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A PARAMETRIC STUDY ON VIBRATING CLAMPED ELLIPTICAL PLATES WITH VARIABLE THICKNESS

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1. INTRODUCTION

Plates of various shapes and of non-uniform thickness are widely used in engineering structures. Elliptical plates commonly used as cover plates for cut-outs in such engineering systems, too, have attracted attention of researchers over the years. Although there have been a lot of published work on vibration of clamped elliptical plates in recent years, some of which are listed in references [1–6], quite little has been reported for elliptical plates with variable thickness e.g., references [1–4]. Singh and Tyagi [1] used Galerkin's method to obtain frequencies of symmetric transverse vibrations of a clamped elliptic plate with parabolically varying thickness. Singh and Chakraverty [2] used boundary characteristic orthogonal polynomials in two dimensions to study transverse vibration of elliptical plates with variable thickness. The method employed was the Rayleigh–Ritz method this time. Again, Singh and Saxena [4] used the Rayleigh–Ritz method to find the first three frequencies and mode shapes for free flexural vibration of a plate in the form of a quadrant of an ellipse with linear and quadratic thickness variations. They gave results for 27 different boundary conditions. Olhoff [7] had earlier determined the shape of a circular plate so that its first natural frequency of transverse vibrations became optimal. He investigated three different boundary conditions. Hinton *et al*., [8] using the FEM and an automated optimization approach together, obtained similar results to those given by Olhoff [7]. Since then the optimal design of vibrating elliptical plates has not been possible yet. So the current parametric study is a little step towards filling that gap. The effect of parabolic variation of thickness on the frequency parameter of clamped elliptic plates has been investigated. Two different approximate methods, the moment method and the Rayleigh–Ritz method, whose trial functions are also different from each other are used to solve the problem. Note that the moment method is a fundamental numerical method in applied electromagnetics while it is rarely preferred in the field of applied mechanics. First, the fundamental frequency of clamped elliptical plates with constant thickness for aspect ratios up to 0·1 is calculated by the two methods. Then, the thickness function is assumed to be of parabolic variation and

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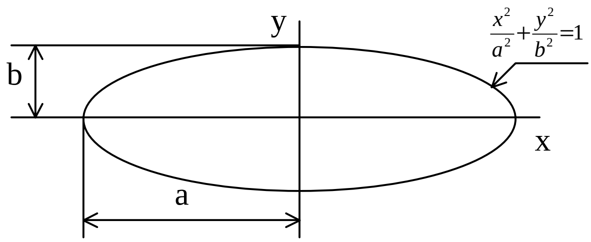

Figure 1. Elliptical plate geometry in Cartesian co-ordinates.

the volumes of the plates are equalized to get results comparable with those in the literature for a variety of aspect ratios. Finally, the effect of variation of thickness on the frequency parameter has been investigated.

2. BASIC EQUATIONS

Assuming that the middle surface of an isotropic plate having no in-plane forces is plane and coincides with the x–y plane of the Cartesian co-ordinate system (Figure 1), then the basic equation governing the transverse vibration of the plate with variable thickness is given by the following [9]:

$$\nabla^2(D\nabla^2 w) - (1-\upsilon)\left[\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right] + \rho h\frac{\partial^2 w}{\partial t^2} = 0, \quad (1)$$

where w, υ, ρ, h, t denote the displacement, the Poisson ratio, mass density per unit volume, the variable thickness of the plate and time respectively. Also appearing in equation (1) is the flexural rigidity D defined by

$$D = Eh^3/[12(1-\upsilon^2)] \quad (2)$$

where E is Young's modulus.

Introducing $\bar{x} = x/a$, $\bar{y} = y/b$ and $K = b/a$, where a and b are the semi-major and semi-minor axes of the ellipse, respectively, the above equation becomes

