Journal of Sound and Vibration (2002) **254**(1), 189–201
doi:10.1006/jsvi.2001.4055, available online at http://www.idealibrary.com on IDEAL®

CONTROL VOLUME AND SYSTEM FORMULATIONS FOR TRANSLATING MEDIA AND STATIONARY MEDIA WITH MOVING BOUNDARIES

W. D. ZHU

Department of Mechanical Engineering, University of Maryland Baltimore County, Baltimore, MD 21250, *U.S.A. E-mail*: *wzhu@umbc.edu*

(*Received* 10 *September* 2001, *and in final form* 16 *October* 2001)

1. INTRODUCTION

The dynamics of translating media, such as high-speed magnetic tapes, band saws, and transport cables, concern the motions of constituent particles instantaneously located within a specified spatial domain [1]. While stationary media with constant length are composed of the same particles at all times, the constituent particles of translating media and stationary media with moving boundaries change with time. Through direct differentiation, Miranker [2] first calculated the rate of change of total mechanical energy of a string translating between two fixed supports. He found that the energy of the string is not conserved and there is a periodic transfer of energy between portions of the string inside and outside the boundaries. However, an error in sign led to an extra integral term in the resulting expression for the rate of change of energy. Through the use of the Reynolds transport theorem, Wickert and Mote [3] presented an extended energy analysis for the translating string and tensioned beam models. It was shown that the rates of change of total mechanical energies of translating media equal the net rates of work done by their internal forces or moments at the boundaries. More recently, Renshaw *et al.* [4] examined the distinction between the Lagrangian and Eulerian functionals defined over a set of material particles and a spatial domain respectively. With a Lagrangian functional and its rate of change defined at time $t = 0$, the rates of change of energies in reference [3] correspond to those of the Lagrangian energy functionals in reference [4]. Direct differentiation of energy in reference [2] would have resulted in the rate of change of the Eulerian energy functional for the translating string in reference [4].

We distinguish here the rates of change of energies of translating media from control volume and system viewpoints. Translating strings and tensioned beams with constant and variable lengths are considered. We also address the related problems of stationary strings and tensioned beams with one and two boundaries moving at arbitrarily prescribed speeds. Correspondences between translating media with variable length and stationary media with a moving boundary, and between translating media with constant length and stationary media with both boundaries moving at the same speed, are established. Effects of boundary conditions on the energy and stability characteristics of these systems are demonstrated.

0022-460X/02/$35.00

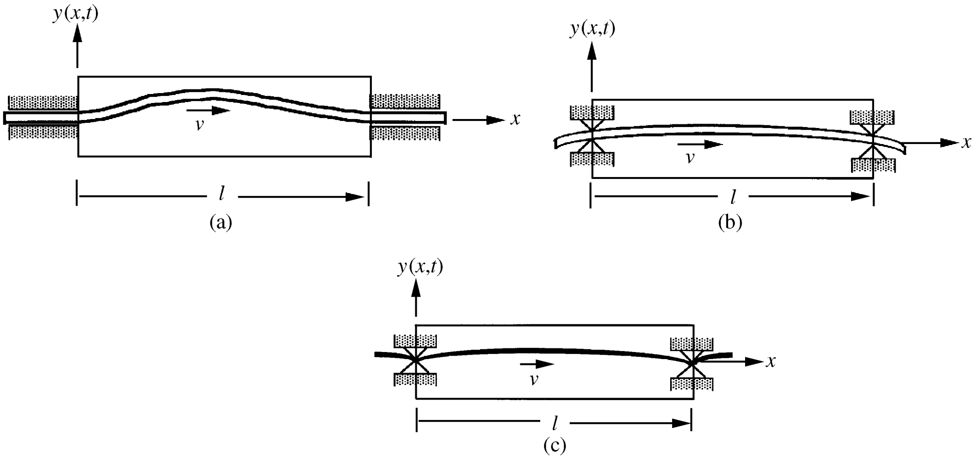

Figure 1. Translating media with constant length at time t: (a) beam with fixed ends, (b) beam with pinned ends, (c) string. In each case the control volume and system at time t coincide and are enclosed in a box. While the slopes are continuous at the two boundaries in (a) and (b), they can be discontinuous in (c).

2. TRANSLATING MEDIA WITH CONSTANT LENGTH

The linear equation describing the transverse vibration of a beam in Figure 1(a,b), translating with constant velocity v between two supports of distance, l, is

$$\rho(y_{tt} + 2vy_{xt} + v^2 y_{xx}) - Py_{xx} + EIy_{xxxx} = 0, \tag{1}$$

where lettered subscripts denote partial differentiation, $y(x, t)$ is the transverse displacement of the beam particle instantaneously located at spatial position $x(0 < x < l)$ at time t, P is the tension in the beam, ρ is the mass per unit length of the beam, and EI is its flexure rigidity. The boundary conditions of the beam with fixed ends, as shown in Figure 1(a), are

$$y = y_x = 0 \quad \text{at } x = 0 \text{ and } x = l, \tag{2}$$

and those of the beam with pinned ends, as shown in Figure 1(b), are

$$y = y_{xx} = 0 \quad \text{at } x = 0 \text{ and } x = l. \tag{3}$$

The linear equation describing the transverse vibration of the translating string in Figure 1(c) is given by equation (1) with $EI = 0$ and associated boundary conditions

