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VIBRATION OF TWISTED AND CURVED CYLINDRICAL PANELS WITH VARIABLE THICKNESS

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A numerical procedure, with an exact strain–displacement relationship of twisted and curved cylindrical panels having variable thickness derived by considering the Green strain tensor on general shell theory, is presented using the principle of virtual work and the Rayleigh–Ritz method with algebraic polynomials as in-plane and transverse displacement functions. The accuracy and applicability of the procedure are verified by comparing the present results with previous experimental and theoretical results for several panels. The effects of variation ratio of thickness in chordwise and lengthwise directions, twist, and curvature both in two directions aforementioned on vibrations of cylindrical panels are studied in detail, and typical vibration mode shapes are plotted to demonstrate the effects.

1. INTRODUCTION

Shells have become very important structure components in many engineering applications such as wings, helicoidal fan blades and bodies of aircraft in aerospace, blades of turbo machines in machinery engineering, and spheroidal, paraboloidal and toroidal shells in civil engineering. It is impossible to obtain exactly theoretical solutions by any method because shells usually have a complicated profile and their status can only be expressed by differential or integral formulas approximately, and increasing applications and high performance in engineering are demanded, which are the reasons why many researchers are interested in the study on dynamics of shells.

There were many outstanding researches for the vibrations of shells introduced in references [1–3] crowded with the available literature. Leissa and their co-operators are representatives of researchers on this problem in the 1980s. As a physical modal type of rotating turbo machinery blades, twisted plates, shallow cylindrical shells, doubly curved shallow shells, and variable thickness and curvature shells were studied [4–10]. It is worth noting the work done by Liew, Lim and their coworkers since 1990s. By introducing *pb*-2 shape functions as displacement functions which accommodate various boundary conditions easily, twisted trapezoidal plates, cantilevered rectangular shallow shells with variable thickness, doubly curved shallow shells of curvilinear planform, doubly tapered cylindrical shallow shells, variable thickness ellipsoidal dish and twisted shallow conical shells with variable thickness were studied in detail by first order, higher order and other theories [11–21] where the effects of parameters, such as twist and variable thickness, and various boundary constraints on vibration characteristics are studied, and other studies on this topic can be seen from a review [3]. It is known that most of the shells aforementioned

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were assumed to be shallow shells with a rectangular planform, therefore, the numerical analysis procedures were inadequate for vibration analysis of deep shells, and there were a few researches related to twisted deep shells with variable thickness. Establishing an exact strain–displacement relationship for shells based on general shell theory and formulating governing equations of vibration by the principle of virtual work and the Rayleigh–Ritz method, curved and twisted plates, twisted cylindrical panels, and curved and twisted cylindrical panels were studied, some specimens were tested and the comparison between theoretical and experimental results was also carried out [22–25]. It is noted that the numerical analysis procedure is based on general shell theory, therefore, there is no limitation of application.

For the purpose of forwarding the work [25], in this paper, an exact strain–displacement relationship for twisted and curved cylindrical panels is adopted, and the eigenvalue equation for free vibrations of twisted and curved cylindrical panels with variable thickness is formulated by the principle of virtual work and the Rayleigh–Ritz method with displacement functions expressed as general algebraic polynomials, where the vectors of two displacement components in plane are not mutual. By the parametric study, the vibration characteristics represented by vibration frequency parameters and mode shapes of the panels affected by the parameters, such as variation ratios of thickness and curvature both in chordwise and lengthwise directions, and twist, are studied.

