Journal of Sound and Vibration (2002) **254**(1), 37–49
doi:10.1006/jsvi.2001.4078, available online at http://www.idealibrary.com on IDEAL®

RANDOM RESPONSE OF PREISACH HYSTERETIC SYSTEMS

Z. G. YING AND W. Q. ZHU

Department of Mechanics, Zhejiang University, Hangzhou 310027, *People's Republic of China. E-mail: wqzhu@yahoo.com*

Y. Q. NI

Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong

AND

J. M. KO

Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong

(*Received* 26 *March* 2001, *and in final form* 15 *August* 2001)

An approximate method for analyzing the response of Preisach hysteretic systems with non-local memory under stationary Gaussian excitation is proposed. The covariance matrix equation of system response is derived. The cross correlation function of Preisach hysteretic force and response in the covariance equation is evaluated based on the switching probability analysis and the Gaussian approximation of response process and an explicit expression for the cross correlation function is given for the case of symmetric Preisach weighting function. It is shown that the numerical result obtained by using the proposed method is in good agreement with that from digital simulation.

1. INTRODUCTION

Non-linear hysteresis behavior exists widely in mechanical and structural systems [1–4], where the restoring force depends on not only the instantaneous deformation but also the past history of deformation [5]. Furthermore, there has been an increasing interest recently in using smart materials [6–8] such as piezoceramics, shape memory alloys, and electro-/magneto-rheological fluids, which exhibit significant hysteresis. Various analytical models [9–12] have been proposed for representing the hysteretic constitutive relationship. However, almost all hysteresis models used in mechanical and structural disciplines can only represent hysteresis with local memory. They cannot be used to describe the complicated hysteresis behavior such as the crossing of minor loops which can arise in real materials. In order to more accurately capture such complicated constitutive behavior with non-local memory, a Preisach integral model [1–4] has been developed in physics and mathematics but little work has been done in the context of mechanical and structural engineering until recently.

In the fields of mechanical and structural engineering, the dynamic loading acting on hysteretic systems is usually random in nature. For strongly non-linear hysteretic systems, it

0022-460X/02/$35.00

is extremely difficult to analytically determine the exact random response and thus some approximate solution techniques have been developed, including the equivalent linearization method [13–17] and the stochastic averaging method [18–24]. The random dynamic responses of bilinear and Bouc–Wen hysteretic systems have been studied by using these two methods [5, 25–30]. However, only the mean output of the Preisach model to stochastic input has been investigated [31, 32].

In the present paper, the random response of Preisach hysteretic systems is investigated. The covariance matrix equation of system response is derived. The cross correlation functions of Preisach hysteretic force and response in the covariance equation are obtained based on the switching probability analysis and the Gaussian approximation of response process. The switching probabilities are calculated by using the mathematical machinery of exit problem, and, in particular, approximate results are obtained for the case of symmetric Preisach weighting function. Finally, an example of Preisach hysteretic system with symmetric weighting function under Gaussian white noise excitation is presented to illustrate the application of the proposed method.

2. PREISACH HYSTERESIS MODEL

The Preisach model for hysteresis is expressed in terms of the following integral:

$$z = \iint_{\alpha \leqslant \beta} \mu(\alpha, \beta) \hat{\gamma}_{\alpha\beta}(x) \, \mathrm{d}\alpha \, \mathrm{d}\beta, \tag{1}$$

where z and x denote hysteretic force and displacement, respectively, $\mu(\alpha, \beta)$ is a weighting function called Preisach function, α and β are integral variables in a limiting triangle S on Preisach plane (α, β) (see Figure 1), and $\hat{\gamma}_{\alpha\beta}(x)$ is called relay hysteresis operator as shown in Figure 2. The operator takes the value $+1$ or -1 corresponding to "up" or "down" positions of the relay, respectively, and is represented by the equation

$$\hat{\gamma}_{\alpha\beta}(x) = \begin{cases} +1, & \text{ascending } x > \alpha \text{ or descending } x > \beta, \\ -1, & \text{ascending } x < \alpha \text{ or descending } x < \beta. \end{cases} \tag{2}$$

The Preisach hysteresis model in equation (1), expressed as the superposition of a continuous family of elementary rectangular loops, can be interpreted in terms of the

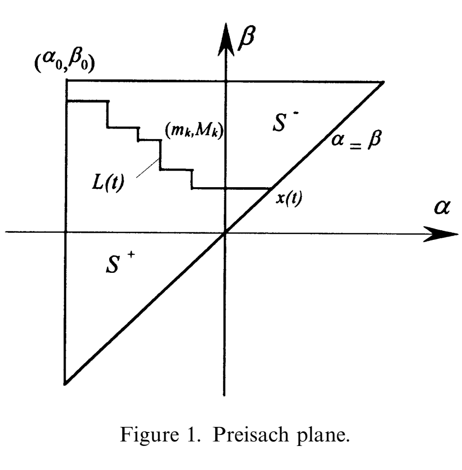

Figure 1. Preisach plane.

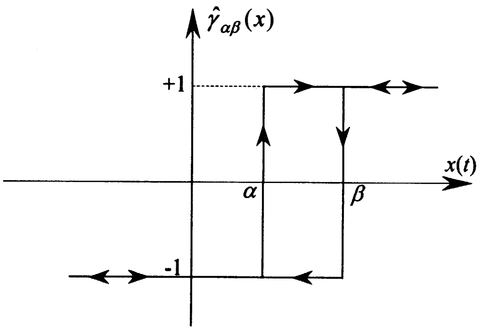

Figure 2. Relay hysteresis operator.

spectral decomposition of a complicated hysteretic constitutive law into the simplest relay hysteresis operators. The Preisach hysteresis behavior is completely characterized by the weighting function $\mu(\alpha, \beta)$. For an arbitrary displacement $x(t)$, the hysteretic force $z(t)$ can be determined by the weighting function together with a staircase line L which divides the Preisach plane into two parts, S^+ and S^-, corresponding to the 'up' and 'down' positions of the relay. Each vertex of interface line L is associated with past local minimum m_k or maximum M_k of the displacement ($k = 1, 2, \ldots$). Therefore, the Preisach hysteretic force depends on not only the instantaneous displacement but also the non-local displacement history in terms of selective memory, i.e., it has the characteristics of non-local memory. Note that the hysteretic force $z(t)$ is independent of the magnitude of velocity $\dot{x}$.

