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STOCHASTIC AVERAGING OF DUHEM HYSTERETIC SYSTEMS

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The response of Duhem hysteretic system to externally and/or parametrically non-white random excitations is investigated by using the stochastic averaging method. A class of integrable Duhem hysteresis models covering many existing hysteresis models is identified and the potential energy and dissipated energy of Duhem hysteretic component are determined. The Duhem hysteretic system under random excitations is replaced equivalently by a non-hysteretic non-linear random system. The averaged Itô's stochastic differential equation for the total energy is derived and the Fokker–Planck–Kolmogorov equation associated with the averaged Itô's equation is solved to yield stationary probability density of total energy, from which the statistics of system response can be evaluated. It is observed that the numerical results by using the stochastic averaging method is in good agreement with that from digital simulation.

1. INTRODUCTION

Non-linear hysteretic behavior exists widely in mechanical and structural systems [1–4], where the restoring force depends not only on the instantaneous deformation but also on the past history of deformation [5]. In addition, there has been an increasing interest recently in using smart materials [6–8] such as piezoceramics, shape memory alloys, and electro-/magneto-rheological fluids, which exhibit significant hysteresis. Various analytical models including bilinear model and Bouc–Wen model [9–12] have been proposed for representing the hysteretic constitutive relationship. However, the hysteresis models often used in dynamic analysis are relatively simple and they cannot represent some complicated hysteresis behaviors. In order to capture more accurately the complicated constitutive behavior including soft-hardening features, etc., a Duhem differential model [1–4] has been developed recently. This model is much more flexible and versatile than those often used in dynamic analysis and thus it deserves further investigation and application in dynamic analysis.

In mechanical and structural engineering fields, dynamic loading such as wind or earthquake ground motion is usually random in nature. For strongly non-linear hysteretic systems, it is extremely difficult to analytically determine the exact random response. So several approximate solution techniques have been developed, including the equivalent linearization method [13–17] and the stochastic averaging method [18–23]. Besides, an approximate technique for solving the stationary Fokker–Planck–Kolmogorov (FPK) equation recently developed by Er [24] may also be applied. The random responses of bilinear and Bouc–Wen hysteretic systems have been studied by using these two methods [5, 25–30]. However, few works on random dynamics of the Duhem hysteretic system with local memory have been reported.

In the present paper, a class of antisymmetric integrable Duhem hysteresis models is identified and the potential energy and dissipated energy by integrable Duhem hysteretic component are determined. A Duhem hysteretic system under externally and/or parametrically non-white random excitations is replaced equivalently by a non-hysteretic non-linear random system. The averaged Itô's stochastic differential equation for total energy is derived by using the stochastic averaging method. The FPK equation associated with the averaged Itô equation is solved to yield stationary probability density of total energy and the statistics of system response. Finally, an example of antisymmetric Duhem hysteretic system under non-white random excitation is presented to illustrate the application of the stochastic averaging method.

It may be pertinent to explain why we chose the stochastic averaging method for studying the random response of Duhem hysteretic systems. It has been shown by Pradlwarter and Schuëller [31] that for hysteretic systems subjected to random excitation, the equivalent linearization method usually yields excellent mean square velocity, good mean square displacement but inaccurate probability densities of velocity and displacement and thus the reliability. The stochastic averaging method, however, generally yields quite good results for mean square displacement, mean square velocity as well as probability densities of displacement and velocity [5, 27, 28, 30, 31]. On the other hand, only one-dimensional averaged FPK equation has to be solved in using the stochastic averaging method for Duhem hysteretic systems subject to random excitation with Kanai–Tajimi spectral density and the stationary solution of the FPK equation can be obtained readily, while five-dimensional FPK equations have to be solved if the approximate method proposed by Er [24] is used since a first order differential equation is needed to describe the hysteresis and a second order linear filter to produce the random excitation. Furthermore, the former method can be used to predict both stationary and non-stationary response while the latter only the stationary response. Besides, there are general formulae in the former method for certain class of problems and it is routine to apply the method to a specific problem while experience is necessary for choosing the parameters of the approximate probability density in using the latter method for a specific problem.

2. INTEGRABLE DUHEM HYSTERESIS MODEL

The Duhem model for hysteresis is governed by the following first order differential equation:

$$\dot{z} = g[x, z, \operatorname{sgn}(\dot{x})]\dot{x} = g_1(x, z)\dot{x}_+ - g_2(x, z)\dot{x}_-$$

$$= \begin{cases} g_1(x, z)\dot{x}, & \dot{x} > 0, \\ g_2(x, z)\dot{x}, & \dot{x} < 0, \end{cases} \tag{1a}$$

$$\dot{x}_+ = (|\dot{x}| + \dot{x})/2, \qquad \dot{x}_- = (|\dot{x}| - \dot{x})/2, \tag{1b}$$

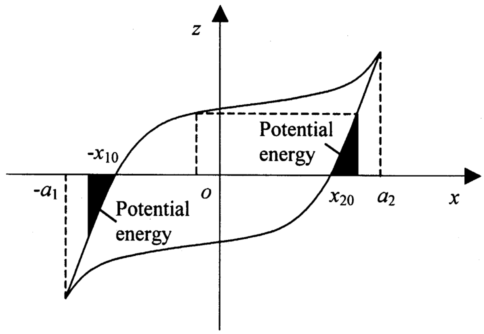

Figure 1. A representative of Duhem hysteresis loop.