$$\frac{1}{b^4}\left\{D\left(K^4\frac{\partial^4 w}{\partial\bar{x}^4} + 2K^2\frac{\partial^4 w}{\partial\bar{x}^2\partial\bar{y}^2} + \frac{\partial^4 w}{\partial\bar{y}^4}\right) + 2K^4\frac{\partial D}{\partial\bar{x}}\frac{\partial^3 w}{\partial\bar{x}^3} + 2K^2\frac{\partial D}{\partial\bar{x}}\frac{\partial^3 w}{\partial\bar{x}\partial\bar{y}^2}\right.$$
$$+ 2K^2\frac{\partial D}{\partial\bar{y}}\frac{\partial^3 w}{\partial\bar{x}^2\partial\bar{y}} + 2\frac{\partial D}{\partial\bar{y}}\frac{\partial^3 w}{\partial\bar{y}^3} + K^4\frac{\partial^2 D}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{x}^2} + K^2\frac{\partial^2 D}{\partial\bar{y}^2}\frac{\partial^2 w}{\partial\bar{x}^2} + K^2\frac{\partial^2 D}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{y}^2}$$
$$\left.+ \frac{\partial^2 D}{\partial\bar{y}^2}\frac{\partial^2 w}{\partial\bar{y}^2} - K^2(1-\upsilon)\left[\frac{\partial^2 D}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{y}^2} - 2\frac{\partial^2 D}{\partial\bar{x}\partial\bar{y}}\frac{\partial^2 w}{\partial\bar{x}\partial\bar{y}} + \frac{\partial^2 D}{\partial\bar{y}^2}\frac{\partial^2 w}{\partial\bar{x}^2}\right]\right\} + \rho h\frac{\partial^2 w}{\partial t^2} = 0. \quad (3)$$

Assuming simple harmonic motion where one seeks the solution of the form

$$w(\bar{x},\bar{y},t) = w(\bar{x},\bar{y})e^{i\omega t}, \quad (4)$$

where ω is the natural angular frequency.

The plate thickness, on the other hand, is assumed to vary according to the following relation:

$$h(\bar{x},\bar{y}) = ch_0[\alpha + \beta(\bar{x}^2 + \bar{y}^2)], \quad (5)$$

where h_0 is the thickness of an elliptic plate with constant thickness, α is a parameter defining the constant part of thickness, β is the taper parameter controlling the variation of thickness and c is a parameter ensuring that all the plates considered are of equal volume defined as

$$c = 2/(2\alpha + \beta). \tag{6}$$

The volume of an elliptical plate with constant thickness is $\pi a b h_0$ and note that this corresponds to the following which gives the volume of an elliptical plate with varying thickness in dimensionless form:

$$V = 4\int_0^1 \left\{ \int_0^{\sqrt{1-\bar{x}^2}} \left\{ \int_{-h/2}^{h/2} \mathrm{d}z \right\} \mathrm{d}y \right\} \mathrm{d}x = 4\int_0^1 \left\{ \int_0^{\sqrt{1-\bar{x}^2}} h(\bar{x}, \bar{y})\, \mathrm{d}\bar{y} \right\} \mathrm{d}\bar{x} = \pi c h_0(\alpha + \beta/2) = \pi h_0. \tag{7}$$

How the definite integral is taken will be explained in detail later.

The flexural rigidity now may be rewritten as

$$D(\bar{x}, \bar{y}) = D_0 c^3 H, \tag{8}$$

where $D_0 = Eh_0^3/[12(1-\upsilon^2)]$ and $H = [\alpha + \beta(\bar{x}^2 + \bar{y}^2)]^3$.

Substituting equations (4) and (8) into equation (3) gives

$$\begin{aligned} &\frac{D_0 c^3}{b^4}\Bigg\{\left(K^4 \frac{\partial^4 w}{\partial \bar{x}^4} + 2K^2 \frac{\partial^4 w}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 w}{\partial \bar{y}^4}\right) + 2K^4 \frac{\partial H}{\partial \bar{x}} \frac{\partial^3 w}{\partial \bar{x}^3} + 2K^2 \frac{\partial H}{\partial \bar{x}} \frac{\partial^3 w}{\partial \bar{x} \partial \bar{y}^2} \\ &\quad + 2K^2 \frac{\partial H}{\partial \bar{y}} \frac{\partial^3 w}{\partial \bar{x}^2 \partial \bar{y}} + 2\frac{\partial H}{\partial \bar{y}} \frac{\partial^3 w}{\partial \bar{y}^3} + K^4 \frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{x}^2} + K^2 \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{x}^2} \\ &+ K^2 \frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} + \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{y}^2} - K^2(1-\upsilon)\left[\frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} - 2\frac{\partial^2 H}{\partial \bar{x} \partial \bar{y}} \frac{\partial^2 w}{\partial \bar{x} \partial \bar{y}} + \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{x}^2}\right]\Bigg\} \\ &- \rho\omega^2 h w = 0. \end{aligned} \tag{9}$$

3. SOLUTION BY THE MOMENT METHOD

First the moment method, one of the well-known (e.g., reference [10] and references cited therein)—although rarely applied in mechanical engineering problems—weighted residual methods, is used to solve the above equation. A three-term deflection function satisfying the geometric boundary conditions of zero edge deflection and zero slope at the edge is chosen:

$$w = (a_1 + a_2\phi + a_3\phi^2)\phi^2, \tag{10}$$

where $\phi = \bar{x}^2 + \bar{y}^2 - 1$.