3. MEAN SQUARE RESPONSE OF PREISACH HYSTERETIC SYSTEMS

Consider the response of a Preisach hysteretic system to random excitation governed by the following equation:

$$\ddot{X} + 2\zeta\dot{X} + g(X) + Z(X, \dot{X}) = f(t), \tag{3}$$

where X denotes displacement, ζ is viscous damping coefficient, $g(X)$ is a non-linear restoring force with $g(-X) = -g(X)$, Z denotes Preisach hysteretic force governed by equation (1), and $f(t)$ represents an external random excitation.

Under the excitation of stationary Gaussian process with zero mean, the mean stationary response of Preisach hysteretic system (3) is equal to zero since $g(0) = 0$ and the hysteretic force Z in equation (1) approaches zero [31]. By introducing state vector $\mathbf{Y} = [X, \dot{X}]^{\mathrm{T}}$ and rewriting second order differential equation (3) in the form of first order differential equations for the state vector, the following covariance matrix equation of system response can be obtained:

$$\begin{aligned}\dot{\mathbf{W}}(t) &= E[\dot{\mathbf{Y}}(t)\mathbf{Y}^{\mathrm{T}}(t)] + E[\mathbf{Y}(t)\dot{\mathbf{Y}}^{\mathrm{T}}(t)] \\ &= \mathbf{U}(t) + \mathbf{U}^{\mathrm{T}}(t) + \mathbf{V}(t) + \mathbf{V}^{\mathrm{T}}(t) + \mathbf{D}_F(t),\end{aligned} \tag{4}$$

where $E[\cdot]$ denotes expectation operator

$$\mathbf{W} = E[\mathbf{Y}\mathbf{Y}^{\mathrm{T}}] = \begin{bmatrix} E[X^2] & E[X\dot{X}] \\ E[\dot{X}X] & E[\dot{X}^2] \end{bmatrix}, \tag{5a}$$

$$\mathbf{U} = \begin{bmatrix} E[X\dot{X}] & E[\dot{X}^2] \\ -E[Xg] - 2\zeta E[X\dot{X}] & -E[\dot{X}g] - 2\zeta E[\dot{X}^2] \end{bmatrix}, \tag{5b}$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 \\ -E[ZX] & -E[Z\dot{X}] \end{bmatrix}, \quad \mathbf{D}_F = \begin{bmatrix} 0 & -E[Xf] \\ -E[Xf] & -2E[\dot{X}f] \end{bmatrix}. \tag{5c}$$

For the stationary Gaussian white noise excitation $f(t)$ with intensity D, the correlation matrix $\mathbf{D}_F$ of excitation and response is of the form

$$D_F = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}. \tag{6}$$

The covariance matrix $\mathbf{W}$ of the stationary response is a constant matrix and equation (4) becomes

$$\mathbf{U} + \mathbf{U}^{\mathrm{T}} + \mathbf{V} + \mathbf{V}^{\mathrm{T}} + \mathbf{D}_F = 0. \tag{7}$$

In the case of linear restoring force, i.e., $g(X) = kX$, k is a constant, the correlation matrix $\mathbf{U}$ is of the form

$$\mathbf{U} = \mathbf{AW}, \tag{8a}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k & -2\zeta \end{bmatrix}. \tag{8b}$$

By using equation (1) the cross correlation functions $E[ZX]$ and $E[Z\dot{X}]$ of hysteretic force and response in correlation matrix $\mathbf{V}$ can be expressed as follows:

$$E[ZX] = \iint_{\alpha \leqslant \beta} \mu(\alpha, \beta) E[\hat{\gamma}_{\alpha\beta}(X)X] \, \mathrm{d}\alpha \, \mathrm{d}\beta, \tag{9}$$

$$E[Z\dot{X}] = \iint_{\alpha \leqslant \beta} \mu(\alpha, \beta) E[\hat{\gamma}_{\alpha\beta}(X)\dot{X}] \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{10}$$

Since the elementary hysteresis operator $\hat{\gamma}_{\alpha\beta}(X)$ takes the value either $+1$ or -1, the correlation function $E[\hat{\gamma}_{\alpha\beta}(X)X]$ can be obtained as follows:

$$\begin{aligned} E[\hat{\gamma}_{\alpha\beta}(X)X] &= E[+X]P\{\hat{\gamma}_{\alpha\beta}(X) = +1\} + E[-X]P\{\hat{\gamma}_{\alpha\beta}(X) = -1\} \\ &= (E[X|_{X \geqslant \alpha}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \alpha\} \\ &\quad + E[X|_{X \geqslant \beta}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \beta\})P\{\hat{\gamma}_{\alpha\beta}(X) = +1\} \\ &\quad + (E[-X|_{X \leqslant \alpha}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \alpha\} \\ &\quad + E[-X|_{X \leqslant \beta}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \beta\})P\{\hat{\gamma}_{\alpha\beta}(X) = -1\} \\ &= (E[X|_{X \geqslant 0} - |X||_{X \in (0,\alpha)}]q_\alpha \\ &\quad + E[X|_{X \geqslant 0} - |X||_{X \in (0,\beta)}]q_\beta)P\{\hat{\gamma}_{\alpha\beta}(X) = +1\} \end{aligned}$$