2. STRAIN FORMULAS

A twisted and curved cylindrical thin panel with variable thickness is shown in Figure 1, where two co-ordinate systems are introduced. In a right-hand co-ordinate system (x, y, z') with unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, where x is a curvilinear axis in a lengthwise direction and a twisting center axis, z'-axis takes in a radial direction where the cylindrical arc is divided into two parts equally. In another cylindrical co-ordinate system, θ is an angle measured from the z'-axis and s-axis takes in a circumferential direction on the midsurface of the panel. Parameters Ω_1, r, l and b are a central angle, the radius of a reference, a length, an arc length on a cross-section perpendicular to the x-axis respectively. h is a two-dimensional function denoting the thickness in the normal of a reference, k is a twist angle per unit length around the x-axis, $1/R$ and Ω_2 express a curvature of the x-axis ($\Omega_2 = l/R$), e is a distance between the origins O and O_1 of the two co-ordinate systems, and z-axis is perpendicular to the midsurface with the outward direction considered positive. ϕ is an angle between the y-axis and the radial direction of the x-axis at a fixed end of the panel.

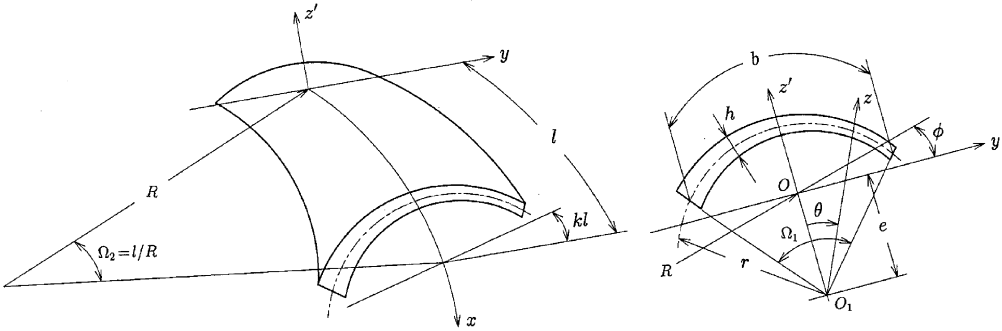

Figure 1. A twisted and curved cylindrical panel with variable thickness.

Selecting a midsurface of the panel as a reference, an arbitrary point outside the midsurface is considered after deformation, which can be expressed by a position vector **r** as

$$\mathbf{r} = \mathbf{r}_c^{(0)} + z\frac{1}{B}(-ek\sin\theta\,\mathbf{i}_1 + A\sin\theta\,\mathbf{i}_2 A\cos\theta\,\mathbf{i}_3) + \mathscr{U}$$

$$= \mathbf{r}^{(0)} + \mathscr{U}, \tag{1}$$

where $\mathbf{r}_c^{(0)}$ is a position vector corresponding to the point on the midsurface, z denotes the position of the point in normal direction referred to the midsurface, $\mathbf{r}^{(0)}$ is a position vector of the point before deformation and $\mathscr{U}$ is the displacement vector whose components are defined as follows if the thin panel is considered [25]:

$$U = \left(1 + z\frac{1}{B}h_4\right)u - z\frac{ek}{B^3 r}h_3 v - z\frac{1}{B^2}\frac{\partial w}{\partial x} + z\frac{k}{B^2 r}(e\cos\theta - r)\frac{\partial w}{\partial \theta},$$

$$V = -z\frac{k}{B}\left[1 + \frac{ek^2}{B^2 R}p\sin\theta(e\cos\theta - r)\right]u + \left\{1 + z\frac{1}{Br}\left[A + \frac{ek^2}{B^2}h_3(e\cos\theta - r)\right]\right\}v$$

$$+ z\frac{k}{B^2}(e\cos\theta - r)\frac{\partial w}{\partial x} - z\frac{1}{r}\left[1 + \frac{k^2}{B^2}(e\cos\theta - r)^2\right]\frac{\partial w}{\partial \theta},$$

$$W = w, \tag{2}$$

where u, v and w denote the displacements of a point on the midsurface in $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions which are defined as follows:

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}_c^{(0)}}{\partial x}, \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}_c^{(0)}}{r\partial \theta}, \qquad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}, \tag{3}$$

and the quantities in the above equations and others are defined in Appendix A.