$$y = 0 \quad \text{at } x = 0 \text{ and } x = l. \tag{4}$$

In each case the control volume at time t is defined as the spatial domain $0 \leqslant x \leqslant l$, and the system concerned consists of material particles of fixed identity, occupying the spatial domain $[0, l]$ at time t (see Figure 1). Since the control volume and system contain the same particles, the total mechanical energies in the two approaches are equal at time t:

$$E_{cv}(t) = E_{syst}(t) = E_r + E_v(t) = \int_0^l \varepsilon_r \, dx + \int_0^l \varepsilon_v(x, t) \, dx = \int_0^l \varepsilon(x, t) \, dx, \tag{5}$$

where E_r and $E_v(t)$ are the energies associated with the rigid-body translation and transverse vibration of the media, respectively, $\varepsilon(x, t)$ is the total energy density,

$$\varepsilon_r = \tfrac{1}{2}\rho v^2 \tag{6}$$

is the energy density associated with the rigid-body translation, and

$$\varepsilon_v(x, t) = \tfrac{1}{2}\rho(y_t + vy_x)^2 + \tfrac{1}{2}Py_x^2 + \tfrac{1}{2}EIy_{xx}^2 \tag{7}$$

is the energy density associated with the transverse vibration of translating beams. The energy density associated with the transverse vibration of the translating string is given by equation (7) with $EI = 0$. Note that E_{cv} corresponds to the Eulerian energy functional in reference [4], and while the notion of E_{syst} is similar to that of the Lagrangian energy functional, the Lagrangian energy functional in reference [4] is defined at $t = 0$.

Differentiating E_{cv} in equation (5) and using equation (6) yields

$$\dot{E}_{cv}(t) = \int_0^l \frac{\partial \varepsilon_v(x, t)}{\partial t}\,dx. \tag{8}$$

Substituting equation (7) into equation (8), followed by the use of the governing equation (1), integration by parts, and application of the boundary conditions in equation (2), yields [4]

$$\dot{E}_{cv}(t) = \tfrac{1}{2}vEIy_{xx}^2(x, t)|_0^l \tag{9}$$

for the translating beam with fixed ends. Similarly,

$$\dot{E}_{cv}(t) = \tfrac{1}{2}v(P - \rho v^2)y_x^2(x, t)|_0^l - EIvy_x(x, t)y_{xxx}(x, t)|_0^l \tag{10}$$

for the translating beam with pinned ends, and $\dot{E}_{cv}$ for the translating string is given by equation (10) with $EI = 0$ [4].

At time $t + \Delta t$, while the control volume remains unchanged in each case, the system of material particles has translated a distance $v\Delta t$, as shown in Figure 2. The total mechanical energy of the system of material particles, occupying the spatial domain $[v\Delta t, l + v\Delta t]$, is

$$E_{syst}(t + \Delta t) = E_{cv}(t + \Delta t) - v\Delta t\varepsilon(0, t + \Delta t) + v\Delta t\varepsilon(l, t + \Delta t). \tag{11}$$

The limit of

$$\frac{E_{syst}(t + \Delta t) - E_{syst}(t)}{\Delta t} = \frac{E_{cv}(t + \Delta t) - E_{cv}(t)}{\Delta t} - v\varepsilon(0, t + \Delta t) + v\varepsilon(l, t + \Delta t) \tag{12}$$

as $\Delta t \to 0$, where use has been made of equations (5) and (11), results in the Reynolds transport theorem for a translating medium with constant length [3]:

$$\dot{E}_{syst}(t) = \dot{E}_{cv}(t) - v\varepsilon(0, t) + v\varepsilon(l, t). \tag{13}$$

Substituting equations (6), (7), and (9) into equation (13) and using equation (2) yields [3]

$$\dot{E}_{syst}(t) = EIy_{xx}^2(x, t)|_0^l \tag{14}$$

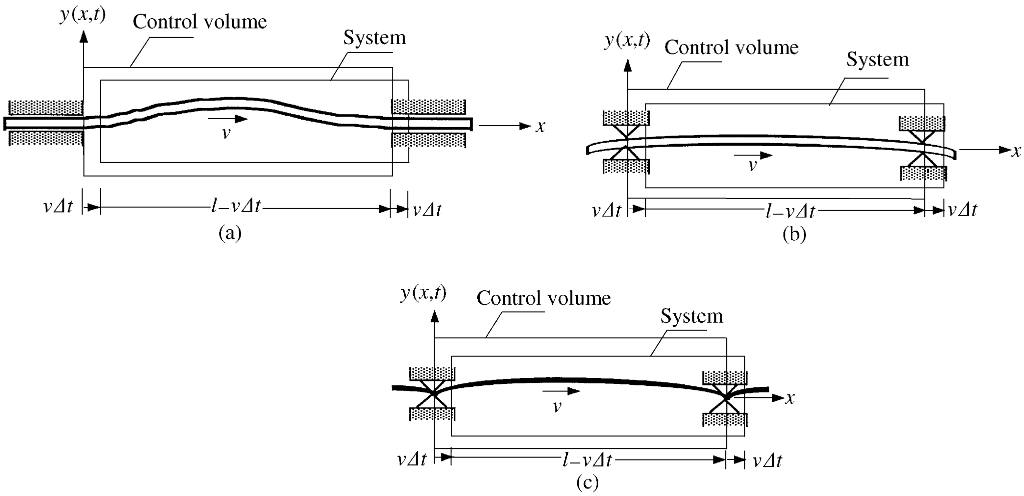

Figure 2. Control volumes and systems of translating media with constant length at time $t + \Delta t$: (a)–(c) as in Figure 1.

for the translating beam with fixed ends. Similarly,

$$\dot{E}_{syst}(t) = [Pvy_x^2(x, t) - EIvy_x(x, t)y_{xxx}(x, t)]|_0^l \tag{15}$$

for the translating beam with pinned ends, and $\dot{E}_{syst}(t)$ for the translating string is given by equation (15) with $EI = 0$ [3]. Equations (14) and (15) state that the rate of change of total mechanical energy of the system of material particles equals the net rate of work done by the nonconservative forces or moments on it at time t [3].

