled elastic stiffness matrix, conservative part of the geometric and non-conservative part of the geometric stiffness matrices, mass matrix and eigenvector respectively.

Equation (11) can be solved to obtain the frequencies ω with the variation of intermediate follower force P and the critical load P_{cr} is that load where two frequencies coalesce.

3. NUMERICAL RESULTS AND DISCUSSION

Figure 1 shows a uniform cantilever column with an intermediate follower force. Using the formulation given in the previous section, the critical load parameters $\lambda_{cr}(= P_{cr}a^2/\pi^2 EI)$ where P_{cr} is the critical load which is obtained for various L/r and a/L ratios. The finite element idealization, 16 equal length elements above the load point and 16 equal length elements below the load point are found to give converged results up to four significant figures for all the a/L values considered in the present study. For $a/L = 1{\cdot}0$, i.e., for the tip concentrated follower force, 16 equal length elements give the same accuracy mentioned above and for this case, the present results match very well with those given in reference [10].

The values of critical load parameter λ_{cr}, coalescence frequency Ω_c ($= \rho A\omega_c^2 L^4/EI$, where ω_c is coalescence frequency) and the coalescing vibration modes are given in Table 1 for various values of L/r and a/L. $L/r = 1000$ represents the results for a slender column without the effects of shear and rotary inertia. For $L/r = 100$, the effects of shear and rotary inertia become predominant, for example, for $a/L = 0{\cdot}2$. This shows that even for a relatively slender column, the position of the intermediate follower force is important.

In other words, what matters for the present problem is not the global slenderness ratio of the column ($L/r = 100$) but the local slenderness ratio ($a/r = 20$). The variation of λ_{cr} for $L/r = 1000$ and 100 is first decreasing and then increasing as a/L decreases. This sort of

TABLE 1

Buckling load parameter λ_{cr} and the coalescence frequency parameter Ω_c for different L/r of a cantilever column subjected to an intermediate follower force

L/r	a/L	λ_{cr}	Ω_c	Coalescing modes
1000	1·0	2·032	121·3	1 and 2
	0·8	1·301	114·5	1 and 2
	0·6	0·9347	108·5	1 and 2
	0·4	1·039	144·5	1 and 2
	0·2	1·214	5155	3 and 4
100	1·0	2·010	119·5	1 and 2
	0·8	1·290	113·5	1 and 2
	0·6	0·9257	107·2	1 and 2
	0·4	1·014	140·9	1 and 2
	0·2	1·159	1035	3 and 4
50	1·0	1·949	114·4	1 and 2
	0·8	1·256	109·1	1 and 2
	0·6	0·8996	103.8	1 and 2
	0·4	0·9469	132·9	1 and 2
	0·2	0·8936	1835	2 and 3
25	1·0	1·740	97·49	1 and 2
	0·8	1·136	94·01	1 and 2
	0·6	0·8079	90·50	1 and 2
	0·4	0·7492	107·0	1 and 2
	0·2	0·4676	954·3	2 and 3
10	1·0	1·017	46·62	1 and 2
	0·8	0·6849	46·16	1 and 2
	0·6	0·4691	45·06	1 and 2
	0·4	0·3079	44·40	1 and 2
	0·2	0·1170	116·6	2 and 3

TABLE 2

Buckling load parameter λ_{cr} for an intermediate follower force and conservative force for $L/r = 25$

a/L	λ_{cr} for follower force, Present study	λ_{cr} for conservative force, reference [8]
1·0	1·7400	0·2467
0·8	1·1360	0·2452
0·6	0·8079	0·2417
0·4	0·7492	0·2324
0·2	0·4676	0·1938

trend is not unexpected in the case of stability of columns subjected to follower forces and is shown in reference [6], while studying the stability behaviour of spring-hinged cantilever columns. The trend for Ω_c is also similar. It is interesting to note that coalescing modes of vibration are 1 and 2 up to $a/L = 0{\cdot}4$ and for $a/L = 0{\cdot}2$, these are 3 and 4.

For $a/r = 50$, the trend is increasing with decreasing a/L, whereas the trend in Ω_c is as discussed for $L/r = 1000$ and 100. However, the coalescing vibration modes for $a/L = 0{\cdot}2$

are 2 and 3 instead of 3 and 4. λ_{cr} monotonically decreases with decreasing a/L for $L/r = 25$ and 10, but the trend in Ω_c is as discussed for $L/r = 50$ and the coalescing vibration modes are 2 and 3 for these cases of L/r for $a/L = 0{\cdot}2$.

The effect of shear deformation and rotary inertia is highly predominant in the case of an intermediate follower force when compared to its counterpart of an intermediate conservative (Euler) force. This fact is shown by comparing the present result with those of reference [8]. In Table 2, the comparison is given. For a typical L/r, say 25 the λ_{cr} for $a/L = 1{\cdot}0$ and 0·2 differ by 272% in the case of a follower force, whereas the difference in λ_{cr} (with the same definition) is 27·33% in the case of conservative force.

4. CONCLUDING REMARKS

The stability behaviour of uniform short cantilever columns subjected to an intermediate concentrated follower force (Beck type) is investigated in this paper using the versatile finite element method. Obviously for this problem, the instability is of flutter type and hence the stability parameter can be obtained by applying dynamic criterion. Since short columns are considered in this paper, the effects of shear deformation and rotary inertia is included in the formulation. Numerical results are obtained for various slenderness ratios of the column and for several positions of the intermediate follower force. The numerical results reveal trends that are specific to non-conservative problem. The coalescing modes of vibration when the follower force is very near to the support are seen to be higher modes. The effects of shear deformation and rotary inertia are found to be significantly higher in the case of the intermediate follower force when compared to a similar problem with intermediate conservative force (Euler type).

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