By partial differentiation of the vectors before and after deformation with respect to x, $r\theta$ and z, the covariant base vectors $\mathbf{g}_i$ $(i = 1,2,3)$ and $\mathbf{G}_j$ $(j = 1,2,3)$ are obtained, and the Green strain tensors f_{ij} $(i, j = 1,2,3)$ with respect to general curvilinear co-ordinate system can be given by

$$2f_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j - \mathbf{g}_i \cdot \mathbf{g}_j \quad (i,j = 1,2,3). \tag{4}$$

Due to a demand of elastic theory, herein a local orthogonal co-ordinate system (ξ, η, ζ) to the point is introduced, and the strains ε_{ij} $(i,j = \xi, \eta, \zeta)$ are presented as follows:

$$\varepsilon_{\xi\xi} = \frac{1}{F}\mathbf{Z}\mathbf{G}_{x1}\mathbf{U} + \frac{1}{2}(\mathbf{G}_{x2}\mathbf{U})^{\mathrm{T}}(G_{x2}\mathbf{U}), \qquad \varepsilon_{\eta\eta} = \frac{1}{F}\mathbf{Z}\mathbf{G}_{\theta 1}\mathbf{U} + \frac{1}{2}(\mathbf{G}_{\theta 2}\mathbf{U})^{\mathrm{T}}(\mathbf{G}_{\theta 2}\mathbf{U}),$$

$$\gamma_{\xi\eta} = \frac{1}{F}\mathbf{Z}\mathbf{G}_{x\theta}\mathbf{U} + (\mathbf{G}_{x2}\mathbf{U})^{\mathrm{T}}(\mathbf{G}_{\theta 2}\mathbf{U}), \qquad \varepsilon_{\zeta\zeta} = 0, \qquad \gamma_{\xi\zeta} = 0, \qquad \gamma_{\eta\zeta} = 0, \tag{5}$$

where matrices $\mathbf{Z}$, $\mathbf{G}_{x1}$, $\mathbf{G}_{\theta 1}$, $\mathbf{G}_{x\theta}$, $\mathbf{G}_{x2}$, $\mathbf{G}_{\theta 2}$ and U are expressed as

$$\mathbf{Z}^{\mathrm{T}} = \begin{bmatrix} 1 \\ z/l \\ z^2/l^2 \end{bmatrix}, \quad \mathbf{G}_{x1} = \begin{bmatrix} {}_{x1}\mathbf{G}_{1,i} \\ {}_{x1}\mathbf{G}_{2,i} \\ {}_{x1}\mathbf{G}_{3,i} \end{bmatrix}, \quad G_{\theta 1} = \begin{bmatrix} {}_{\theta 1}\mathbf{G}_{1,i} \\ {}_{\theta 1}\mathbf{G}_{2,i} \\ {}_{\theta 1}\mathbf{G}_{3,i} \end{bmatrix}, \quad G_{x\theta} = \begin{bmatrix} {}_{x\theta}\mathbf{G}_{1,i} \\ {}_{x\theta}\mathbf{G}_{2,i} \\ {}_{x\theta}\mathbf{G}_{3,i} \end{bmatrix}, \quad (i = 1, \ldots, 12),$$

$$\mathbf{G}_{x2} = \left[0 \quad 0 \quad -h_2 \quad 0 \quad 0 \quad \frac{ek}{B^2 r} h_1 \quad 0 \quad 0 \quad 0 \quad \frac{1}{B} \quad -\frac{k}{Br} e_2 \quad 0 \right],$$

$$\mathbf{G}_{\theta 2} = \left[0 \quad 0 \quad \frac{k}{B} h_3 \quad 0 \quad 0 \quad -\frac{A}{Br} \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{r} \quad 0 \right],$$

$$\mathbf{U}^{\mathrm{T}} = \left[\frac{\partial u}{\partial x} \quad \frac{\partial u}{r\partial \theta} \quad u \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{r\partial \theta} \quad v \quad \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{r^2 \partial \theta^2} \quad \frac{\partial^2 w}{r\partial x \partial \theta} \quad \frac{\partial w}{\partial x} \quad \frac{\partial w}{r\partial \theta} \quad w \right], \tag{6}$$

the non-zero elements in matrices $\mathbf{G}_{x1}$, $\mathbf{G}_{\theta 1}$ and $\mathbf{G}_{x\theta}$ are defined in Appendix B.