3. TRANSLATING MEDIA WITH VARIABLE LENGTH

The linear equation describing the transverse vibration of a translating tensioned beam in Figure 3(a,b) with linearly varying length $l(t) = l_0 + vt$, where l_0 is the initial length and v is the constant velocity, is given by equation (1) where $0 < x < l(t)$. A positive and negative v indicates extension and retraction of the beam respectively. The boundary conditions of the beam with fixed ends, as shown in Figure 3(a), are

$$y = y_x = 0 \text{ at } x = 0 \quad \text{and} \quad x = l(t), \tag{16}$$

and those of the beam with pinned ends, as shown in Figure 3(b), are

$$y = y_{xx} = 0 \text{ at } x = 0 \quad \text{and} \quad x = l(t). \tag{17}$$

The linear equation describing the transverse vibration of the translating string in Figure 3(c) is given by equation (1) with $EI = 0$ and associated boundary conditions

$$y = 0 \text{ at } x = 0 \quad \text{and} \quad x = l(t). \tag{18}$$

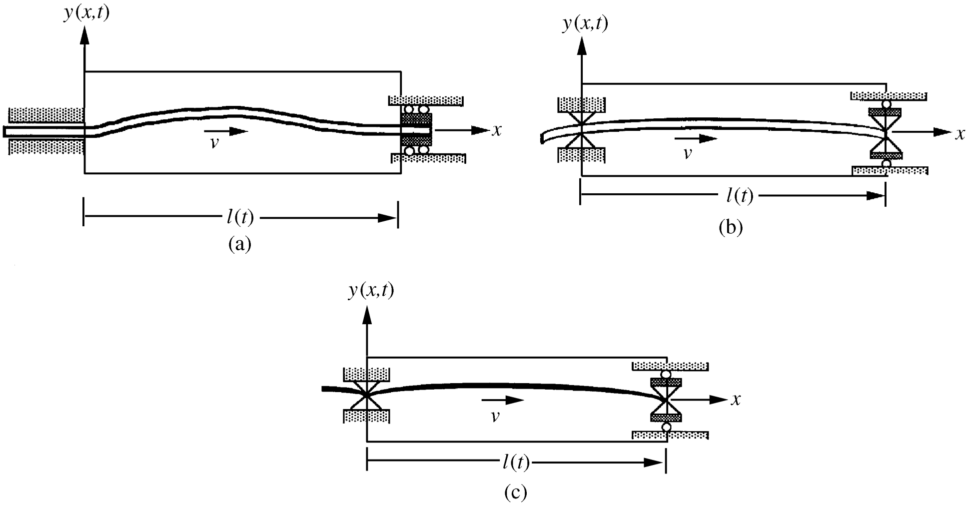

Figure 3. Translating media with variable length at time t: (a) beam with fixed ends, (b) beam with pinned ends, (c) string. In each case the control volume and system at time t coincide and are enclosed in a box. While the slope is continuous at the boundary $x = 0$ in (a) and (b), it can be discontinuous in (c).

In each case the control volume at time t is defined as the spatial domain $0 \leqslant x \leqslant l(t)$, formed instantaneously by the translating medium between the two boundaries, and the system concerned consists of material particles of fixed identity, occupying the spatial domain $[0, l(t)]$ at time t (see Figure 3). The total mechanical energies in the two approaches at time t are

$$E_{cv}(t) = E_{syst}(t) = E_r(t) + E_v(t) = \int_0^{l(t)} \varepsilon_r \, \mathrm{d}x + \int_0^{l(t)} \varepsilon_v(x, t) \, \mathrm{d}x = \int_0^{l(t)} \varepsilon(x, t) \, \mathrm{d}x, \qquad (19)$$

where ε_r and ε_v are given in section 2. While E_r is constant in equation (5), it depends on time in equation (19).

Differentiating E_{cv} in equation (19) using Leibnitz's rule and equation (6) yields

$$\dot{E}_{cv}(t) = \dot{E}_r(t) + \dot{E}_v(t), \qquad (20)$$

where

$$\dot{E}_r(t) = \frac{1}{2}\rho v^3, \qquad \dot{E}_v(t) = \int_0^{l(t)} \frac{\partial \varepsilon_v(x, t)}{\partial t} \, \mathrm{d}x + v\varepsilon_v(l(t), t). \qquad (21, 22)$$

Differentiating the boundary conditions at $x = l(t)$ in equation (16) yields

$$y_t(l(t), t) = -\, v y_x(l(t), t) = 0, \quad y_{xt}(l(t), t) = -\, v y_{xx}(l(t), t). \qquad (23)$$