The residual ε_R, which will later be obtained by substituting equation (10) into equation (9), is defined by the following:

$$\begin{aligned} &c^3\Bigg\{\left(K^4 \frac{\partial^4 w}{\partial \bar{x}^4} + 2K^2 \frac{\partial^4 w}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 w}{\partial \bar{y}^4}\right) + 2K^4 \frac{\partial H}{\partial \bar{x}} \frac{\partial^3 w}{\partial \bar{x}^3} + 2K^2 \frac{\partial H}{\partial \bar{x}} \frac{\partial^3 w}{\partial \bar{x} \partial \bar{y}^2} + 2K^2 \frac{\partial H}{\partial \bar{y}} \frac{\partial^3 w}{\partial \bar{x}^2 \partial \bar{y}} \\ &\quad + 2\frac{\partial H}{\partial \bar{y}} \frac{\partial^3 w}{\partial \bar{y}^3} + K^4 \frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{x}^2} + K^2 \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{x}^2} + K^2 \frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} + \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{y}^2} \\ &\quad - K^2(1-\upsilon)\left[\frac{\partial^2 H}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} - 2\frac{\partial^2 H}{\partial \bar{x} \partial \bar{y}} \frac{\partial^2 w}{\partial \bar{x} \partial \bar{y}} + \frac{\partial^2 H}{\partial \bar{y}^2} \frac{\partial^2 w}{\partial \bar{x}^2}\right]\Bigg\} - \lambda^* w = \varepsilon_R, \end{aligned} \tag{11}$$

where λ^* is defined as $\omega^2 b^4 \rho h_0/D_0$.

The three moment equations are then found from the following:

$$\int_0^1 \left\{ \int_0^{\sqrt{1-\bar{x}^2}} \varepsilon_R \bar{x}^i \, \mathrm{d}\bar{y} \right\} \mathrm{d}\bar{x} = 0, \quad i = 0, 2 \text{ and } 4. \tag{12}$$

The reason for choosing even values of i is that if odd values of i are included in equation (12), the method does not yield any results. In fact, this is not surprising since integration of $\varepsilon_R \bar{x}^i$ over the whole plate area will result in zero which will cause the determinant of the coefficient matrix to be zero when odd values of i are used. Also, note that there is no difference in results if $\bar{y}^i$ is put in place of $\bar{x}^i$ due to the symmetry of the problem. One may also ask what happens if ϕ^i is put in the place of $\bar{x}^i$. This case coincides with Galerkin's procedure with a three-term deflection function if values of i are taken to be 2, 3 and 4. Results of this case (for an elliptical plate with constant thickness) are exactly the same as those given in references [1, 6], where Galerkin's method with up to a 10-term deflection function and the Rayleigh-Ritz method with a three-term deflection function are used respectively. Results of the other two cases when $i = 0, 1, 2$ and $1, 2, 3$ are almost the same as above.

The definite integration over the plate area is a difficult task, so thanks to the following formula, as given in references [1, 2] makes it possible to evaluate the exact integral.

$$\iint_R \bar{x}^p \bar{y}^q \phi^r \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} = (-1)^r \Gamma\frac{p+1}{2} \Gamma\frac{r+1}{2} \Gamma(r+1) \bigg/ \Gamma\left(\frac{p+q}{2} + r + 2\right), \tag{13}$$

where p and q are even integers and p, q and r are greater than -1.

It is now a generalized eigenvalue problem:

$$[\mathbf{A} - \lambda^* \mathbf{B}]\{a_j\} = 0, \tag{14}$$

where **A** and **B** are not symmetric matrices both of which are of the order three.

For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero:

$$|\mathbf{A} - \lambda^* \mathbf{B}| = 0. \tag{15}$$

Solution of equation (15) leads to a characteristic equation involving a polynomial of third degree in λ^*, the smallest root of which corresponds to the square of the fundamental frequency.