where z denotes hysteretic force; x denotes displacement; g_1 and g_2 are two continuous functions of displacement and hysteretic force. According to Duhem model (1a) and (1b), the hysteretic force is determined by g_1 for $\dot{x} > 0$ and g_2 for $\dot{x} < 0$. The corresponding hysteresis loop in the (x, z) plane consists of two parts, i.e., ascending line $z_1(x)$ for $\dot{x} > 0$ and descending line $z_2(x)$ for $\dot{x} < 0$. Both ascending line and descending line are independent of the magnitude of velocity $\dot{x}$. The hysteretic force on ascending line or descending line depends on not only the instantaneous displacement but also the local displacement history since the last changes in velocity direction, but is independent of the displacement history before that change. Thus, the Duhem hysteresis model has the characteristics of local memory.

The potential energy $U(x)$ deposited in a hysteresis component can be expressed by the following integrals for $\dot{x} > 0$:

$$U(x) = \int_{-x_{10}}^{x} z_1(x_1)\,\mathrm{d}x_1, \quad -a_1 \leqslant x \leqslant -x_{10}, \tag{2a}$$

$$U(x) = \int_{x_{20}}^{z_2^{-1}[z_1(x)]} z_2(x_1)\,\mathrm{d}x_1, \quad -x_{10} \leqslant x \leqslant a_2, \tag{2b}$$

where a_1 and a_2 are negative and positive displacement amplitudes, respectively; x_{10} and x_{20} are residual hysteresis displacements, see Figure 1. The potential energy by hysteresis component for $\dot{x} < 0$ can be expressed similarly. The area of hysteresis loop, A_r, is equal to the energy dissipated in one cycle by the hysteresis component, i.e.,

$$A_r = \oint z(x)\,\mathrm{d}x = \int_{-a_1}^{a_2} z_1(x)\,\mathrm{d}x + \int_{a_2}^{-a_1} z_2(x)\,\mathrm{d}x. \tag{3}$$

Most hysteresis models have antisymmetric hysteresis loops. For antisymmetric Duhem hysteresis model, $g_2(x, z) = g_1(-x, -z)$, $z_1(x) = -z_2(-x)$, $a_1 = a_2 = a$, $x_{10} = x_{20} = x_0$. In this case, the hysteretic force can be separated into an elastic part and an inelastic part, i.e., $z_1 = z^e + z_1^p$, $z_2 = z^e + z_2^p$ (superscripts e and p denote elastic part and inelastic part respectively). Then the potential energy (2a), (2b) and the dissipated energy (3) are represented as follows:

$$U(x) = \int_0^x z^e(x_1)\,\mathrm{d}x_1 + \int_{-x_0}^{x} z_1^p(x_1)\,\mathrm{d}x_1, \quad -a \leqslant x \leqslant -x_0, \tag{4a}$$

$$U(x) = \int_0^x z^e(x_1)\,\mathrm{d}x_1 + \int_{x_0}^{(z_2^p)^{-1}[z_1^p(x)]} z_2^p(x_1)\,\mathrm{d}x_1, \quad -x_0 \leqslant x \leqslant a, \tag{4b}$$

$$A_r = \oint z^p(x)\,\mathrm{d}x = 2\int_{-a}^{a} z_1(x)\,\mathrm{d}x. \tag{5}$$

In the general Duhem hysteresis model, there exists a class of hysteresis functions g_1 and g_2, for example, the ascending function of the form

$$g_1(x, z) = \mathrm{d}z_0(x)/\mathrm{d}x + h_z(z - z_0)h_0(x), \tag{6}$$

where $z_0(x)$, $h_0(x)$ and $h_z(z - z_0)$ are arbitrary continuous and differentiable functions such that the Duhem hysteresis equations (1a) and (1b) are analytically integrable. Integrating equation (1) with equation (6) yields the ascending line

$$G_{1z}(z - z_0) = G_{1x}(x), \qquad G_{1z}(z - z_0) = \int_0^{z - z_0} \frac{\mathrm{d}u}{h_z(u)}, \qquad G_{1x}(x) = \int_{-x_0}^{x} h_0(u)\,\mathrm{d}u \tag{7}$$

and the hysteretic force for $\dot{x} > 0$ is expressed as

$$z(x) = z_0(x) + G_{1z}^{-1}[G_{1x}(x)], \quad \dot{x} > 0. \tag{8a}$$

The expression of hysteretic force for $\dot{x} < 0$ is then obtained by using the antisymmetric relation as follows:

$$z(x) = -z_0(-x) - G_{1z}^{-1}[G_{1x}(-x)], \quad \dot{x} < 0. \tag{8b}$$

The class of hysteresis expressed by equations such as equations (8a) and (8b) is called the integrable Duhem hysteresis. The integrable Duhem hysteresis model is quite general. It includes many existing hysteresis models, such as the Bouc–Wen model [9, 10] and the Yar–Hammond bilinear model [12], etc., as special cases. Thus, the class of integrable Duhem hysteresis models is quite versatile.