3. EIGENVALUE EQUATION FOR FREE VIBRATION

The general form of the principle of virtual work for the free vibration of twisted and curved cylindrical thin panels [25] is

$$\iiint_{\mathscr{A}} (\sigma_{\xi\xi}\delta\varepsilon_{\xi\xi} + \sigma_{\eta\eta}\delta\varepsilon_{\eta\eta} + \tau_{\xi\eta}\delta\gamma_{\xi\eta}) F \,\mathrm{d}x\, r\,\mathrm{d}\theta\,\mathrm{d}z - \iiint_{\mathscr{A}} \rho\omega^2 \mathscr{U}\delta\mathscr{U} F \,\mathrm{d}x\, r\,\mathrm{d}\theta\,\mathrm{d}z = 0. \tag{7}$$

Substituting stress–strain relationship, strain–displacement relationship and displacements into equation (7), integrating with respect to z and neglecting the terms containing z^i where i is > 3, it can be rewritten as

$$\iint_{\mathscr{S}} \delta\mathbf{U}^{\mathrm{T}} \left(\mathbf{G}_{x1}^{\mathrm{T}} \mathbf{D} \mathbf{G}_{x1} + {}_{v}\mathbf{G}_{x1}^{\mathrm{T}} \mathbf{D} \mathbf{G}_{\theta 1} + {}_{v}\mathbf{G}_{\theta 1}^{\mathrm{T}} \mathbf{D} \mathbf{G}_{x1} + \mathbf{G}_{\theta 1}^{\mathrm{T}} \mathbf{D} \mathbf{G}_{\theta 1} + \frac{1-v}{2} \mathbf{G}_{x\theta}^{\mathrm{T}} \mathbf{D} \mathbf{G}_{x\theta} \right) \mathbf{U} \frac{1}{\mathbf{B}} \,\mathrm{d}x\, r\,\mathrm{d}\theta$$

$$- \iint_{\mathscr{S}} \rho\omega^2 \{ [B^2 + k^2 (\mathrm{e}\cos\theta - r)^2]\, u\delta u + k\, (\mathrm{e}\cos\theta - r)(v\delta u + u\delta v)$$

$$+ v\delta v + w\delta w \} Bh \,\mathrm{d}x\, r\,\mathrm{d}\theta = 0, \tag{8}$$

where matrix $\mathbf{D}$ is defined by

$$\mathbf{D} = \int_{-h/2}^{h/2} \frac{E}{1-v^2} \mathbf{Z}^{\mathrm{T}} \mathbf{Z} \frac{\mathbf{1}}{\mathbf{1} + z/l\bar{p}_1} \,\mathrm{d}z \doteq \int_{-h/2}^{h/2} \frac{E}{1-v^2} \mathbf{Z}^{\mathrm{T}} \mathbf{Z} \left(1 - \frac{z}{l} \bar{p}_1 \right) \mathrm{d}z$$

$$= D_0 \begin{bmatrix} \frac{12}{h_0^2}\alpha & -\frac{\bar{p}_1}{l^2}\alpha^3 & \frac{1}{l^2}\alpha^3 \\ & \frac{1}{l^2}\alpha^3 & 0 \\ Sym. & & 0 \end{bmatrix}, \quad D_0 = \frac{Eh_0^3}{12(1-v^2)}, \; h = h_0\,\alpha, \; \bar{p}_1 = p_1 \mathrm{l}. \tag{9}$$

E, v and ρ are Young's modulus, the Poisson ratio and a specific weight of a material, respectively, ω is an angular frequency, and h_0 and α are a reference thickness and a function denoting the variation ratio of thickness.