Substituting equation (7) into equation (22), followed by the use of equation (1), integration by parts, and application of equations (16) and (23), yields

$$\dot{E}_v(t) = -\,\tfrac{1}{2} v EI y_{xx}^2(0, t) \qquad (24)$$

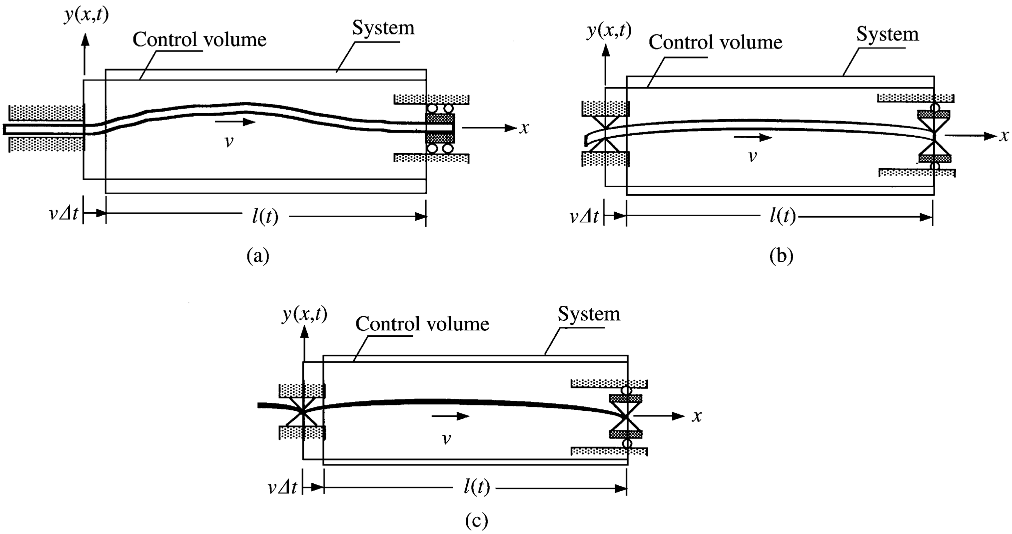

Figure 4. Control volumes and systems of translating media with variable length at time $t + \Delta t$: (a)–(c) as in Figure 3.

for the translating beam with fixed ends. Similarly,

$$\dot{E}_v(t) = -\tfrac{1}{2}v[P - \rho v^2]y_x^2(0, t) + EIvy_x(0, t)y_{xxx}(0, t) \tag{25}$$

for the translating beam with pinned ends, and $\dot{E}_v$ for the translating string is given by equation (25) with $EI = 0$. Note that $\dot{E}_v$ in equations (24) and (25) does not depend explicitly on the boundary conditions at $x = l(t)$.

While $\dot{E}_{cv}$ describes the instantaneous growth and decay of total mechanical energy of a translating medium with variable length, $\dot{E}_v$ can characterize its dynamic stability. Because the translating medium under consideration gains and loses mass during extension ($v > 0$) and retraction ($v < 0$), respectively, E_r increases and decreases accordingly, as observed from equation (21). On the other hand, the energy of vibration of the beam with a fixed end at $x = 0$ decreases and increases monotonically during extension and retraction, respectively, as observed from equation (24). When $|v| < \sqrt{P/\rho}$, the wave speed, the same behavior is predicted for the translating string. The energy of vibration of the string about its trivial equilibrium increases and decreases monotonically during extension and retraction, respectively, when $|v| > \sqrt{P/\rho}$, and remains unchanged when $|v| = \sqrt{P/\rho}$. Due to sign-indefiniteness of the second term on its right-hand side, general stability characteristics of the beam with a pinned end at $x = 0$ cannot be readily inferred from equation (25).

At time $t + \Delta t$, the control volume becomes the spatial domain $[0, l(t) + v\Delta t]$ in each case, and the system of material particles has translated a distance $v\Delta t$, as shown in Figure 4. The total mechanical energy of the system of material particles, occupying the spatial domain $[v\Delta t, l(t) + v\Delta t]$, is

$$E_{syst}(t + \Delta t) = E_{cv}(t + \Delta t) - v\Delta t\varepsilon(0, t + \Delta t). \tag{26}$$

Using equations (19) and (26) yields, as $\Delta t \to 0$, the Reynolds transport theorem for a translating medium with variable length [5]:

$$\dot{E}_{syst}(t) = \dot{E}_{cv}(t) - v\varepsilon(0, t). \tag{27}$$

Substituting equations (6), (7), (20), (21), and (24) into equation (27) and using equations (16) and (23) yields

$$\dot{E}_{syst}(t) = -\, vEIy_{xx}^2(0, t) \tag{28}$$

for the translating beam with fixed ends. Similarly,

$$\dot{E}_{syst}(t) = -\, vPy_x^2(0, t) + vEIy_x(0, t)y_{xxx}(0, t) \tag{29}$$

for the translating beam with pinned ends, and $\dot{E}_{syst}(t)$ for the translating string is given by equation (29) with $EI = 0$. The right-hand side of equation (28) represents the rate of work done by the bending moment at the left of the system at time t; the rate of work done by the shear force at the left end vanishes [5]. The rates of work done by the bending moment and shear force at the right end of the system also vanish, because its absolute angular and transverse velocities are

$$\frac{\mathrm{D}y_x(l(t), t)}{\mathrm{D}t} = \frac{\mathrm{D}y(l(t), t)}{\mathrm{D}t} = 0, \tag{30}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + v\partial/\partial x$ is the material time derivative. Similarly, the right-hand side of equation (29) represents the net rate of work done by the transverse component of the tension and the shear force at the left end of the system at time t, with its absolute transverse velocity given by

$$\frac{\mathrm{D}y(0, t)}{\mathrm{D}t} = y_t(0, t) + vy_x(0, t) = vy_x(0, t). \tag{31}$$