3. STOCHASTIC AVERAGING OF DUHEM HYSTERETIC SYSTEMS UNDER NON-WHITE RANDOM EXCITATIONS

Consider a Duhem hysteretic system subjected to random excitations. The equation of motion is of the form

$$\ddot{X} + 2\zeta\dot{X} + Z(X, \dot{X}) = f_j(X, \dot{X})\xi_j(t), \quad j = 1, 2, \ldots, m, \tag{9}$$

where X denotes displacement; ζ is viscous damping coefficient; Z denotes Duhem hysteretic force governed by equations (1a) and (1b); $f_j(X, \dot{X})$ represent the amplitudes of external and/or parametric random excitations, which are continuous and differentiable functions of displacement and velocity; $\xi_j(t)$ are wide-band stationary random excitations with zero mean and correlation function $R_{jk}(\tau) = E[\xi_j(t)\xi_k(t + \tau)]$; m is the number of random excitations.

The Duhem hysteretic system (9) with equations (1a) and (1b) under random excitations can be replaced by the following non-hysteretic non-linear random system:

$$\ddot{X} + [2\zeta + 2\zeta_1(H)]\dot{X} + \partial U(X)/\partial X = f_j(X, \dot{X})\xi_j(t), \tag{10}$$

where

$$H = \dot{x}^2/2 + U(x) \tag{11}$$

is the total energy of the system; U is the potential energy deposited in the hysteresis component, which can be represented by equations (2a) and (2b), or (4a) and (4b); $2\zeta_1$ is the equivalent quasi-linear damping coefficient, which can be evaluated by using the following formula:

$$2\zeta_1(H) = \frac{A_r}{2\int_{-a_1}^{a_2}\sqrt{2H - 2U(x)}\,\mathrm{d}x}. \tag{12}$$

To apply the stochastic averaging method [23] to the equivalent non-linear system (10), let

$$\mathrm{sgn}(X)\sqrt{U(X)} = \sqrt{H}\cos\varphi, \quad \dot{X} = -\sqrt{2H}\sin\varphi \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{13}$$

Equation (10) is transformed into the following first order differential equations for the total energy and phase:

$$\dot{H} = -2H\sin^2\varphi\,[2\zeta + 2\zeta_1(H)] - \sqrt{2H}\sin\varphi f_j(H, \varphi)\,\xi_j(t), \tag{14a}$$

$$\dot{\varphi} = \frac{1}{\sqrt{2H}}\left[-\sqrt{2H}\sin\varphi\cos\varphi\,(2\zeta + 2\zeta_1(H)) + \frac{\partial U(X)/\partial X}{\cos\varphi}\right] - \frac{\cos\varphi}{\sqrt{2H}} f_j(H, \varphi)\xi_j(t). \tag{14b}$$

Under the condition that damping and excitation are weak, the total energy is slowly varying process and can be approximated as a Markov diffusion process. Performing time averaging of equation (14a) yields the following Itô equation:

$$\mathrm{d}H = m(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}B(t), \tag{15}$$

where $B(t)$ is a unit Wiener process; drift coefficient $m(H)$ and diffusion coefficient $\sigma^2(H)$ are expressed as

$$m(H) = \Big\langle -2H\sin^2\varphi[2\zeta + 2\zeta_1(H)] + \int_{-\infty}^{0}\Big[(\sqrt{2H}\sin\varphi f_j(H, \varphi))_{t+\tau}\frac{\partial}{\partial H}(\sqrt{2H}\sin\varphi f_k(H, \varphi))_t + \left(\frac{\cos\varphi}{\sqrt{2H}} f_j(H, \varphi)\right)_{t+\tau}\frac{\partial}{\partial\varphi}(\sqrt{2H}\sin\varphi f_k(H, \varphi))_t\Big] R_{jk}(\tau)\,\mathrm{d}\tau\Big\rangle_t, \tag{16a}$$

$$\sigma^2(H) = \left\langle\int_{-\infty}^{+\infty}(\sqrt{2H}\sin\varphi f_j(H, \varphi))_{t+\tau}(\sqrt{2H}\sin\varphi f_k(H, \varphi))_t\, R_{jk}(\tau)\,\mathrm{d}\tau\right\rangle_t, \quad j, k = 1, 2, \ldots, m \tag{16b}$$

in which $\langle\cdot\rangle_t$ represents time averaging. For certain constant H, responses X and $\dot{X}$ can be treated as periodic functions. Then the integrands in equations (16a) and (16b) can be expanded as the following Fourier series:

$$\sin\varphi f_j(H,\varphi) = \frac{a_{j0}^{(1)}}{2} + \sum_{i=1}^{\infty}\left(a_{ji}^{(1)}\cos\frac{2\pi it}{T} + a_{ji}^{(2)}\sin\frac{2\pi it}{T}\right), \tag{17a}$$

$$\cos\varphi f_j(H,\varphi) = \frac{b_{j0}^{(1)}}{2} + \sum_{i=1}^{\infty}\left(b_{ji}^{(1)}\cos\frac{2\pi it}{T} + b_{ji}^{(2)}\sin\frac{2\pi it}{T}\right), \tag{17b}$$

$$2H\sin\varphi\frac{\partial f_j(H,\varphi)}{\partial H} = \frac{c_{j0}^{(1)}}{2} + \sum_{i=1}^{\infty}\left(c_{ji}^{(1)}\cos\frac{2\pi it}{T} + c_{ji}^{(2)}\sin\frac{2\pi it}{T}\right), \tag{17c}$$

$$\sin\varphi\frac{\partial f_j(H,\varphi)}{\partial\varphi} = \frac{d_{j0}^{(1)}}{2} + \sum_{i=1}^{\infty}\left(d_{ji}^{(1)}\cos\frac{2\pi it}{T} + d_{ji}^{(2)}\sin\frac{2\pi it}{T}\right). \tag{17d}$$