With the purpose of parametric study, the following non-dimensional quantities are introduced.

$$\bar{x} = \frac{x}{l}, \quad \bar{u} = \frac{u}{l}, \quad \bar{v} = \frac{v}{l}, \quad \bar{w} = \frac{w}{l}, \quad \bar{R} = \frac{R}{l}, \quad \bar{r} = \frac{r}{l}, \quad \bar{e} = \frac{e}{r}, \quad K = kl. \tag{10}$$

Although it is possible to define a non-linear thickness function $h(\bar{x}, \theta)$, herein, considering the complicated profile of a panel having a twist and double curvature, which is defined as a linear variation of thickness in two directions:

$$h(\bar{x}, \theta) = h_0\left[1{\cdot}0 - (1{\cdot}0 - \delta_{\bar{x}})\bar{x} - (1{\cdot}0 - \delta_\theta)\left(\frac{1}{2} + \frac{\theta}{\Omega_1}\right)\right], \tag{11}$$

where $\delta_{\bar{x}}$ and δ_θ are variation ratios in lengthwise and chordwise directions. Because of a thickness $h(\bar{x}, \theta) \geqslant 0$, the two variation ratios would obey the following:

$$\delta_{\bar{x}} + \delta_\theta \geqslant 1{\cdot}0. \tag{12}$$

The Rayleigh–Ritz method, with two-dimensional polynomial functions satisfying the geometric boundary conditions $\bar{u} = 0$, $\bar{v} = 0$, $\bar{w} = 0$ and $\partial\bar{w}/\partial\bar{x} = 0$ at $\bar{x} = 0$ given as follows, is used:

$$\bar{u} = \sum_{i=1}^{N_{\bar{u}}} \sum_{j=0}^{M_{\bar{u}}} a_{ij}\, \bar{x}^i\theta^j, \qquad \bar{v} \sum_{k=1}^{N_{\bar{v}}} \sum_{l=0}^{M_{\bar{v}}} b_{kl}\, \bar{x}^k\theta^l, \qquad \bar{w} = \sum_{m=2}^{N_{\bar{w}}} \sum_{n=0}^{M_{\bar{w}}} c_{mn}\, \bar{x}^m\theta^n, \tag{13}$$

where a_{ij}, b_{kl} and c_{mn} are unknown coefficients, N_i and M_i $(i = \bar{u}, \bar{v}, \bar{w})$ are the maximum powers of $\bar{x}$ and θ in the displacement polynomial functions, respectively.

Substituting equations (10), (11) and (13) into equation (8) and integrating, the eigenvalue equation for free vibration of a twisted and curved cylindrical panel with variable thickness is yielded,

$$\begin{bmatrix} \mathbf{A}_{11} - \lambda^2\mathbf{B}_{11} & \mathbf{A}_{12} - \lambda^2\mathbf{B}_{12} & \mathbf{A}_{13} \\ & \mathbf{A}_{22} - \lambda^2\mathbf{B}_{22} & \mathbf{A}_{23} \\ Sym. & & \mathbf{A}_{33} - \lambda^2\mathbf{B}_{33} \end{bmatrix} \mathbf{q} = 0, \tag{14}$$

where $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ $(i,j = 1, 2, 3)$ are sub-stiffness and sub-mass matrices of the panel, respectively, vector $\mathbf{q}$ composed by unknown coefficients a_{ij}, b_{kl} and c_{mn} are related to vibration modes, and λ is a non-dimensional frequency parameter defined as