$$+ (E[-X|_{X \leqslant 0} - |X||_{X \in (0,\alpha)}]q_\alpha$$

$$+ E[-X|_{X \leqslant 0} - |X||_{X \in (0,\beta)}]q_\beta)P\{\hat{\gamma}_{\alpha\beta}(X) = -1\}$$

$$= (E[X|_{X \geqslant 0}] - E[|X||_{X \in (0,\alpha)}])q_\alpha$$

$$+ (E[X|_{X \geqslant 0} - E[|X||_{X \in (0,\beta)}])q_\beta$$

$$= \tfrac{1}{2}E[|X|] - E[|X||_{X \in (0,\alpha)}]q_\alpha - E|X||_{X \in (0,\beta)}]q_\beta, \quad (11)$$

where $P\{\cdot\}$ denotes probability operator. The notations $q_\alpha = P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \alpha\}$ and $q_\beta = P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \beta\}$ denote the probability of $\hat{\gamma}_{\alpha\beta}(X)$ switching from -1 to $+1$ at $X = \alpha$ and from $+1$ to -1 at $X = \beta$ respectively. There exist the probability relations $P\{\hat{\gamma}_{\alpha\beta}(X) = +1\} + P\{\hat{\gamma}_{\alpha\beta}(X) = -1\} = 1$ and $q_\alpha + q_\beta = 1$. Similarly, correlation function

$$E[\hat{\gamma}_{\alpha\beta}(X)\dot{X}] = E[+\dot{X}]P\{\hat{\gamma}_{\alpha\beta}(X) = +1\} + E[-\dot{X}]P\{\hat{\gamma}_{\alpha\beta}(X) = -1\}$$

$$= (E[\dot{X}|_{X \geqslant \alpha, \dot{X} \geqslant 0}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \alpha\}$$

$$+ E[\dot{X}|_{X \geqslant \beta, \dot{X} \leqslant 0}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \beta\})P\{\hat{\gamma}_{\alpha\beta}(X) = +1\}$$

$$+ (E[-\dot{X}|_{X \leqslant \alpha, \dot{X} \geqslant 0}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \alpha\}$$

$$+ E[-\dot{X}|_{X \leqslant \beta, \dot{X} \leqslant 0}]P\{\hat{\gamma}_{\alpha\beta} \textit{ switching at } \beta\})P\{\hat{\gamma}_{\alpha\beta}(X) = -1\}$$

$$= (E[|\dot{X}||_{X \geqslant 0, \dot{X} \geqslant 0} - \mathrm{sgn}(\alpha)|\dot{X}||_{X \in (0,\alpha), \dot{X} \geqslant 0}]q_\alpha$$

$$+ E[-|\dot{X}||_{X \geqslant 0, \dot{X} \leqslant 0} + \mathrm{sgn}(\beta)|\dot{X}||_{X \in (0,\beta), \dot{X} \leqslant 0}]q_\beta)P\{\hat{\gamma}_{\alpha\beta}(X) = +1\}$$

$$+ (E[-|\dot{X}||_{X \leqslant 0, \dot{X} \geqslant 0} - \mathrm{sgn}(\alpha)|\dot{X}||_{X \in (0,\alpha), \dot{X} \geqslant 0}]q_\alpha$$

$$+ E[|\dot{X}||_{X \leqslant 0, \dot{X} \leqslant 0} + \mathrm{sgn}(\beta)|\dot{X}||_{X \in (0,\beta), \dot{X} \leqslant 0}]q_\beta)P\{\hat{\gamma}_{\alpha\beta}(X) = -1\}$$

$$= \mathrm{sgn}(-\alpha)E[|\dot{X}||_{X \in (0,\alpha), \dot{X} \geqslant 0}]q_\alpha + \mathrm{sgn}(\beta)E[|\dot{X}||_{X \in (0,\beta), \dot{X} \leqslant 0}]q_\beta$$

$$= \tfrac{1}{2}\mathrm{sgn}(-\alpha)E[|\dot{X}||_{X \in (0,\alpha)}]q_\alpha + \tfrac{1}{2}\mathrm{sgn}(\beta)E|[\dot{X}||_{X \in (0,\beta)}]q_\beta. \quad (12)$$

For a stationary Gaussian random excitation, the response of the equivalent linear system of equation (3) would be Gaussian. Under the assumption of Gaussian response, the mean absolute displacement and velocity in equations (11) and (12) can be evaluated in terms of mean square response as follows:

$$E[|X|] = \sqrt{\frac{2E[X^2]}{\pi}}, \quad (13a)$$

$$E[|X||_{X \in (0,\alpha)}] = \sqrt{\frac{E[X^2]}{2\pi}}\left(1 - \exp\left\{-\frac{\alpha^2}{2E[X^2]}\right\}\right), \quad (13b)$$

$$E[|X||_{X \in (0,\beta)}] = \sqrt{\frac{E[X^2]}{2\pi}}\left(1 - \exp\left\{-\frac{\beta^2}{2E[X^2]}\right\}\right), \quad (13c)$$

$$E[|\dot{X}||_{X\in(0,\alpha)}] = \sqrt{\frac{E[\dot{X}^2]}{2\pi}}\, erf\left(\frac{\alpha}{\sqrt{2E[X^2]}}\right)\mathrm{sgn}(\alpha), \tag{13d}$$

$$E[|\dot{X}||_{X\in(0,\beta)}] = \sqrt{\frac{E[\dot{X}^2]}{2\pi}}\, erf\left(\frac{\beta}{\sqrt{2E[X^2]}}\right)\mathrm{sgn}(\beta), \tag{13e}$$

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-u^2}\,\mathrm{d}u, \tag{13f}$$

where $erf(\cdot)$ is the error function which is an odd function, i.e., $erf(-x) = -erf(x)$, and has the properties $erf(0) = 0$ and $erf(\pm\infty) = \pm 1$.