By substituting equations (17a)–(17b) into equations (16a)–(16b) and by performing the time averaging and then converting the expressions into space averaging ones, the following averaged drift and diffusion coefficients are obtained:

$$\begin{aligned} m(H) = {} & -\frac{A_r}{T} - \frac{4\zeta}{T}\int_{-a}^{a}\sqrt{2H-2U(x)}\,\mathrm{d}x + \frac{\pi}{4}(a_{j0}^{(1)}a_{k0}^{(1)} + a_{j0}^{(1)}c_{k0}^{(1)} + b_{j0}^{(1)}b_{k0}^{(1)} \\ & + b_{j0}^{(1)}d_{k0}^{(1)})\Phi_{jk}^{(1)}(0) + \frac{\pi}{2}\sum_{i=1}^{\infty}[(a_{ji}^{(1)}a_{ki}^{(1)} + a_{ji}^{(2)}a_{ki}^{(2)} + a_{ji}^{(1)}c_{ki}^{(1)} + a_{ji}^{(2)}c_{ki}^{(2)} \\ & + b_{ji}^{(1)}b_{ki}^{(1)} + b_{ji}^{(2)}b_{ki}^{(2)} + b_{ji}^{(1)}d_{ki}^{(1)} + b_{ji}^{(2)}d_{ki}^{(2)})\Phi_{jk}^{(1)}(2\pi\mathrm{i}/T) \\ & + (a_{ji}^{(2)}c_{ki}^{(1)} - a_{ji}^{(1)}c_{ki}^{(2)} + b_{ji}^{(2)}d_{ki}^{(1)} - b_{ji}^{(1)}d_{ki}^{(2)})\Phi_{jk}^{(2)}(2\pi\mathrm{i}/T)], \end{aligned} \tag{18a}$$

$$\sigma^2(H) = \pi H a_{j0}^{(1)}a_{k0}^{(1)}\Phi_{jk}^{(1)}(0) + 2\pi H\sum_{i=1}^{\infty}[(a_{ji}^{(1)}a_{ki}^{(1)} + a_{ji}^{(2)}a_{ki}^{(2)})\Phi_{jk}^{(1)}(2\pi\mathrm{i}/T)], \tag{18b}$$

where the Fourier coefficients are represented in terms of the space integrals as follows:

$$a_{ji}^{(1)} = \frac{2}{T}\int_0^T \sin\varphi f_j(H,\varphi)\cos\frac{2\pi\mathrm{i}t}{T}\,\mathrm{d}t = -\frac{\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}[f_j(x,\dot{x}) - f_j(x,-\dot{x})]\cos\frac{2\pi\mathrm{i}t(x)}{T}\,\mathrm{d}x, \tag{19a}$$

$$a_{ji}^{(2)} = \frac{2}{T}\int_0^T \sin\varphi f_j(H,\varphi)\sin\frac{2\pi\mathrm{i}t}{T}\,\mathrm{d}t = -\frac{\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}[f_j(x,\dot{x}) + f_j(x,-\dot{x})]\sin\frac{2\pi\mathrm{i}t(x)}{T}\,\mathrm{d}x, \tag{19b}$$

$$b_{ji}^{(1)} = \frac{2}{T}\int_0^T \cos\varphi f_j(H,\varphi)\cos\frac{2\pi it}{T}\,dt$$

$$= \frac{\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a} \text{sgn}(x)\sqrt{U(x)}\,[f_j(x,\dot{x}) + f_j(x,-\dot{x})]\cos\frac{2\pi it(x)}{T}\frac{dx}{\dot{x}}, \tag{19c}$$

$$b_{ji}^{(2)} = \frac{2}{T}\int_0^T \cos\varphi f_j(H,\varphi)\sin\frac{2\pi it}{T}\,dt$$

$$= \frac{2}{T\sqrt{H}}\int_{-a}^{a} \text{sgn}(x)\sqrt{U(x)}\,[f_j(x,\dot{x}) - f_j(x,-\dot{x})]\sin\frac{2\pi it(x)}{T}\frac{dx}{\dot{x}}, \tag{19d}$$

$$c_{ji}^{(1)} = \frac{2}{T}\int_0^T 2H\sin\varphi\frac{\partial f_j(H,\varphi)}{\partial H}\cos\frac{2\pi it}{T}\,dt = -\frac{2\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}\left[\frac{\partial(f_j(x,\dot{x}) - f_j(x,-\dot{x}))}{\partial x}\right.$$

$$\left.\times\frac{U(x)}{\partial U(x)/\partial x} + \frac{\partial(f_j(x,\ \dot{x}) + f_j(x,-\dot{x}))}{\partial \dot{x}}\frac{\dot{x}^2}{2}\right]\cos\frac{2\pi it(x)}{T}\,dx, \tag{19e}$$

$$c_{ji}^{(2)} = \frac{2}{T}\int_0^T 2H\sin\varphi\frac{\partial f_j(H,\varphi)}{\partial H}\sin\frac{2\pi it}{T}\,dt = -\frac{2\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}\left[\frac{\partial(f_j(x,\dot{x}) + f_j(x,-\dot{x}))}{\partial x}\right.$$