$$\lambda^2 = \frac{\rho\, h_0\, \omega^2\, l^4}{D_0}. \tag{15}$$

4. NUMERICAL RESULTS AND DISCUSSIONS

In the present numerical procedure, the maximum powers of $\bar{x}$ and θ in in-plane and transverse displacement polynomial functions, and integral points with respect to $\bar{x}$ and θ are the two factors which govern the accuracy of the approximate solutions. As a matter of experience, two-dimensional Gauss–Legendre numerical integral scheme with 16 points is enough. The emphasis is the effect of the maximum powers of $\bar{x}$ and θ in the displacement polynomial functions on the convergence of the solutions. The vibration characteristics of twisted and curved cylindrical panels with variable thickness are represented by the first eight frequency parameters λ and some corresponding mode shapes in this paper. The present results compared with the previous experimental and theoretical results are carried out in order to verify the present procedure, and numerous models with different geometric parameters, such as twist, curvature, are studied by the present procedure. Therefore, the effects of twist, curvature and a variation ratio of thickness in chordwise and lengthwise directions on the vibration characteristics are revealed. The Poisson ratio v of isotropic materials is 0·3 in here.

4.1 CONVERGENCE STUDIES

A model with a severe profile, such as a large twist, large curvature in both chordwise and lengthwise directions and a special variation ratio of thickness, is taken to be considered in the convergence of the first ten frequency parameters λ, namely, $\Omega_1 = 60^\circ$, $\Omega_2 = 90^\circ$, $K = 60^\circ$, $\phi = 0^\circ$, an aspect ratio $l/b = 2$, a thickness ratio $b/h_0 = 25$ and two cases of $\delta_{\bar{x}} = 1{\cdot}00$, $\delta_\theta = 0{\cdot}00$ and $\delta_{\bar{x}} = 0{\cdot}00$, $\delta_\theta = 1{\cdot}00$. The same number of terms and the same maximum powers of $\bar{x}$ and θ in the displacement polynomial functions are used in the two cases. For the in-plane displacements $\bar{u}$ and $\bar{v}$, the same maximum powers of $\bar{x}$ and θ are assumed, or $N_{\bar{u}} = N_{\bar{v}}$ and $M_{\bar{u}} = M_{\bar{v}}$. The number of terms in the displacement polynomial functions can be obtained by the parameters $N_i, M_i\,(i = \bar{u}, \bar{v}, \bar{w})$ in equation (13).

In Table 1, the seven different sets of the parameters $N_i, M_i\,(i = \bar{u}, \bar{v}, \bar{w})$ are cited to demonstrate how the first ten frequency parameters λ vary with the maximum powers of $\bar{x}$ and θ, or the number of terms in the three displacement polynomial functions. It can be seen that the results greatly differ for the cases where there are almost the same number of terms in $\bar{u}$, $\bar{v}$ and $\bar{w}$, such as (48/48/63) and (49/49/63), but the different maximum powers of $\bar{x}$ and θ, such as (6, 7; 6, 7; 8, 8) and (7, 6; 7, 6; 8, 8). The lower frequency parameters λ show better convergence with the number of terms than the higher ones, which means that it is not necessary for lower frequency parameters to use a large number of terms in the displacement polynomial functions, otherwise, a large amount of calculations will produce calculation error and accumulating errors. Therefore, it is important for the convergence and accuracy to optimize the combination of the maximum powers of $\bar{x}$ and θ in the displacement polynomial functions.