The frequency parameter is then found to be

$$\lambda^2 = \sqrt{\lambda^*}. \tag{16}$$

4. SOLUTION BY THE RAYLEIGH–RITZ METHOD

The well-known expression, in Cartesian co-ordinates, for the strain energy of a plate in bending is as follows [9, 11]:

$$U = \frac{1}{2} \iint_S D(x, y) \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \mathrm{d}x \, \mathrm{d}y, \tag{17}$$

where S represents the area of the plate.

Since the same notation as the moment method is used, some of the details are omitted here.

$$U = \frac{1}{2b^4} \iint_S D(\bar{x}, \bar{y}) \left\{ \left(K^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \frac{\partial^2 w}{\partial \bar{y}^2} \right)^2 - 2(1-\nu) K^2 \left[\frac{\partial^2 w}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} - \left(\frac{\partial^2 w}{\partial \bar{x} \partial \bar{y}} \right)^2 \right] \right\} \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}, \tag{18}$$

$$U = \frac{1}{2b^4} D_0 \iint_S c^3 [\alpha + \beta(\bar{x}^2 + \bar{y}^2)]^3 \left\{ \left(K^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \frac{\partial^2 w}{\partial \bar{y}^2} \right)^2 - 2(1-\nu) K^2 \left[\frac{\partial^2 w}{\partial \bar{x}^2} \frac{\partial^2 w}{\partial \bar{y}^2} - \left(\frac{\partial^2 w}{\partial \bar{x} \partial \bar{y}} \right)^2 \right] \right\} \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}. \tag{19}$$

The kinetic energy of the plate on the other hand is given by [11]

$$T = \frac{1}{2} \rho \omega^2 \iint_S h(\bar{x}, \bar{y}) w^2(\bar{x}, \bar{y})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}.$$

$$T = \frac{1}{2} \rho \omega^2 h_0 \iint_S c[\alpha + \beta(\bar{x}^2 + \bar{y}^2)] w^2(\bar{x}, \bar{y})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}. \tag{20}$$

Note that a different deflection function which satisfies the geometric boundary conditions is chosen now:

$$W = (a_{00} + a_{20}\bar{x}^2 + a_{02}\bar{y}^2)\phi^2. \tag{21}$$

It should be explained here that there is no other reason for choosing a different type of deflection function than trying to contribute to the available literature. Otherwise, it might be chosen the same as equation (10) but then the results would not be any better—at least for the constant thickness case—than those obtained by using equation (21), which can be seen in Table 1 by comparing with the results of reference [6], where the Rayleigh–Ritz method with the same deflection function as equation (10) is used, with results of the present work.

The total energy functional, F, is given by

$$F = U - T. \tag{22}$$

TABLE 1

Comparison of frequency parameters ($\lambda^2 = \omega b^2 \sqrt{\rho h_0 / D_0}$) *of clamped elliptical plates with constant thickness* ($\alpha = 1{\cdot}0$, $\beta = 0{\cdot}0$ *and* $\upsilon = 0{\cdot}3$)

b/a	Ref. [9]	Ref. [6]	Relative error (%)	Moment method	Relative error (%)	Rayleigh–Ritz meth.	Relative error(%)
1·0	10·216	10·216	0·0	10·217	0·01	10·217	0·01
0·5	6·845	6·936	1·3	7·932	15·9	6·849	0·06
0·4	6·504	6·657	2·4	7·683	18·1	6·514	0·15
0·2	5·996	6·343	5·8	7·391	23·3	6·058	1·03
0·1	5·831	6·277	7·6	7·326	25·6	5·941	1·89

In order to find a least upper bound on the frequency equation (21) is substituted into equation (22) and then it is minimized with respect to the coefficients a_{ij}, which finally gives three homogeneous equations in a_{ij}:

$$[\mathbf{A} - \lambda^* \mathbf{B}]\{a_{ij}\} = 0, \tag{23}$$

where λ^* is defined as $\omega^2 b^4 \rho h_0 / D_0$, and **A** and **B** are symmetric matrices both of which are of the order three and **B** is positive definite.

For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero:

$$|\mathbf{A} - \lambda^* \mathbf{B}| = 0. \tag{24}$$

Similarly, solution of equation (24) leads to a characteristic equation involving a polynomial of third degree in λ^*, the smallest root of which corresponds to the square of the fundamental frequency.