$$\left.\times\frac{U(x)}{\partial U(x)/\partial x} + \frac{\partial(f_j(x,\ \dot{x}) - f_j(x,-\dot{x}))}{\partial \dot{x}}\frac{\dot{x}^2}{2}\right]\sin\frac{2\pi it(x)}{T}\,dx, \tag{19f}$$

$$d_{ji}^{(1)} = \frac{2}{T}\int_0^T \sin\varphi\frac{\partial f_j(H,\varphi)}{\partial\varphi}\cos\frac{2\pi it}{T}\,dt = -\frac{2}{T\sqrt{H}}\int_{-a}^{a}\text{sgn}(x)\sqrt{U(x)}\left[\frac{\partial(f_j(x,\dot{x}) + f_j(x,-\dot{x}))}{\partial x}\right.$$

$$\left.\times\frac{\dot{x}}{\partial U(x)/\partial x} - \frac{\partial(f_j(x,\ \dot{x}) + f_j(x,-\dot{x}))}{\partial \dot{x}}\right]\cos\frac{2\pi it(x)}{T}\,dx, \tag{19g}$$

$$d_{ji}^{(2)} = \frac{2}{T}\int_0^T \sin\varphi\frac{\partial f_j(H,\varphi)}{\partial\varphi}\sin\frac{2\pi it}{T}\,dt = -\frac{2}{T\sqrt{H}}\int_{-a}^{a}\text{sgn}(x)\sqrt{U(x)}\left[\frac{\partial(f_j(x,\dot{x}) - f_j(x,-\dot{x}))}{\partial x}\right.$$

$$\left.\times\frac{\dot{x}}{\partial U(x)/\partial x} - \frac{\partial(f_j(x,\ \dot{x}) - f_j(x,-\dot{x}))}{\partial \dot{x}}\right]\sin\frac{2\pi it(x)}{T}\,dx, \quad i = 0,1,2,\ldots, \quad j = 1,2,\ldots,m \tag{19h}$$

and

$$\dot{x} = \sqrt{2H - 2U(x)}, \quad t(x) = \int_{-a}^{x}\frac{dx_1}{\sqrt{2H - 2U(x_1)}}, \quad T = 2\int_{-a}^{a}\frac{dx}{\sqrt{2H - 2U(x)}}, \tag{20}$$

$$\Phi_{jk}^{(1)}(\omega) = \frac{1}{\pi}\int_{-\infty}^{0} R_{jk}(\tau)\cos\omega\tau\,d\tau, \quad \Phi_{jk}^{(2)}(\omega) = \frac{1}{\pi}\int_{-\infty}^{0} R_{jk}(\tau)\sin\omega\tau\,d\tau. \tag{21}$$

The FPK equation associated with the averaged Itô equation (15) for the total energy is

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial H}[m(H)p] - \frac{1}{2}\frac{\partial^2}{\partial H^2}[\sigma^2(H)p] = 0, \tag{22}$$

where probability density $p = p(H,\ t)$ [or $p = p(H,\ t|H_0, t_0)$] for the initial condition $p(H_0, t_0)$ [or $p = \delta(H - H_0)$]. The stationary probability density of total energy can be obtained from solving FPK equation (22) with $\partial p/\partial t = 0$ as follows:

$$p(H) = \frac{C}{\sigma^2(H)} \exp\left\{\int_0^H \frac{2m(y)}{\sigma^2(y)}\,\mathrm{d}y\right\}, \tag{23}$$

where C is a normalization constant. Then the response statistics of system (9) can be evaluated. For example, the mean square displacement is represented by

$$E[X^2] = \int_0^\infty \frac{p(H)}{T(H)}\,\mathrm{d}H \times 2\int_{-a}^{a} \frac{x^2\,\mathrm{d}x}{\sqrt{2H - 2U(x)}}, \tag{24}$$

where $E[\cdot]$ denotes expectation operator.

4. EXAMPLE AND NUMERICAL RESULTS

Consider a Duhem hysteretic system subjected to random excitation $(e_1 + e_2X)\xi(t)$. The non-white stationary excitation $\xi(t)$ has the following Kanai–Tajimi spectral density [32, 33]:

$$\Phi_1(\omega) = \frac{1 + 4\zeta_g^2(\omega/\omega_g)^2}{[1 - (\omega/\omega_g)^2]^2 + 4\zeta_g^2(\omega/\omega_g)^2} S_0, \tag{25}$$

where S_0, ω_g and ζ_g are the spectral parameters determined by the intensity and site characteristics of excitation. The ascending function of antisymmetric integrable Duhem hysteresis model with non-linear elasticity is taken as

$$g_1(x, z) = k_1 + 3k_3x^2 + \frac{\gamma}{\beta}[\alpha - \beta(z - k_1x - k_3x^3)], \tag{26}$$

where k_1 and k_3 are linear and non-linear stiffnesses, respectively; α, β and γ are hysteresis constants. g_1 in equation (26) is a special case of that in equation (6). So the hysteretic force z is obtained from equations (8a), (8b) and (26):

$$z_1 = k_1x + k_3x^3 + \frac{1}{\beta}[1 - \mathrm{e}^{-\gamma(x + x_0)}], \quad \dot{x} > 0, \tag{27a}$$

$$z_2 = k_1x + k_3x^3 - \frac{1}{\beta}[1 - \mathrm{e}^{\gamma(x + x_0)}], \quad \dot{x} < 0. \tag{27b}$$