TABLE 1

Convergence of λ with the maximum powers of $\bar{x}$ and θ in $\bar{u}$, $\bar{v}$ and $\bar{w}$ ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60^\circ$, $\Omega_2 = 90^\circ$, $K = 60^\circ$, $\phi = 0^\circ$)

		$N_{\bar{u}}, M_{\bar{u}}$		6, 7	6, 7	7, 6	7, 6	7, 6	7, 7	7, 7
		$N_{\bar{v}}, M_{\bar{v}}$		6, 7	6, 7	7, 6	7, 6	7, 6	7, 7	7, 7
		$N_{\bar{w}}, M_{\bar{w}}$		7, 8	8, 8	7, 6	8, 7	8, 8	8, 7	8, 8
$\delta_{\bar{x}}$	δ_θ	Terms		48/48/54	48/48/63	49/49/42	49/49/56	49/49/63	56/56/56	56/56/63
0·00	1·00	λ_i	1	8·3739	8·3732	8·3790	8·3771	8·3764	8·3740	8·3724
			2	17·843	17·842	17·862	17·856	17·855	17·840	17·839
			3	30·823	30·820	30·678	30·565	30·542	30·486	30·468
			4	34·959	34·946	35·137	34·961	34·922	34·734	34·642
			5	48·709	48·669	45·486	45·363	45·258	45·092	45·058
			6	60·533	60·454	60·638	59·439	58·959	58·466	58·198
			7	87·954	87·766	70·632	69·956	69·842	69·762	69·692
			8	89·121	88·824	86·908	85·112	84·385	84·575	83·758
			9	104·81	104·54	103·03	102·09	101·87	101·59	101·15
			10	131·20	129·05	114·49	111·71	111·33	111·31	110·58
1·00	0·00	λ_i	1	4·8180	4·8145	4·8174	4·8137	4·8136	4·8113	4·8112
			2	17·841	17·816	17·723	17·715	17·714	17·709	17·709
			3	28·425	28·388	28·292	28·255	28·255	28·253	28·253
			4	46·312	46·283	45·730	45·702	45·701	45·690	45·690
			5	58·938	58·734	57·804	57·784	57·783	57·760	57·759
			6	66·551	66·224	64·984	64·938	64·936	64·863	64·862
			7	80·711	80·190	79·195	78·457	78·456	78·376	78·375
			8	91·113	90·457	86·226	85·681	85·677	85·518	85·517
			9	103·16	102.97	99·656	99·402	99·396	99·231	99·229
			10	123·16	122·91	108·41	108·07	108·06	107·12	107·11

It is known that there is a good rate of convergence for the first ten frequency parameters λ in the case of (7, 7; 7, 7; 8, 8) or (56/56/63), which is used for analyzing vibrations of twisted and curved cylindrical panels with variable thickness in this paper.

4.2. COMPARISON WITH AVAILABLE RESULTS

A cantilevered cylindrical panel with linear chordwise taper was studied by experimental and theoretical methods [10, 12, 26–28]. Although there were no twist and curvature in the

TABLE 2

Frequencies (Hz) *of a cantilevered cylindrical panel with chordwise taper* ($a = 12$ in, $b = 12$ in, $h_0 = 0{\cdot}048$ in, $R_y = 30$ in, $\delta_x = 1{\cdot}0, \delta_y = 3{\cdot}4375$)

f_i	Present	FEM [27]	FEM [28]	Exp. [26, 27]	Reference [10]	Reference [12]
1	77·880	78·855	78·99	76·4	78·296	77·066
2	113·59	114·10	114·5	108	113·95	111·26
3	206·74	211·55	210·8	202	210·37	206·23
4	250·36	258·09	263·0	253	254·86	247·57
5	364·56	370·60	373·8	364	368·21	360·61
6	440·42	452·32	454·6	426	465·53	439·46
7	453·61	480·52	496·8	465	499·68	449·38
8	572·21	581·32	587·3	572	586·03	564·86
9	675·21	690·11	695·6	677	730·28	669·66
10	707·52	755·03	779·2	692	757·44	691·34

TABLE 3

Comparison of λ *for the panel with taper thickness* ($\nu = 0{\cdot}3$, $a/b = 1{\cdot}0$, $b/h_0 = 100$, $b/R_y = 0{\cdot}5$)