4. SWITCHING PROBABILITY

The switching probabilities q_α and q_β in equations (11) and (12) can be calculated by using the mathematical machinery of an exit problem [32]. Since $q_\alpha + q_\beta = 1$, only one switching probability, for example q_α, needs to be calculated. Consider the time evolution of response process. The switching probability q_α is the sum of the probabilities of disjoint events of even and odd switching numbers and thus, for different initial states it can be expressed as follows:

$$q_\alpha(t) = \begin{cases} \frac{1}{2}P_0^+(t) + \sum_{i=1}^{\infty} P_{2i}^+(t) & \text{for } \hat{\gamma}_{\alpha\beta}(x_0) = +1, \\ \frac{1}{2}P_0^-(t) + \sum_{i=0}^{\infty} P_{2i+1}^-(t) & \text{for } \hat{\gamma}_{\alpha\beta}(x_0) = -1, \end{cases} \tag{14}$$

where $P_j^\pm(t)$ are the probabilities of j switchings during time interval $(0, t)$ for initial states $\hat{\gamma}_{\alpha\beta}(x_0) = \pm 1$, i.e.,

$$P_j^\pm(t) = P\left\{\begin{matrix} j \text{ switchings during time} \\ \text{interval } (0,t)|\hat{\gamma}_{\alpha\beta}(x_0) = \pm 1 \end{matrix}\right\} \quad (j = 0, 1, 2, \ldots). \tag{15}$$

$\hat{\gamma}_{\alpha\beta}(X)$ switching at α (or β) takes place at the moment when the response process X starting from point x_0 or β (or α) exits from semi-infinite interval $(-\infty, \beta)$ [or $(\alpha, +\infty)$]. Based on the mathematical machinery of this exit problem, the switching probability $P_j^\pm$ can be expressed as follows:

$$P_{2i}^+(t) = \rho_{\beta 0}(t, x_0) * \rho_\alpha(t, \beta) * [\rho_\beta(t, \alpha) * \rho_\alpha(t, \beta) *]^{2i-2} P_\beta(t, \alpha), \tag{16a}$$

$$P_{2i+1}^-(t) = \rho_{\alpha 0}(t, x_0) * [\rho_\beta(t, \alpha) * \rho_\alpha(t, \beta) *]^{2i} P_\beta(t, \alpha). \tag{16b}$$

where "*" denotes convolution operator, $\rho_\alpha(t, \beta)$ [$\rho_\beta(t, \alpha)$] represents the probability density as a function of time t for the event of only one $\hat{\gamma}_{\alpha\beta}(X)$ switching at $\alpha(\beta)$ for response process X starting from point $\beta(\alpha)$, $\rho_{\alpha 0}(t, \beta)$ [$\rho_{\beta 0}(t, \alpha)$] is the probability density function of one $\hat{\gamma}_{\alpha\beta}$ switching at $\alpha(\beta)$ with starting point x_0, and $P_\beta(t, \alpha)$ represents the probability function of no $\hat{\gamma}_{\beta\alpha}(X)$ switching at β for response process X starting from point α, or the probability of no exiting from semi-infinite interval $(\alpha, +\infty)$ during time interval $(0, t)$. Substituting equations (16a) and (16b) into equation (14) leads to

$$q_\alpha(t) = \begin{cases} \frac{1}{2} P_{\beta 0}(t, x_0) + \sum_{i=1}^{\infty} \rho_{\beta 0}(t, x_0) * \rho_\alpha(t, \beta) * [\rho_\beta(t, \alpha) * \rho_\alpha(t, \beta) *]^{2i-2} P_\beta(t, \alpha), & \hat{\gamma}_{\alpha\beta}(x_0) = +1, \\ \frac{1}{2} P_{\alpha 0}(t, x_0) + \sum_{i=0}^{\infty} \rho_{\alpha 0}(t, x_0) * [\rho_\beta(t, \alpha) * \rho_\alpha(t, \beta) *]^{2i} P_\beta(t, \alpha), & \hat{\gamma}_{\alpha\beta}(x_0) = -1, \end{cases} \tag{17}$$

where $P_{\alpha 0}(t, x_0)$ $[P_{\beta 0}(t, x_0)]$ is the probability function of no $\hat{\gamma}_{\alpha\beta}$ switching at $\alpha(\beta)$ for starting point x_0. According to the exit theory of stochastic process, no switching probability functions $P_\alpha(t, x)$ and $P_\beta(t, x)$ are governed by the following backward Kolmogorov equation:

$$\frac{\partial P_c}{\partial t} = m(x) \frac{\partial P_c}{\partial x} + \frac{\sigma^2(x)}{2} \frac{\partial^2 P_c}{\partial x^2} \tag{18a}$$

with initial and boundary conditions

$$P_c(0, x) = 1, \qquad P_c(t, c) = 0, \tag{18b}$$

where c stands for α or β; m and σ are the drift and diffusion coefficients of response process X. Under the Gaussian assumption of response, the drift and diffusion coefficients are evaluated in terms of mean square response as follows:

$$m(X) = \frac{-D}{2E[X^2]} x, \qquad \sigma^2(X) = D. \tag{19}$$

By applying the Laplace transform to equation (18a) with equation (18b), the transformed probability function of no switching is obtained as follows:

$$\tilde{P}_c(s, x) = \frac{1}{s} \left[1 - e^{(x^2 - c^2)/4E[X^2]} \frac{D_{-2sE[X^2]/D}(x/\sqrt{E[X^2]})}{D_{-2sE[X^2]/D}(c/\sqrt{E[X^2]})} \right], \tag{20}$$

where $D_{-n}(x)$ is a parabolic cylinder function with constant n; $c = \alpha$ or β. Then the transformed probability density function of one switching is

$$\rho_c(s, x) = e^{(x^2 - c^2)/4E[X^2]} \frac{D_{-2sE[X^2]/D}(x/\sqrt{E[X^2]})}{D_{-2sE[X^2]/D}(c/\sqrt{E[X^2]})}. \tag{21}$$