The frequency parameter is then found to be

$$\lambda^2 = \sqrt{\lambda^*}. \tag{25}$$

5. PRESENTATION AND DISCUSSION OF RESULTS

First, in order to check the accuracy of the present work the frequency parameters of clamped elliptical plates with constant thickness for aspect ratios from 1 to 0·1 are calculated. The results are compared with some of the literature in Table 1. Note that results obtained by the Rayleigh–Ritz method correlates well with reference [9] for all aspect ratios, while the moment method is not sufficiently accurate as b/a decreases.

In order to be able to compare the results of the present work with others available in the literature a little manipulation had to be made to equations (3) and (18), since in references [1–3, 5, 9], where the Galerkin method, the Rayleigh–Ritz method with characteristic orthogonal polynomials, collocation method with a five-term deflection function, the pb-2 Rayleigh–Ritz method and a minimal energy method with a five-term deflection function were used, respectively; frequency parameters of clamped elliptical plates are given as $\lambda^2 = \omega a^2 \sqrt{\rho h_0 / D_0}$. The results of this comparison can be seen in Table 2.

TABLE 2

Comparison of frequency parameters ($\lambda^2 = \omega a^2 \sqrt{\rho h_0 / D_0}$) of clamped elliptical plates with constant thickness ($\alpha = 1{\cdot}0$, $\beta = 0{\cdot}0$ and $\upsilon = 0{\cdot}3$)

b/a	1·0	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2
Reference [9]	10·216	11·443	13·229	15·928	20·195	27·378	40·649	69·163	149·89
Reference [1]	10·216	—	13·246	—	20·337	27·743	41·605	—	158·59
Reference [3]	10·205	—	13·226	—	20·363	27·815	41·588	70·292	—
Reference [2]	10·216	—	13·229	—	20·195	27·377	40·646	—	149·66
Reference [5]	10·217	—	13·230	—	—	27·388	40·663	—	—
Moment meth.	10·217	11·939	14·287	17·686	22·942	31·726	48·020	83·405	188·26
Rayleigh–Ritz	10·217	11·444	13·231	15·931	20·201	27·395	40·713	69·461	151·48

TABLE 3

Comparison of frequency parameters ($\lambda^2 = \omega a^2\sqrt{\rho h_0/D_0}$) *of clamped elliptical plates with varying thickness, which are not of equal volume* ($\alpha = c = 1{\cdot}0$)

β	$b/a \rightarrow$	1·0	0·8	0·6	0·4	0·2
0·8	Reference [1]	15·815	20·501	31·450	64·266	244·74
	Moment meth.	16·116	22·838	36·942	77·590	—
	Rayleigh–Ritz	16·070	20·808	31·768	64·160	240·40
0·6	Reference [1]	14·428	18·703	28·696	58·651	223·39
	Moment meth.	14·576	20·619	33·323	69·962	—
	Rayleigh–Ritz	14·557	18·849	28·772	58·066	217·24
0·4	Reference [1]	13·032	16·895	25·927	53·005	201·93
	Moment meth.	13·087	18·465	29·799	62·523	—
	Rayleigh–Ritz	13·084	16·941	25·857	52·145	194·74
0·2	Reference [1]	11·629	15·076	23·141	47·324	180·34
	Moment meth.	11·640	16·362	26·351	55·233	—
	Rayleigh–Ritz	11·642	15·074	23·009	46·375	172·94
0·0	Reference [1]	10·216	13·246	20·337	41·605	158·59
	Moment meth.	10·217	14·287	22·942	48·020	—
	Rayleigh–Ritz	10·217	13·231	20·201	40·713	151·48

Before carrying out a parametric study in order to see the effect of α and β on the frequency parameter of clamped elliptical plates of equal volume, which is achieved by changing α from 1·0 to 0·0 and β from 0·0 to 1·0, results of the present work are again compared with reference [1]. Having seen that results for the case $c = 1{\cdot}0$ correlate well enough with reference [1], which is shown in Table 3, the parametric study, whose results are shown in Table 4, was performed. It is interesting to see that whatever β is when $\alpha \rightarrow 0{\cdot}0$ the frequency parameters of clamped elliptical plates increase rapidly. Results for the case of circular plates are given in detail in Table 5. Results of this case for different aspect ratios on the other hand are presented in Table 6. Note that the optimal shape of a clamped circular plate given by references [7, 8], where the thickness variation is represented by a 4-point-cubic spline, is convex. In the present work, on the other hand, the thickness function is concave since it is of parabolic variation. Although one may have doubts about the validity of results obtained by the moment method for elliptical plates, it should be kept in mind that for the case of circular plates results of the two different methods nearly coincide and that as b/a decreases the trends of results are quite similar to each other.