The Duhem hysteresis model (27a) and (27b) can be used to describe soft-hardening hysteresis characteristics such as that in wire-cable vibration isolator [34]. The potential energy and dissipated energy of hysteresis component in one cycle are

$$U(x) = \frac{1}{2}k_1 x^2 + \frac{1}{4}k_3 x^4 + \frac{1}{\beta}(x + x_0) + \frac{1}{\beta\gamma}[\mathrm{e}^{-\gamma(x+x_0)} - 1], \quad -a \leqslant x \leqslant -x_0, \tag{28a}$$

$$U(x) = \frac{1}{2}k_1 x^2 + \frac{1}{4}k_3 x^4 + \frac{1}{\beta\gamma}[1 - \mathrm{e}^{-\gamma(x+x_0)}] - \frac{1}{\beta\gamma}\ln[2 - \mathrm{e}^{-\gamma(x+x_0)}], \quad -x_0 \leqslant x \leqslant a, \tag{28b}$$

$$A_r = \frac{4}{\beta\gamma}(1 + a\gamma) - \frac{4}{\beta\gamma}\,\mathrm{e}^{\gamma(a-x_0)}, \tag{29}$$

where the residual hysteresis displacement x_0 and displacement amplitude a for certain H are determined by

$$x_0 = -a + \frac{1}{\gamma}\ln\frac{1 + \mathrm{e}^{2a\gamma}}{2}, \tag{30a}$$

$$H = \frac{1}{2}k_1 a^2 + \frac{1}{4}k_3 a^4 - \frac{1}{\beta}(a - x_0) + \frac{1}{\beta\gamma}[\mathrm{e}^{\gamma(a-x_0)} - 1]. \tag{30b}$$

The averaged Itô's stochastic differential equation for total energy H is of the form of equation (15), where the drift and diffusion coefficients are

$$\begin{aligned} m(H) = &-\frac{A_r}{T} - \frac{4\zeta}{T}\int_{-a}^{a}\sqrt{2H - 2U(x)}\,\mathrm{d}x + \frac{\pi}{4}(b_0^{(1)}b_0^{(1)} + b_0^{(1)}d_0^{(1)})\,\Phi_1(0) \\ &+ \frac{\pi}{2}\sum_{i=1}^{\infty}(a_i^{(2)}a_i^{(2)} + a_i^{(2)}c_i^{(2)} + b_i^{(1)}b_i^{(1)} + b_i^{(1)}d_i^{(1)})\,\Phi_1\left(\frac{2\pi i}{T}\right), \end{aligned} \tag{31a}$$

$$\sigma^2(H) = 2\pi H\sum_{i=1}^{\infty}a_i^{(2)}a_i^{(2)}\Phi_1\left(\frac{2\pi i}{T}\right) \tag{31b}$$

with the Fourier coefficients

$$a_i^{(2)} = -\frac{2\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}f(x)\sin\left[\frac{2\pi i}{T}\int_{-a}^{x}\frac{\mathrm{d}x_1}{\sqrt{2H - 2U(x_1)}}\right]\mathrm{d}x, \tag{32a}$$

$$b_i^{(1)} = \frac{4}{T\sqrt{H}}\int_{-a}^{a}\frac{\mathrm{sgn}(x)\sqrt{U(x)}}{\sqrt{2H - 2U(x)}}f(x)\cos\left[\frac{2\pi i}{T}\int_{-a}^{x}\frac{\mathrm{d}x_1}{\sqrt{2H - 2U(x_1)}}\right]\mathrm{d}x, \tag{32b}$$

$$c_i^{(2)} = -\frac{4\sqrt{2}}{T\sqrt{H}}\int_{-a}^{a}\frac{U(x)}{U'(x)}f'(x)\sin\left[\frac{2\pi i}{T}\int_{-a}^{x}\frac{\mathrm{d}x_1}{\sqrt{2H - 2U(x_1)}}\right]\mathrm{d}x, \tag{32c}$$

$$d_i^{(1)} = -\frac{4}{T\sqrt{H}}\int_{-a}^{a}\frac{\mathrm{sgn}(x)\sqrt{U(x)}}{U'(x)}\sqrt{2H - 2U(x)}f'(x)\cos\left[\frac{2\pi i}{T}\int_{-a}^{x}\frac{\mathrm{d}x_1}{\sqrt{2H - 2U(x_1)}}\right]\mathrm{d}x. \tag{32d}$$

For an external random excitation $e_1\xi(t)$, coefficients $c_i^{(2)} = d_i^{(1)} = 0$. The stationary probability density $p(H)$ is of the form of equation (23) and the mean square displacement can be evaluated by using equation (24).