The transformed switching probability $\tilde{q}_\alpha(s)$ can be obtained as follows:

$$\begin{aligned} \tilde{q}_\alpha(s) &= \frac{1}{2} \tilde{P}_{\beta 0}(s, x_0) + \sum_{i=1}^{\infty} \tilde{\rho}_{\beta 0}(s, x_0) \tilde{\rho}_\alpha(s, \beta) [\tilde{\rho}_\beta(s, \alpha) \tilde{\rho}_\alpha(s, \beta)]^{2i-2} \tilde{P}_\beta(s, \alpha) \\ &= \frac{1}{2} \tilde{P}_{\beta 0}(s, x_0) + \frac{\tilde{\rho}_{\beta 0}(s, x_0) \tilde{\rho}_\alpha(s, \beta)}{1 - \tilde{\rho}_\alpha(s, \beta) \tilde{\rho}_\beta(s, \alpha)} \tilde{P}_\beta(s, \alpha) \\ &= \frac{1 - \tilde{\rho}_{\beta 0}(s, x_0)}{2s} + \frac{\tilde{\rho}_{\beta 0}(s, x_0) \tilde{\rho}_\alpha(s, \beta) [1 - \tilde{\rho}_\beta(s, \alpha)]}{s[1 - \tilde{\rho}_\alpha(s, \beta) \tilde{\rho}_\beta(s, \alpha)]} \end{aligned} \tag{22a}$$

for the initial state $\hat{\gamma}_{\alpha\beta}(x_0) = +1$ and

$$\begin{aligned}\tilde{q}_\alpha(s) &= \frac{1}{2}\tilde{P}_{\alpha 0}(s, x_0) + \sum_{i=0}^{\infty} \tilde{\rho}_{\alpha 0}(s, x_0)[\tilde{\rho}_\beta(s, \alpha)\tilde{\rho}_\alpha(s, \beta)]^{2i}\tilde{P}_\beta(s, \alpha) \\ &= \frac{1}{2}\tilde{P}_{\alpha 0}(s, x_0) + \frac{\tilde{\rho}_{\alpha 0}(s, x_0)}{1 - \tilde{\rho}_\alpha(s, \beta)\tilde{\rho}_\beta(s, \alpha)}\tilde{P}_\beta(s, \alpha) \\ &= \frac{1 - \tilde{\rho}_{\alpha 0}(s, x_0)}{2s} + \frac{\tilde{\rho}_{\alpha 0}(s, x_0)[1 - \tilde{\rho}_\beta(s, \alpha)]}{s[1 - \tilde{\rho}_\alpha(s, \beta)\tilde{\rho}_\beta(s, \alpha)]}\end{aligned} \tag{22b}$$

for the initial state $\hat{\gamma}_{\alpha\beta}(x_0) = -1$. The switching probability $q_\alpha(t)$ is obtained from the inverse Laplace transform of $\tilde{q}_\alpha(s)$ and the stationary switching probability by letting $t \to \infty$. By substituting q_α into expressions (9)–(12) and then into equation (5c), the mean square stationary response can be obtained by solving equation (7).

In fact, most Preisach hysteresis has wiping-out and congruency properties [1–3] such that the weighting function $\mu(\alpha, \beta)$ possesses mirror symmetry with respect to the line $\alpha + \beta = 0$ on the Preisach plane, i.e., $\mu(-\beta, -\alpha) = \mu(\alpha, \beta)$. For the response process with zero mean, the symmetric weighting function means that possible $\hat{\gamma}_{\alpha\beta}(X)$ switching events appear in couples, and the two switching probabilities of each couple are almost equal. Thus, by using the relation $q_\alpha + q_\beta = 1$ the switching probabilities are obtained as $q_\alpha \cong q_\beta \cong 1/2$. The corresponding correlation functions $E[\hat{\gamma}_{\alpha\beta}(X)X]$ in equation (11) and $E[\hat{\gamma}_{\alpha\beta}(X)\dot{X}]$ in equation (12) become

$$\begin{aligned}E[\hat{\gamma}_{\alpha\beta}(X)X] &= \tfrac{1}{2}(E[|X|] - E[|X||_{X\in(0,\alpha)}] - E[|X||_{X\in(0,\beta)}]) \\ &= \sqrt{\frac{E[X^2]}{8\pi}}(\mathrm{e}^{-\alpha^2/2E[X^2]} + \mathrm{e}^{-\beta^2/2E[X^2]}),\end{aligned} \tag{23}$$

$$\begin{aligned}E[\hat{\gamma}_{\alpha\beta}(X)\dot{X}] &= \tfrac{1}{4}E[|\dot{X}||_{X\in(\alpha,\beta)}] \\ &= \sqrt{\frac{E[\dot{X}^2]}{32\pi}}\left[erf\left(\frac{\beta}{\sqrt{2E[X^2]}}\right) - erf\left(\frac{\alpha}{\sqrt{2E[X^2]}}\right)\right].\end{aligned} \tag{24}$$

For the case of unsymmetrical Preisach weighting function (in this case, non-linear restoring force $g(X)$ can also be non-antisymmetrical), the mean displacement response does not vanish even if the stationary excitation has zero mean. In this case one more equation for mean displacement response has to be added except covariance equation (4) or (7) and the mean and mean square responses are obtained by solving these two equations simultaneously. The procedure developed from equations (9) to (22) for evaluating the correlation function of hysteretic force and response holds in this case provided X is replaced by $X - E[X]$. In principle, it is possible to calculate switching probabilities q_α and q_β, especially for the case of stationary response. However, the calculation is complicated and a simplified approach needs to be developed.