In the case of a circular plate an increase of 53% in the frequency parameter is reported in reference [7] (Figure 2), while a similar increase is found by the present work when $\alpha \cong 0{\cdot}22$ and $\beta = 1{\cdot}0$. However, due to structural strength and constructional reasons there may be a constraint upon the rate of h_{origin}/h_{edge}, where h_{origin} is the thickness at the origin and h_{edge} is the thickness of the edge. If, for instance, this constraint is considered to be $h_{origin}/h_{edge} \geqslant 0{\cdot}5$, gains in frequency parameters are presented in Table 7 (the case when $\alpha = \beta$, also seen in Figure 3). Note that due to symmetry, only half of the plate is shown in Figure 3. It is interesting to see that although results of the moment method for the case of constant thickness deviate from the literature as b/a decreases, for the case of varying thickness relative increases or gains in frequency parameters calculated by the moment method are nearly the same order as Rayleigh–Ritz's method.

TABLE 4

Frequency parameters ($\lambda^2 = \omega b^2 \sqrt{\rho h_0/D_0}$) *of clamped elliptical plates with varying thickness when both* α *and* β *are changed from* 1·0 to 0·0 (*MM and R–RM denote for the moment method and the Rayleigh–Ritz method respectively*)

		$b/a = 1·0$		$b/a = 0·8$		$b/a = 0·5$		$b/a = 0·2$	
β	α	MM	R–RM	MM	R–RM	MM	R–RM	MM	R–RM
0·0	1·0	10·217	10·217	9·144	8·468	7·932	6·849	7·391	6·058
	0·8	10·217	10·217	9·144	8·468	7·932	6·849	7·391	6·058
	0·6	10·217	10·217	9·144	8·468	7·932	6·849	7·391	6·058
	0·4	10·217	10·217	9·144	8·468	7·932	6·849	7·391	6·058
	0·2	10·217	10·217	9·144	8·468	7·932	6·849	7·391	6·058
0·2	1·0	10·582	10·583	9·520	8·770	8·289	7·091	7·732	6·286
	0·8	10·666	10·667	9·605	8·839	8·370	7·146	7·809	6·339
	0·6	10·801	10·801	9·742	8·951	8·499	7·237	7·932	6·425
	0·4	11·061	11·060	10·002	9·165	8·741	7·411	8·162	6·592
	0·2	11·805	11·822	10·719	9·797	9·393	7·931	8·776	7·084
	0·0	24·674	27·265	21·727	22·622	18·494	18·493	17·083	16·898
0·4	1·0	10·906	10·903	9·848	9·035	8·598	7·306	8·026	6·491
	0·8	11·061	11·054	10·002	9·160	8·741	7·408	8·162	6·590
	0·6	11·313	11·298	10·249	9·363	8·968	7·574	8·377	6·749
	0·4	11·805	11·777	10·719	9·760	9·393	7·901	8·776	7·061
	0·2	13·189	13·178	11·990	10·923	10·507	8·859	9·813	7·960
	0·0	24·674	25·856	21·727	21·452	18·494	17·533	17·083	16·017
0·6	1·0	11·213	11·198	10·151	9·279	8·878	7·506	8·292	6·684
	0·8	11·437	11·412	10·369	9·457	9·077	7·652	8·480	6·824
	0·6	11·805	11·762	10·719	9·747	9·393	7·891	8·776	7·053
	0·4	12·518	12·445	11·381	10·315	9·977	8·359	9·322	7·496
	0·2	14·379	14·299	13·052	11·854	11·415	9·628	10·651	8·682
	0·0	24·674	25·187	21·727	20·896	18·494	17·075	17·083	15·596
0·8	1·0	11·513	11·478	10·440	9·512	9·142	7·697	8·541	6·869
	0·8	11·805	11·754	10·719	9·741	9·393	7·886	8·776	7·049
	0·6	12·284	12·203	11·166	10·114	9·789	8·194	9·146	7·342
	0·4	13·189	13·063	11·990	10·828	10·507	8·783	9·813	7·896
	0·2	15·373	15·211	13·925	12·611	12·152	10·254	11·329	9·267
	0·0	24·674	24·796	21·727	20·571	18·494	16·808	17·083	15·349
1·0	1·0	11·805	11·749	10·719	9·737	9·393	7·883	8·776	7·047
	0·8	12·166	12·084	11·056	10·014	9·692	8·112	9·056	7·265
	0·6	12·747	12·624	11·590	10·463	10·160	8·483	9·491	7·615
	0·4	13·810	13·629	12·547	11·297	10·984	9·171	10·254	8·261
	0·2	16·203	15·959	14·647	13·232	12·757	10·767	11·883	9·747
	0·0	24·674	24·540	21·727	20·358	18·494	16·632	17·083	15·187