4. STATIONARY MEDIA WITH A MOVING BOUNDARY

The linear equation describing the transverse vibration of a stationary tensioned beam in Figure 5(a,b), with the left end at $x = l(t)$ moving at an arbitrarily prescribed speed $v(t) = \dot{l}(t)$, is

$$\rho y_{tt} - Py_{xx} + EIy_{xxxx} = 0, \tag{32}$$

where $y(x, t)$ is the transverse displacement of the beam particle at position $x(0 < x < l(t))$ at time t, and other variables are the same as those in section 2. A positive and negative $v(t)$ indicates that the left boundary moves instantaneously in the positive and negative direction of the x-axis respectively. The boundary conditions are given by equation (16) for fixed ends, and equation (17) for pinned ends. The linear equation describing the transverse vibration of the stationary string in Figure 5(c) is given by equation (32) with $EI = 0$ and associated boundary conditions given by equation (18). When v is constant, equations (32) and (16)–(18) describe the motions of translating media in Figure 3 relative to the coordinate systems that move with the media, with origins located at their right boundaries and positive x-axes opposite to those of the fixed coordinate systems.

In each case the control volume at time t is defined as the spatial domain $0 \leqslant x \leqslant l(t)$, and the system concerned consists of material particles of fixed identity, occupying the spatial domain $[0, l(t)]$ at time t (see Figure 5). The energies in the two approaches at time t are

$$E_{cv}(t) = E_{syst}(t) = \int_0^{l(t)} \varepsilon_v(x, t)\, \mathrm{d}x, \tag{33}$$

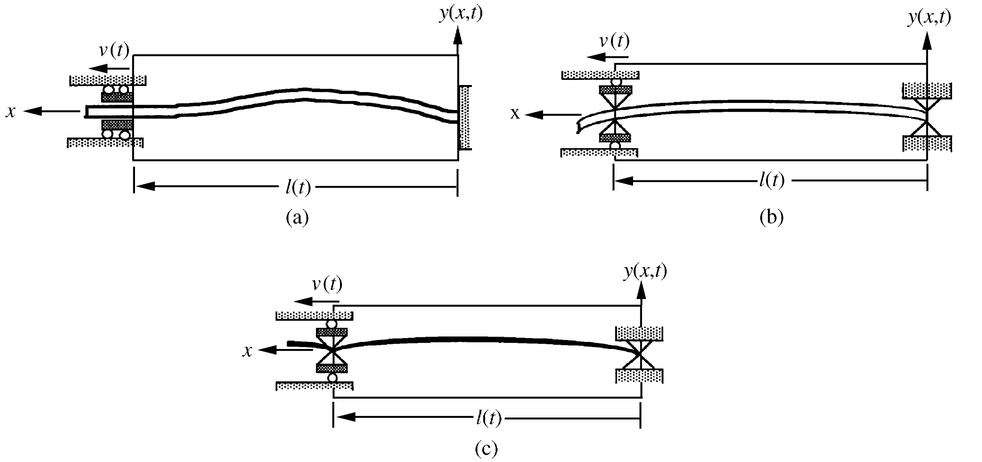

Figure 5. Stationary media with a moving boundary at time t: (a) beam with fixed ends, (b) beam with pinned ends, (c) string. In each case the control volume and system at time t coincide and are enclosed in a box. While the slope is continuous at the boundary $x = l(t)$ in (a) and (b), it can be discontinuous in (c).

where

$$\varepsilon_v(x, t) = \tfrac{1}{2}\rho y_t^2 + \tfrac{1}{2}P y_x^2 + \tfrac{1}{2}EI y_{xx}^2 \tag{34}$$

for the tensioned beams. The energy density for the string is given by equation (34) with $EI = 0$. Differentiating E_{cv} in equation (33) using Leibnitz's rule yields

$$\dot{E}_{cv}(t) = \int_0^{l(t)} \frac{\partial \varepsilon_v(x, t)}{\partial t}\,\mathrm{d}x + v(t)\varepsilon_v(l(t), t). \tag{35}$$

Substituting equation (34) into equation (35), followed by the use of equation (32), integration by parts, and application of equations (16) and (23), yields

$$\dot{E}_{cv}(t) = -\tfrac{1}{2}\, v(t) EI y_{xx}^2(l(t), t) \tag{36}$$

for the tensioned beam with fixed ends. Similarly,

$$\dot{E}_{cv}(t) = -\tfrac{1}{2} v(t)[P - \rho v^2(t)] y_x^2(l(t), t) + v(t) EI y_x(l(t), t) y_{xxx}(l(t), t) \tag{37}$$

for the tensioned beam with pinned ends, and $\dot{E}_{cv}$ for the string is given by equation (37) with $EI = 0$. Note that $\dot{E}_{cv}$ in equations (36) and (37) does not depend explicitly on the boundary conditions at $x = 0$.