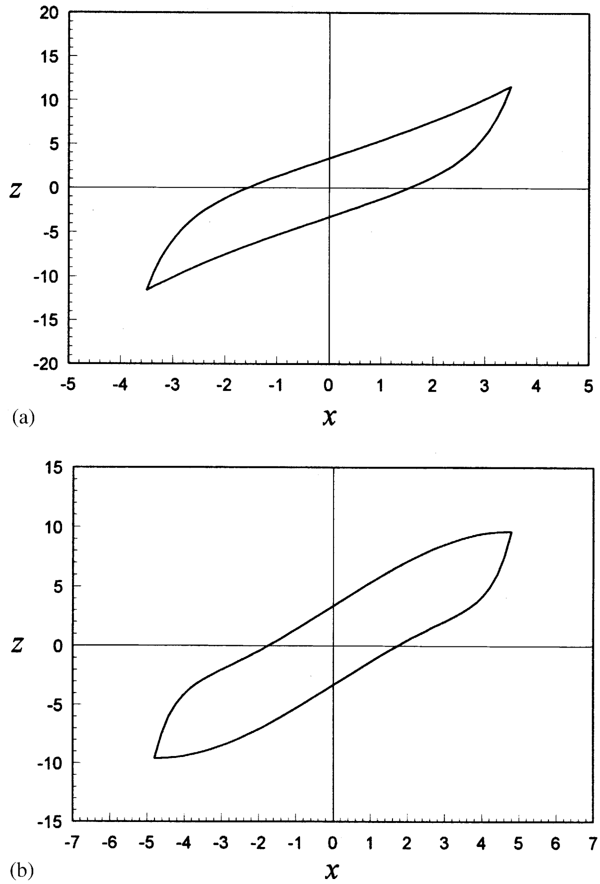

Figure 2. Two Duhem hysteresis loops in the numerical calculation (a) with hardening stiffness, (b) with softening stiffness.

Numerical computation has been made for the following parameter values: $\zeta = 0{\cdot}2$, $k_1 = 5{\cdot}0$ or $2{\cdot}0$, $k_3 = 0{\cdot}03$ or $-0{\cdot}03$, $\beta = 0{\cdot}1$ or $0{\cdot}3$, $\gamma = 3{\cdot}0$ or $2{\cdot}0$, $e_1 = 1$, $e_2 = 0$, $0{\cdot}12$ or $0{\cdot}03$, $S_0 = 1{\cdot}0$, $\omega_g = 3{\cdot}08$, $\zeta_g = 0{\cdot}1$. The hysteresis loops with hardening stiffness ($k_3 = 0{\cdot}03$) and softening stiffness ($k_3 = -0{\cdot}03$) in the numerical calculation are shown in Figure 2. The Fourier coefficients $a_i^{(2)}$ ($i = 1, 2, 3$) and $b_i^{(1)}$ ($i = 0, 1, 2, 3$) in the drift and diffusion coefficients as a function of total energy are shown in Figures 3 and 4 respectively. It is found that these coefficients for $i > 1$ are much smaller than those for $i \leqslant 1$. Thus, neglecting the terms for $i > 3$ in the expressions of drift and diffusion coefficients (31a) and (31b) may yield enough accurate results.

The mean square displacement response of the Duhem hysteretic system (9) to random excitation with spectral density (25) as a function of the excitation intensity is shown in Figures 5–7, where a solid line represents analytical results while a dotted line represents results from digital simulation. Good agreement of the two results is observed for Duhem systems with both hardening and softening stiffness and for both external excitation only and simultaneously external and parametric excitations. Theoretically, the stochastic averaging method is applicable only when the damping is slight and the random excitation is weak. Actually, it can be applied for large ranges of damping and random excitation provided the difference between the energy inputted to the system by random excitation and

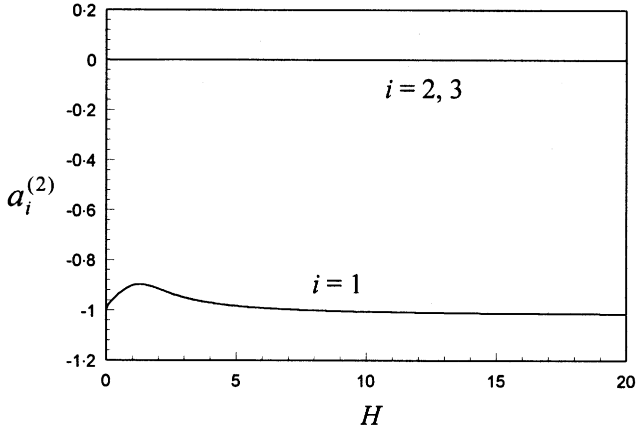

Figure 3. Fourier coefficients $a_i^{(2)}$ versus total energy H ($k_1 = 5{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}1$, $\gamma = 3{\cdot}0$, $e_2 = 0$).

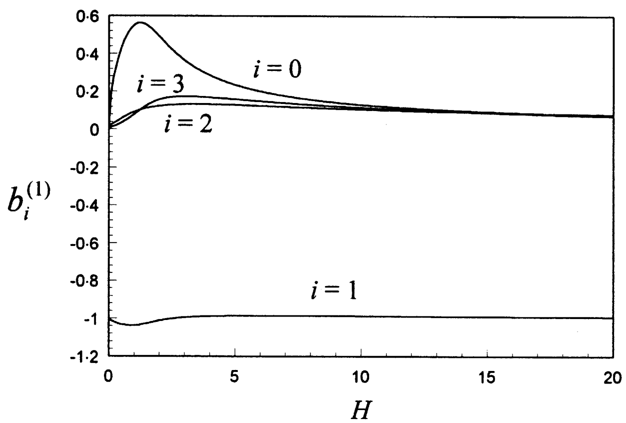

Figure 4. Fourier coefficients $b_i^{(1)}$ versus total energy H ($k_1 = 5{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}1$, $\gamma = 3{\cdot}0$, $e_2 = 0$).