6. CONCLUDING REMARKS

As expected for a given volume it is possible to increase the frequency parameter of clamped circular plates. This is a sharp increase as α gets closer to zero, which does not have a practical value in engineering terms. When the edge thickness is twice as high as the origin thickness relative increases in frequency parameter are found to be around 15% by the two

Table 5

Frequency parameters ($\lambda^2 = \omega b^2\sqrt{\rho h_0/D_0}$) *of clamped circular plates with varying thickness when* $\alpha \to 0{\cdot}0$

	$\alpha \to$	1·0	0·5	0·1	0·05	0·01	0·0
Moment M.	$\beta = 1{\cdot}0$	11·805	13·189	18·837	21·081	23·792	24·674
	$\beta = 0{\cdot}5$	11·061	11·805	16·203	18·837	23·006	24·674
	$\beta = 0{\cdot}1$	10·407	10·582	11·805	13·189	18·837	24·674
R.–Ritz M.	$\beta = 1{\cdot}0$	11·749	13·039	18·595	20·870	23·638	24·540
	$\beta = 0{\cdot}5$	11·053	11·768	16·294	19·146	23·672	25·471
	$\beta = 0{\cdot}1$	10·408	10·584	11·906	13·727	21·540	28·774

Table 6

Relative increases in frequency parameter ($\lambda^2 = \omega b^2\sqrt{\rho h_0/D_0}$) *of clamped elliptical plates with varying thickness, when* $\alpha = 0{\cdot}0$ *and* $\beta = 1{\cdot}0$

b/a	1·0	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
Moment method	24·674	23·168	21·727	20·440	19·356	18·494	17·844	17·384	17·083	16·915
R. increase (%)	141	140	138	136	134	133	132	132	131	131
R.–Ritz method	24·540	22·268	20·358	18·799	17·567	16·632	15·953	15·486	15·187	15·024
R. increase (%)	140	140	140	141	142	143	145	148	151	153

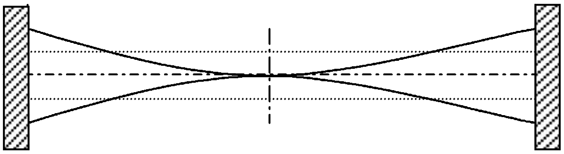

Figure 2. The optimal shape of a clamped circular plate given by references [7, 8].

Table 7

Gains in frequency parameter ($\lambda^2 = \omega b^2\sqrt{\rho h_0/D_0}$) *of clamped elliptical plates, when* $\alpha = \beta = 1{\cdot}0$ ($h_{origin}/h_{edge} = 0{\cdot}5$)

b/a	1·0	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
R.–Ritz	11·749	10·659	9·737	8·976	8·363	7·883	7·513	7·237	7·047	6·936
Gains %	15·0	15·0	15·0	15·0	15·0	15·1	15·3	15·8	16·3	16·7
Moment	11·805	11·267	10·719	10·206	9·759	9·393	9·112	8·910	8·776	8·701
Gains %	15·5	16·5	17·2	17·8	18·2	18·4	18·6	18·7	18·7	18·8

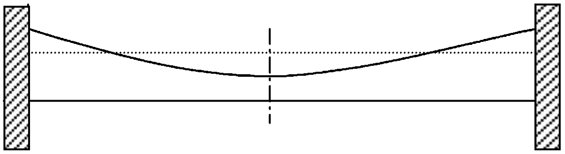

Figure 3. The shape of a clamped plate when $\alpha = \beta = 1{\cdot}0$ ($h_{origin}/h_{edge} = 0{\cdot}5$).

different methods for the circular plate. As the aspect ratio decreases, gains slightly increase for elliptical plates too.

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