The energy of the beam with a fixed end at $x = l(t)$ decreases and increases monotonically when $v(t) > 0$ and $v(t) < 0$, respectively, as observed from equation (36). The same behavior is predicted for the string when $|v(t)| < \sqrt{P/\rho}$, the wave speed. The energy of the string increases and decreases monotonically when $|v(t)| > \sqrt{P/\rho}$, and remains unchanged when $|v| = \sqrt{P/\rho}$. Because $y(l(t), t)$ in equations (36) and (37) corresponds to $y(0, t)$ in equations (24) and (25), equations (36) and (37) are in full agreement with equations (24) and (25),

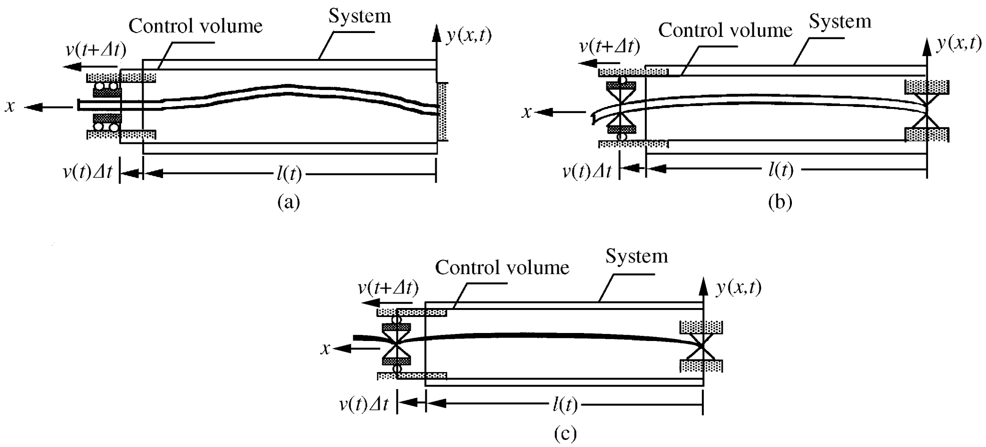

Figure 6. Control volumes and systems of stationary media with a moving boundary at time $t + \Delta t$: (a)–(c) as in Figure 5.

respectively, when v is constant. The dynamic behavior of translating media with variable length is analogous to that of stationary media with a moving boundary [5].

At time $t + \Delta t$, while the system of material particles remains unchanged in each case, the control volume becomes the spatial domain $[0, l(t) + v(t)\Delta t]$, as shown in Figure 6. The energy of the system of material particles at time $t + \Delta t$ can be related to that of constituent particles within the control volume:

$$E_{syst}(t + \Delta t) = E_{cv}(t + \Delta t) - v(t)\Delta t \varepsilon_v(l(t) + v(t)\Delta t, t + \Delta t). \tag{38}$$

Using equations (33) and (38) yields the Reynolds transport theorem for a stationary medium with a moving boundary:

$$\dot{E}_{syst}(t) = \dot{E}_{cv}(t) - v(t)\varepsilon_v(l(t), t). \tag{39}$$

Substituting equations (34) and (36) into equation (39) and using equations (16) and (23) yields

$$\dot{E}_{syst}(t) = -\, v(t) EI y_{xx}^2(l(t), t) \tag{40}$$

for the tensioned beam with fixed ends. Similarly,

$$\dot{E}_{syst}(t) = -\, v(t) P y_x^2(l(t), t) + v(t) EI y_x y_{xxx}(l(t), t) \tag{41}$$

for the tensioned beam with pinned ends, and $\dot{E}_{syst}(t)$ for the string is given by equation (41) with $EI = 0$. Since the system is not moving axially, the angular velocity at its left end at time t is $y_{xt}(l(t), t) = -\, v(t) y_{xx}(l(t), t)$. Hence the right-hand side of equation (40) represents the rate of work done by the bending moment at the left end of the system at time t. Similarly, the right-hand side of equation (41) represents the net rate of work done by the transverse component of the tension and the shear force at the left end of the system at time t, with its transverse velocity given by $y_t(l(t), t) = -\, v y_x(l(t), t)$.

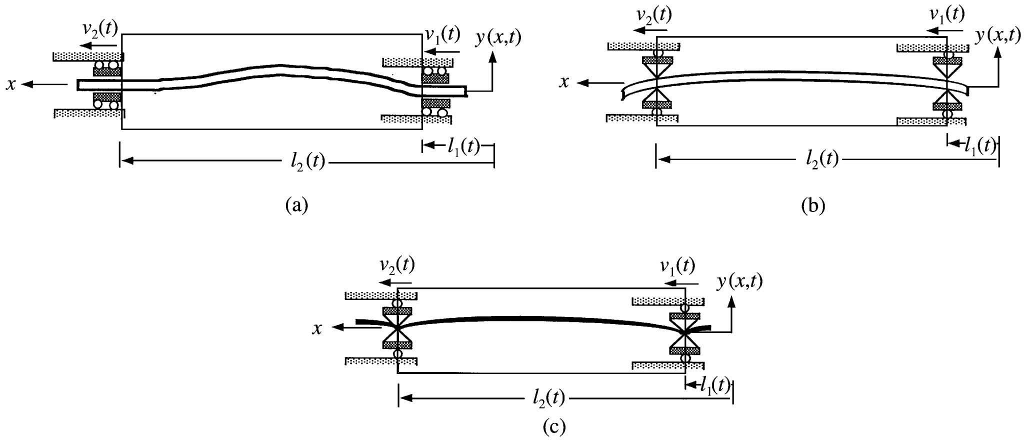

Figure 7. Stationary media with two moving boundaries at time t: (a) beam with fixed ends, (b) beam with pinned ends, (c) string. In each case the control volume and system at time t coincide and are enclosed in a box. While the slopes are continuous at the two boundaries in (a) and (b), it can be discontinuous in (c).