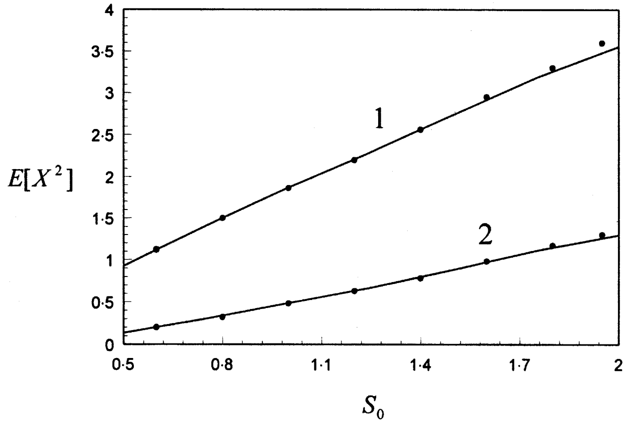

Figure 5. Mean square displacement response of Duhem hysteretic system (9) with hardening stiffness to external random excitation with spectral density (25) versus excitation intensity: —, analytical results; ●, result from digital simulation. Line 1: $k_1 = 2{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}3$, $\gamma = 2{\cdot}0$, $e_1 = 1$, $e_2 = 0$; line 2: $k_1 = 5{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}1$, $\gamma = 3{\cdot}0$, $e_1 = 1$, $e_2 = 0$.

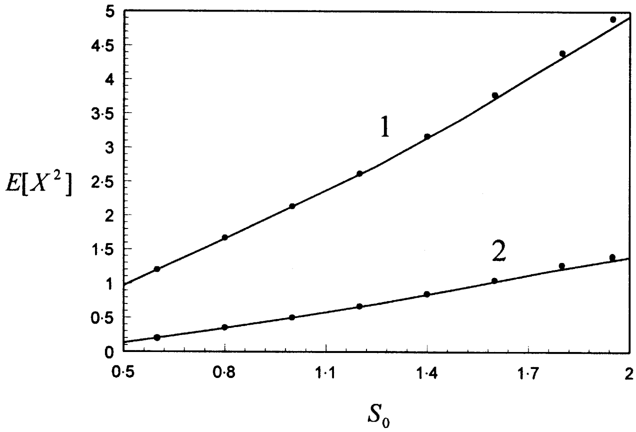

Figure 6. Mean square displacement response of Duhem hysteretic system (9) with softening stiffness to external random excitation with spectral density (25) versus excitation intensity: —, analytical result; ●, result from digital simulation. Line 1: $k_1 = 2{\cdot}0$, $k_3 = -0{\cdot}03$, $\beta = 0{\cdot}3$, $\gamma = 2{\cdot}0$, $e_1 = 1$, $e_2 = 0$; line 2: $k_1 = 5{\cdot}0$, $k_3 = -0{\cdot}03$, $\beta = 0{\cdot}1$, $\gamma = 3{\cdot}0$, $e_1 = 1$, $e_2 = 0$.

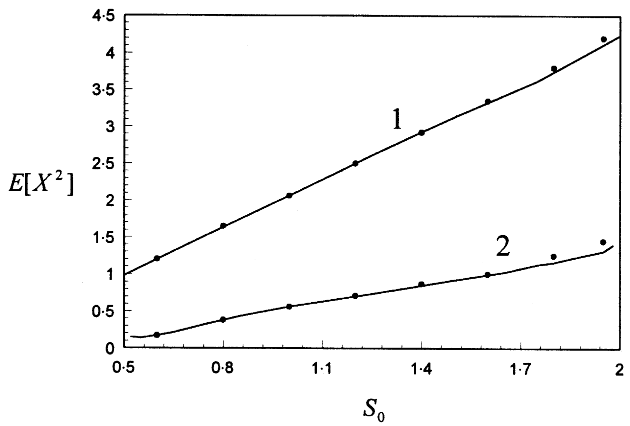

Figure 7. Mean square displacement response of Duhem hysteretic system (9) with hardening stiffness to both external and parametric random excitations with spectral density (25) versus excitation intensity: —, analytical results; ●, result from digital simulation. Line 1: $k_1 = 2{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}3$, $\gamma = 2{\cdot}0$, $e_1 = 1$, $e_2 = 0{\cdot}12$; line 2: $k_1 = 5{\cdot}0$, $k_3 = 0{\cdot}03$, $\beta = 0{\cdot}1$, $\gamma = 3{\cdot}0$, $e_1 = 1$, $e_2 = 0{\cdot}03$.

that outputted from the system by damping during a period of vibration is small compared with the system energy itself. For fixed damping, error increases slightly as the intensity of random excitation increases as can be seen from Figures 5–7 because the energy difference increases.

5. CONCLUSIONS

In the present paper, a class of integrable Duhem hysteresis models, covering many existing hysteresis models, has been proposed to describe general hysteretic constitutive relationship with local memory. The expressions for the potential energy and dissipated energy of Duhem hysteretic component have been obtained so that a Duhem hysteretic system under random excitations can be replaced equivalently by a non-hysteretic non-linear random system. The stochastic averaging method for Duhem hysteretic systems subjected to externally and/or parametrically non-white random excitation has been developed, where time averaging is finally converted into the space averaging for certain total energy. The mean square response obtained by using the proposed method has been compared with that from digital simulation. Taking first few terms of the Fourier series in averaged drift and diffusion coefficients yields quite accurate results.

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