5. EXAMPLE

Consider a Preisach hysteretic system with linear restoring force subjected to Gaussian white noise excitation. Suppose that the Preisach weighting function is symmetric with respect to the line $\alpha + \beta = 0$, i.e.,

$$\mu(\alpha, \beta) = \begin{cases} \dfrac{1}{2\pi\sigma^2} \exp\left\{ -\dfrac{(\alpha + v)^2 + (\beta - v)^2}{2\sigma^2} \right\} & \alpha_0 \leqslant \alpha \leqslant 0, 0 \leqslant \beta \leqslant \beta_0, \\ 0 & \text{elsewhere}, \end{cases} \tag{25}$$

where σ and v are the model parameters governing the width and area of the hysteresis loop; $\alpha_0 = -\beta_0$. By substituting equations (25), (23) and (24) into (9) and (10), the following cross correlation functions $E[ZX)$ and $E(Z\dot{X}]$ of hysteretic force and response are obtained:

$$E[ZX] = \frac{E[X^2]}{\sqrt{32\pi(\sigma^2 + E[X^2])}} e^{-v^2/2(\sigma^2 + E[X^2])} \left\{ \left[erf\left(\frac{v}{\sqrt{2}\sigma}\right) + erf\left(\frac{\beta_0 - v}{\sqrt{2}\sigma}\right) \right] \right.$$
$$\times \left[erf\left(\frac{v}{\sqrt{2}\sigma} \sqrt{\frac{E[X^2]}{\sigma^2 + E[X^2]}} \right) - erf\left(\frac{\alpha_0}{\sqrt{2}\sigma} \sqrt{\frac{\sigma^2 + E[X^2]}{E[X^2]}} \right. \right.$$
$$\left. \left. + \frac{v}{\sqrt{2}\sigma} \sqrt{\frac{E[X^2]}{\sigma^2 + E[X^2]}} \right) \right] + \left[erf\left(\frac{v}{\sqrt{2}\sigma}\right) - erf\left(\frac{\alpha_0 + v}{\sqrt{2}\sigma}\right) \right]$$
$$\times \left[erf\left(\frac{v}{\sqrt{2}\sigma} \sqrt{\frac{E[X^2]}{\sigma^2 + E[X^2]}} \right) + erf\left(\frac{\beta_0}{\sqrt{2}\sigma} \sqrt{\frac{\sigma^2 + E[X^2]}{E[X^2]}} \right. \right.$$
$$\left. \left. \left. - \frac{v}{\sqrt{2}\sigma} \sqrt{\frac{E[X^2]}{\sigma^2 + E[X^2]}} \right) \right] \right\}, \tag{26}$$

$$E[Z\dot{X}] = \sqrt{\frac{E[\dot{X}^2]}{32\pi}} \left\{ \left[erf\left(\frac{v}{\sqrt{2}\sigma}\right) - erf\left(\frac{\alpha_0 + v}{\sqrt{2}\sigma}\right) \right] \right.$$
$$\times \left[erg\left(\frac{\beta_0 - v}{\sqrt{2}\sigma}, v(u) = \frac{\sqrt{2}\sigma u + v}{\sqrt{2E[X^2]}} \right) - erg\left(\frac{-v}{\sqrt{2}\sigma}, v(u) = \frac{\sqrt{2}\sigma u + v}{\sqrt{2E[X^2]}} \right) \right]$$
$$- \left[erf\left(\frac{\beta_0 - v}{\sqrt{2}\sigma}\right) + erf\left(\frac{v}{\sqrt{2}\sigma}\right) \right] \cdot \left[erg\left(\frac{v}{\sqrt{2}\sigma}, v(u) = \frac{\sqrt{2}\sigma u - v}{\sqrt{2E[X^2]}} \right) \right.$$
$$\left. \left. - erg\left(\frac{\alpha_0 + v}{\sqrt{2}\sigma}, v(u) = \frac{\sqrt{2}\sigma u - v}{\sqrt{2E[X^2]}} \right) \right] \right\}, \tag{27}$$

$$erg[x, y(u)] = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} erf[y(u)] \, du. \tag{28}$$

Numerical calculation is performed for the following parameter values: $\alpha_0 = -\beta_0 = -4{\cdot}0$, $\sigma = 0{\cdot}1$, $v = 1{\cdot}0$, $k = 1{\cdot}0$, $\zeta = 0{\cdot}1$ and $D = 2{\cdot}0$. The cross correlation function of hysteretic force and displacement is given in Tables 1 and 2 for various values of parameters σ and v respectively. It is seen that the correlation function values obtained by using the proposed method and from digital simulation are in good agreement. Thus, the inference of equal switching probabilities q_α and q_β in the case of symmetric Preisach

TABLE 1

Cross correlation function for hysteresis parameter $v = 1{\cdot}0$ ($E[ZX]$—analytical value; $E[ZX]_s$—from digital simulation)

σ	$E[ZX]$	$E[ZX]_s$	Error (%)
0·20	0·48376	0·48279	0·20
0·15	0·48388	0·48742	0·73
0·10	0·48393	0·49137	1·51
0·05	0·48394	0·49374	1·98
0·01	0·48394	0·49512	2·26

TABLE 2

Cross correlation function for hysteresis parameter $\sigma = 0{\cdot}1$ ($E[ZX]$—analytical value; $E[ZX]_s$—from digital simulation)

v	$E[ZX]$	$E[ZX]_s$	Error (%)
0·5	0·70150	0·71298	1·61
0·6	0·66433	0·67453	1·51
0·7	0·62292	0·63101	1·28
0·8	0·57834	0·58320	0·83
0·9	0·53166	0·53586	0·78
1·0	0·48393	0·49137	1·51
1·1	0·43615	0·44762	2·56
1·2	0·38921	0·40326	3·48
1·3	0·34390	0·35809	3·96
1·4	0·30087	0·31345	4·01