5. STATIONARY MEDIA WITH TWO MOVING BOUNDARIES

The linear equation describing the transverse vibration of a stationary tensioned beam in Figure 7(a,b), with both ends at $x = l_1(t)$ and $x = l_2(t)$ moving at arbitrarily prescribed speeds, $v_1(t) = \dot{l}_1(t)$ and $v_2(t) = \dot{l}_2(t)$ respectively, is given by equation (32) where $l_1(t) < x < l_2(t)$. A positive and negative speed, $v_1(t)$ or $v_2(t)$, indicates that the corresponding boundary moves instantaneously in the positive and negative direction of the x-axis respectively. The boundary conditions are

$$y = y_x = 0 \quad \text{at } x = l_1(t) \text{ and } l_2(t) \tag{42}$$

for fixed ends, and

$$y = y_{xx} = 0 \quad \text{at } x = l_1(t) \text{ and } l_2(t) \tag{43}$$

for pinned ends. The linear equation describing the transverse vibration of the stationary string in Figure 7(c) is given by equation (32) with $EI = 0$ and associated boundary conditions

$$y = 0 \quad \text{at } x = l_1(t) \text{ and } l_2(t). \tag{44}$$

When $v_1 = v_2 = v$, where v is constant, equations (32) and (42)–(44) describe the motions of translating media in Figure 1, relative to the coordinate systems that move with the media, with origins located at their right boundaries and positive x-axes opposite to those of the fixed coordinate systems.

In each case the control volume at time t is defined as the spatial domain $l_1(t) \leqslant x \leqslant l_2(t)$, and the system concerned consists of material particles of fixed identity, occupying the spatial domain $[l_1(t), l_2(t)]$ at time t (see Figure 7). The energies in the two approaches at time t are

$$E_{cv}(t) = E_{syst}(t) = \int_{l_1(t)}^{l_2(t)} \varepsilon_v(x, t)\, \mathrm{d}x, \tag{45}$$

where ε_v is given in section 4. Differentiating E_{cv} in equation (45) using Leibnitz's rule yields

$$\dot{E}_{cv}(t) = \int_{l_1(t)}^{l_2(t)} \frac{\partial \varepsilon_v(x, t)}{\partial t} \mathrm{d}x + v_2(t)\varepsilon_v(l_2(t), t) - v_1(t)\varepsilon_v(l_1(t), t). \tag{46}$$

Substituting equation (34) into equation (46) and using equation (42) yields

$$\dot{E}_{cv}(t) = -\tfrac{1}{2} v_2(t) EI y_{xx}^2(l_2(t), t) + \tfrac{1}{2} v_1(t) EI y_{xx}^2(l_1(t), t) \tag{47}$$

for the tensioned beam with fixed ends. Similarly,

$$\begin{aligned} \dot{E}_{cv}(t) = &-\tfrac{1}{2} v_2(t)[P - \rho v_2^2(t)] y_x^2(l_2(t), t) + \tfrac{1}{2} v_1(t)[P - \rho v_1^2(t)] y_x^2(l_1(t), t) \\ &+ v_2(t) EI y_x(l_2(t), t) y_{xxx}(l_2(t), t) - v_1(t) EI y_x(l_1(t), t) y_{xxx}(l_1(t), t) \end{aligned} \tag{48}$$

for the tensioned beam with pinned ends, and $\dot{E}_{cv}(t)$ for the string is given by equation (48) with $EI = 0$.

The following stability characteristics of the tensioned beam with fixed ends are inferred from equation (47): when $v_1(t) < 0$ and $v_2(t) > 0$, the energy decreases, and when $v_1(t) > 0$ and $v_2(t) < 0$, the energy increases. The stability characteristics of the string are the same as those of the tensioned beam with fixed ends when $|v_1(t)|$ and $|v_2(t)|$ are smaller than $\sqrt{P/\rho}$, and are reversed when $|v_1(t)|$ and $|v_2(t)|$ exceed $\sqrt{P/\rho}$. Because $y(l_1(t), t)$ and $y(l_2(t), t)$ in equations (47) and (48) correspond to $y(l, t)$ and $y(0, t)$ in equations (9) and (10), respectively, equations (47) and (48) are in full agreement with equations (9) and (10) accordingly, when $v_1(t) = v_2(t) = v$, where v is constant. The dynamic behavior of translating media with constant length is analogous to that of stationary media with both boundaries moving at the same speed.

At time $t + \Delta t$, while the system of material particles remains unchanged in each case, the control volume becomes the spatial domain $[l_1(t) + v_1(t)\Delta t, l_2(t) + v_2(t)\Delta t]$, as shown in Figure 8. The energy of the system of material particles at time $t + \Delta t$ can be related to that of constituent particles within the control volume:

$$\begin{aligned} E_{syst}(t + \Delta t) = E_{cv}(t + \Delta t) &- v_2(t)\Delta t \varepsilon_v(l_2(t) + v_2(t)\Delta t, t + \Delta t) \\ &+ v_1(t)\Delta t \varepsilon_v(l_1(t) + v_1(t)\Delta t, t + \Delta t). \end{aligned} \tag{49}$$