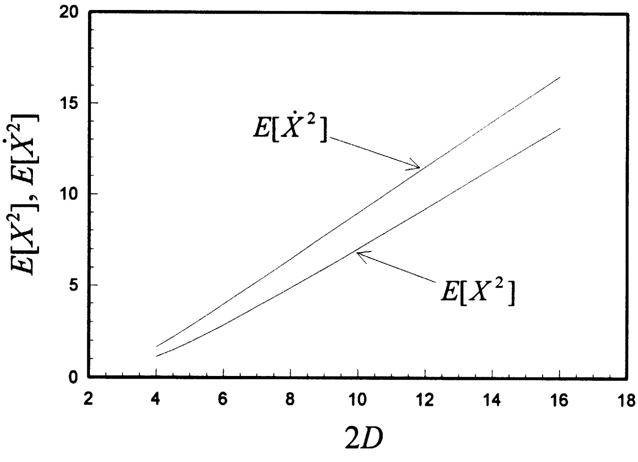

Figure 3. Mean square responses of displacement and velocity versus excitation intensity.

weighting function is reasonable to a significant extent. The mean square responses $E[X^2]$ and $E[\dot{X}^2]$ of the Preisach hysteretic system as functions of excitation intensity obtained by solving equation (7) are shown in Figure 3. Since the accuracy of the solution to equation (7) depends mainly on that of cross correlation function of hysteretic force and response, rather

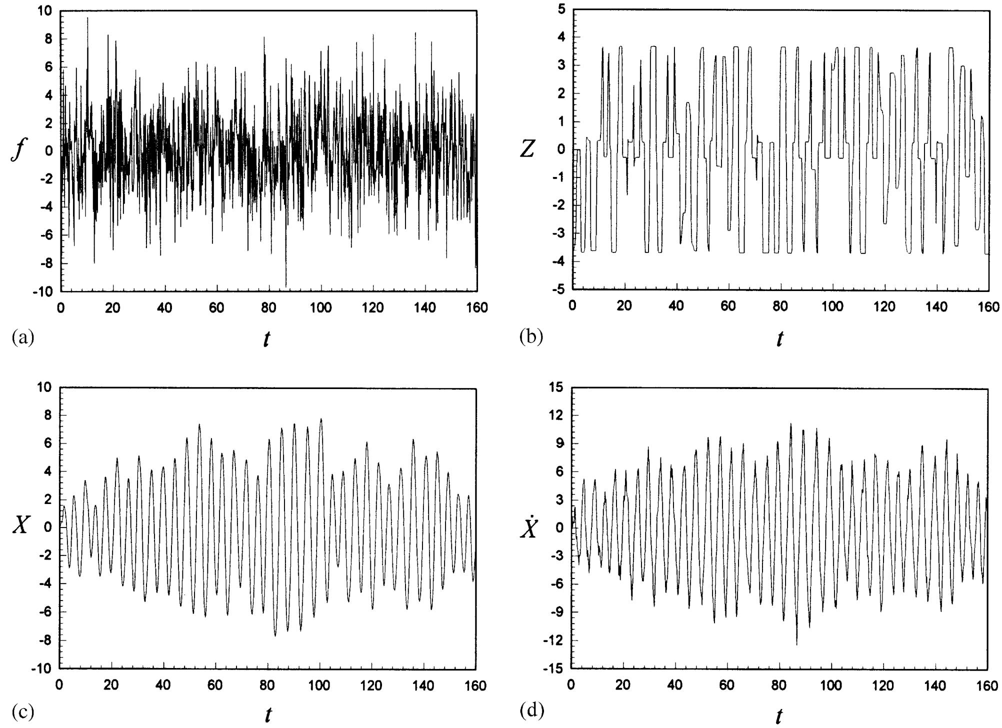

Figure 4. Representative sample functions. (a) Gaussian white noise excitation, (b) hysteretic force, (c) displacement response, (d) velocity response.

accurate mean square responses can be expected based on the results in Tables 1 and 2. A set of representative sample functions of the Gaussian white noise excitation and response are shown in Figure 4. It is seen that the displacement and velocity responses look like narrow band processes while the hysteretic force appears to be a rather complicated process due to its non-local memory.

It is seen from Tables 1 and 2 that the error in the calculation of $E[ZX]$ depends on the values of σ and v. The reasons for these dependences are as follows. It is known from equation (25) that the Preisach weighting function $\mu(\alpha, \beta)$ is a product of two independent Gaussian probability densities for α and β. σ is the variance of both α and β while $-v$ and $+v$ are the means of α and β respectively. In the calculation of $E[ZX]$ in Tables 1 and 2, the same discrete grid was used. For larger σ, this grid is finer and so the error is smaller. For smaller σ, on the contrary, this grid is grosser and so the error is larger. Similarly, the larger error for smaller and larger v is also due to the grosser of the grid in these cases. In fact, there is an optimal value of v for which the error is the smallest. It is $v = 0{\cdot}9$ for Table 2. Obviously, the result of $E[ZX]$ can be improved by using a finer grid.

6. CONCLUSIONS

The Preisach hysteresis model used to model the complicated constitutive relationship with non-local memory can be interpreted in terms of the spectral decomposition of the hysteretic constitutive law into simple hysteresis operators. An approximate method for analyzing the response of Preisach hysteretic systems to stationary Gaussian excitation has been developed based on the switching probability analysis and the Gaussian approximation of the response. The cross correlation function of Preisach hysteretic force and response has been evaluated by using mean square response and switching probabilities of relay hysteresis operators, and in particular, is simplified for the case of symmetric Preisach weighting function. The numerical result for the correlation function obtained by using the proposed method is in good agreement with that from digital simulation and thus rather accurate mean square responses can be expected.

ACKNOWLEDGMENTS

This study is partially supported by the National Natural Science Foundation of China under grant No. 19972059, the Zhejiiang Provincial Natural Science Foundation under grant No. 101046 and Cao Guang Biao Hi-Science-Technology Foundation of Zhejiang University for the first two authors. The financial support from the Hong Kong Polytechnic University through the Area of Strategic Development (ASD) Programme is also gratefully acknowledged.