The Reynolds transport theorem for a stationary medium with two moving boundaries is

$$\dot{E}_{syst}(t) = \dot{E}_{cv}(t) - v_2(t)\varepsilon(l_2(t), t) + v_1(t)\varepsilon(l_1(t), t). \tag{50}$$

Substituting equations (34) and (47) into equation (50) and using equation (42) yields

$$\dot{E}_{syst}(t) = -v_2(t) EI y_{xx}^2(l_2(t), t) + v_1(t) EI y_{xx}^2(l_1(t), t) \tag{51}$$

for the tensioned beam with fixed ends. Similarly,

$$\begin{aligned} \dot{E}_{syst}(t) = &-v_2(t) P y_x^2(l_2(t), t) + v_1(t) P y_x^2(l_1(t), t) \\ &+ v_2(t) EI y_x(l_2(t), t) y_{xxx}(l_2(t), t) - v_1(t) EI y_x(l_1(t), t) y_{xxx}(l_1(t), t) \end{aligned} \tag{52}$$

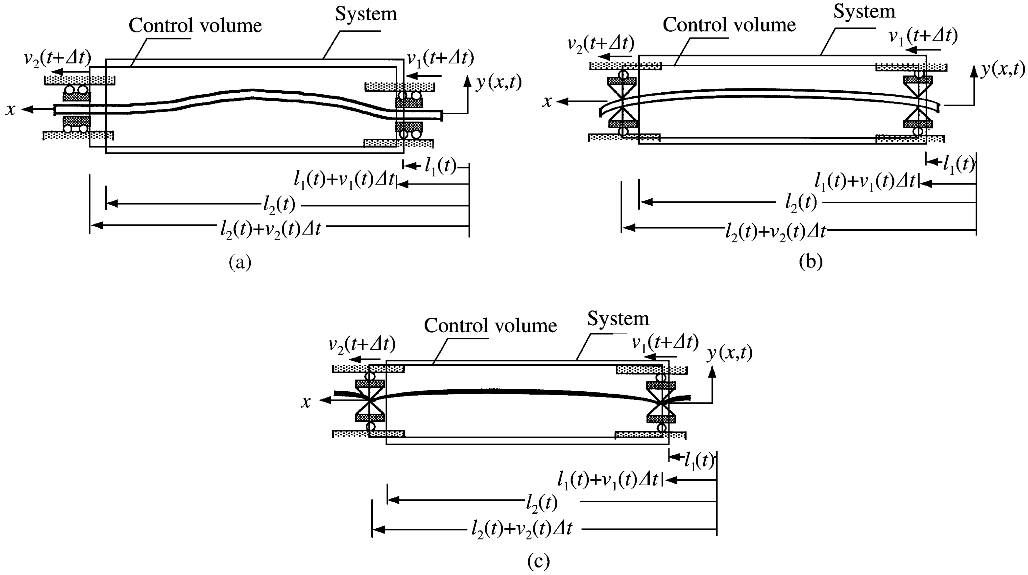

Figure 8. Control volumes and systems of stationary media with two moving boundaries at time $t + \Delta t$: (a)–(c) as in Figure 7.

for the tensioned beam with pinned ends, and $\dot{E}_{syst}(t)$ for the string is given by equation (52) with $EI = 0$. The right-hand side of equation (51) represents the net rate of work done by the bending moments at the two ends of the system at time t, and that of equation (52) the net rate of work done by the transverse components of the tensions and the shear forces at the two ends of the system.

6. CONCLUDING REMARKS

While the equations of motion of translating media are derived in the Eulerian frame of reference and those of stationary media with moving boundaries the Lagrangian frame of reference, their rates of change of energies are distinguished from control volume and system viewpoints. The rates of change of energies from the control volume viewpoint can characterize the dynamic stability of translating media and stationary media with moving boundaries, and the rates of change of total mechanical energies from the system viewpoint establish an instantaneous work and energy relation. While the energies are not conserved due to relative motions between continuous media and their boundaries, conserved functionals from the control volume viewpoint can be constructed following reference [6] and used as Lyapunov functionals. Conserved functionals from the system viewpoint cannot be used as Lyapunov functionals in analyzing the stability of constituent particles within the control volume, in agreement with reference [4]. The boundary conditions at non-dissipative boundaries with no relative motions to the media do not directly affect their stability characteristics.

ACKNOWLEDGMENTS

This material is based on work supported by the National Science Foundation and the Hong Kong Research Grants Council. The author would like to thank Professor Noel Perkins for many valuable discussions.

REFERENCES

1. C. D. Mote Jr 1971 *Shock and Vibration Digest* **4**, 2–11. Dynamic stability of axially moving materials.
2. W. L. Miranker 1960 *IBM Journal of Research and Development* **4**, 36–42. The wave motion in a medium in motion.
3. J. A. Wickert and C. D. Mote Jr 1989 *Journal of the Acoustical Society of America* **85**, 1365–1368. On the energetics of axially moving continua.
4. A. A. Renshaw, C. D. Rahn, J. A. Wickert and C. D. Mote Jr 1998 *American Society of Mechanical Engineers Journal of Vibration and Acoustics* **120**, 634–636. Energetics and conserved functionals for axially moving materials.
5. W. D. Zhu and J. Ni 2000 *American Society of Mechanical Engineers Journal of Vibration and Acoustics* **122**, 295–303. Energetics and stability of translating media with an arbitrarily varying length.
6. C. D. Rahn 1992 *Ph. D. Thesis*, *University of California at Berkeley*. Parametric control of vibration in flexible and axially-moving material systems.