REFERENCES

1. M. A. KRASNOSELSKII and A. V. POKROVSKII 1989 *Systems with Hysteresis*. Berlin: Springer.
2. I. D. MAYERGOYZ 1991 *Mathematical Models of Hysteresis*. New York: Springer-Verlag.
3. A. VISINTIN 1994 *Differential Models of Hysteresis*. Berlin: Springer-Verlag.
4. J. W. MACKI, P. NISTRI and P. ZECCA 1993 *SIAM Reviews* **35**, 94–123. Mathematical models for hysteresis.
5. G. Q. CAI and Y. K. LIN 1990 *Journal of Applied Mechanics, American Society of Mechanical Engineers* **57**, 442–448. On randomly excited hysteretic structures.
6. B. F. SPENCER, S. J. DYKE, M. K. SAIN and J. D. CARLSON 1997 *Journal of Engineering Mechanics, American Society of Civil Engineers* **123**, 230–238. Phenomenological model for magnetorheological dampers.
7. D. HUGHES and J. T. WEN 1997 *Smart Materials and Structures* **6**, 287–300. Preisach modeling of piezoceramic and shape memory alloy hysteresis.
8. M. D. SYMANS and M. C. CONSTANTINOU 1999 *Engineering Structures* **21**, 469–487. Semi-active control systems for seismic protection of structures: a state-of-the art review.
9. R. BOUC 1967 *Proceedings of the Fourth Conference on Non-Linear Oscillation Prague, Czechoslovakia*, 315–315. Forced vibration of mechanical systems with hysteresis.
10. Y. K. WEN 1976 *Journal of Engineering Mechanics Division, American Society of Civil Engineers* **102**, 249–263. Method for random vibration of hysteretic systems.
11. P. K. DAHL 1976 *American Institute of Aeronautics and Astronautics Journal* **14**, 1675–1682. Solid friction damping of mechanical vibrations.
12. M. YAR and J. K. HAMMOND 1987 *Journal of Engineering Mechanics, American Society of Civil Engineers* **113**, 1000–1013. Modeling and response of bilinear hysteretic systems.
13. T. K. CAUGHEY 1963 *Journal of the Acoustical Society of America* **35**, 1706–1711. Equivalent linearization techniques.
14. T. S. ATALIK and S. UTKU 1976 *Earthquake Engineering and Structural Dynamics* **4**, 411–420. Stochastic linearization of multi-degree-of-freedom non-linear systems.
15. J. B. ROBERTS and P. D. SPANOS 1990 *Random Vibration and Statistical Linearization*. Chichester: Wiley.
16. L. SOCHA and T. T. SOONG 1991 *Applied Mechanics Reviews, American Society of Mechanical Engineers* **44**, 399–422. Linearization in analysis of nonlinear stochastic systems.
17. I. ELISHAKOFF and G. FALSON 1993 *Computational Stochastic Mechanics* 175–194. Some recent developments in stochastic linearization technique.
18. R. L. STRATONOVITCH 1963 *Topics in the Theory of Random Noise*. Vol. 1. New York: Gordon and Bresch.
19. R. Z. KHASMINSKII 1966 *Theory of Probability and Applications* **11**, 390–405. A limit theorem for the solutions of differential equations with random right-hand sides.

20. Y. K. LIN 1986 *Probabilistic Engineering Mechanics* **1**, 23–27. Some observations on the stochastic averaging method.
21. J. B. ROBERTS and P. D. SPANOS 1986 *International Journal of Non-Linear Mechanics* **21**, 111–134. Stochastic averaging: an approximate method of solving random vibration problems.
22. W. Q. ZHU 1988 *Applied Mechanics Reviews, American Society of Mechanical Engineers* **41**, 189–199. Stochastic averaging methods in random vibration.
23. W. Q. ZHU 1996 *Applied Mechanics Reviews, American Society of Mechanical Engineers* **49**, 572–580. Recent developments and applications of the stochastic averaging method in random vibration.
24. G. Q. CAI 1995 *Journal of Engineering Mechanics, American Society of Civil Engineers* **121**, 633–639. Random vibration of nonlinear system under nonwhite excitation.
25. T. K. CAUGHEY 1960 *Journal of Applied Mechanics, American Society of Mechanical Engineers* **27**, 649–652. Random vibration of system with bilinear hysteresis.
26. J. B. ROBERTS 1988 in *Nonlinear Stochastic Dynamic Engineering Systems* (F. Ziegler and G. I. Scheuller, editors), 361–379. Berlin: Springer-Verlag. Application of averaging methods to randomly excited hysteretic systems.
27. W. Q. ZHU and Y. LEI 1988 in *Nonlinear Stochastic Dynamic Engineering Systems* (F. Ziegler and G. I. Scheuller, editors), 381–391. Berlin: Springer-Verlag. Stochastic averaging of energy envelope of bilinear hysteretic system.
28. Y. K. WEN 1980 *Journal of Applied Mechanics, American Society of Mechanical Engineers* **47**, 150–154. Equivalent linearization for hysteretic systems under random excitation.
29. W. Q. ZHU and Y. K. LIN 1991 *Journal of Engineering Mechanics, American Society of Civil Engineers* **117**, 1890–1905. Stochastic averaging of energy envelope.
30. M. NOORI, M. DIMENTBERG, Z. HOU, R. CHRISTODOULIDOU and A. ALEXANDROU 1995 *Probabilistic Engineering Mechanics* **10**, 161–170. First-passage study and stationary response analysis of a BWB hysteresis model using quasi-conservative stochastic averaging method.
31. I. D. MAYERGOYZ and C. E. KORMAN 1994 *Journal of Applied Physics* **75**, 5478–5480. The Preisach model with stochastic input as a model for aftereffect.
32. C. E. KORMAN and I. D. MAYERGOYZ 1995 *IEEE Transactions on Magnetics* **31**, 3545–3547. Switching as an exit problem.