			No. of vibration mode							
$\delta_{\bar{x}}$	δ_θ	Method	1	2	3	4	5	6	7	8
0·00	1·0	Reference [12]	13·274	13·423	25·044	28·117	34·848	38·253	41·873	42·403
		Present	13·362	13·399	25·108	28·383	34·889	38·500	43·026	43·029
0·25		Reference [12]	11·137	13·410	30·788	32·371	33·457	35·915	51·078	61·259
		Present	11·240	13·325	30·456	32·645	33·646	35·844	51·074	61·388
0·50		Reference [12]	10·474	15·280	29·674	37·583	37·587	42·701	72·960	73·930
		Present	10·551	15·213	29·910	37·513	37·763	41·855	72·797	73·821
0·75		Reference [12]	10·393	16·592	29·244	39·894	42·666	54·195	82·688	82·957
		Present	10·440	16·651	29·205	40·089	42·528	53·053	82·654	82·995
1·00		Reference [12]	10·588	16·978	30·640	42·205	47·662	65·439	89·715	89·939
		Present	10·612	17·181	30·332	42·384	47·498	64·023	89·709	90·135
1·0	0·0	Reference [12]	8·1412	12·216	19·456	20·429	28·717	29·560	36·373	36·781
		Present	8·1875	12·267	19·529	20·481	28·941	29·517	36·991	37·769
	0·5	Reference [12]	9·0505	14·842	25·984	32·712	40·679	48·854	61·270	74·749
		Present	9·1026	14·876	25·983	32·688	40·705	47·875	61·137	74·056
	1·0	Reference [12]	10·588	16·978	30·640	42·205	47·662	65·439	89·715	89·939
		Present	10·612	17·181	30·332	42·384	47·498	64·023	89·709	90·135
	1·5	Reference [12]	12·496	18·105	35·985	47·240	57·791	80·947	94·829	99·532
		Present	12·490	18·414	35·421	47·299	57·739	79·179	94·874	99·653
	2·0	Reference [12]	14·638	18·774	41·758	51·670	67·462	95·593	100·24	105·85
		Present	14·605	19·130	41·001	51·606	67·566	93·596	99·963	105·61

lengthwise direction, it may be a good example to prove the present numerical procedure. Its material characteristic coefficients are given as follows: Young's modulus $E = 30 \times 10^7$ lb/in^2, the Poisson ratio $\nu = 0{\cdot}3$ and a specific weight $\rho = 0{\cdot}284$ lb/in^3, and geometric parameters cited from the reference are listed in Table 2.

The results obtained by various methods are shown in Table 2 where there are finite element methods with triangular shallow shell elements (125 degrees of freedom (d.o.f.)) and helicoidal shell elements (205 d.o.f.), and the methods based on shallow shell theory by Ritz principle. The present procedure using the principle of virtual work and the Rayleigh–Ritz method is based on general shell theory. It is certain that there are errors in both experiment and theoretical analyses, such as measuring errors of the frequencies and ideal boundary

TABLE 4

Effect of δ_θ on frequency parameters λ of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 30°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_{\bar{x}} = 1{\cdot}00$)

		No. of vibration mode							
K	δ_θ	1	2	3	4	5	6	7	8
0°	0·00	5·2726	13·239	26·543	36·330	42·091	52·425	54·944	63·899
	0·25	5·9768	12·132	27·396	39·215	47·517	57·817	69·307	70·618
	0·50	6·2013	12·565	29·586	43·163	57·560	57·740	73·963	89·660
	0·75	6·3497	13·661	31·830	47·579	67·249	71·284	82·575	100·20
	1·00	6·5046	15·150	33·950	52·501	71·358	76·559	93·667	107·43
	1·25	6·6890	16·900	35·996	57·810	71·192	84·874	105·02	115·25
	1·50	6·9085	18·831	38·040	63·376	70·824	92·681	116·42	123·71
15°	0·00	4·5177	15·067	29·266	35·065	48·622	53·845	59·490	66·607
	0·25	4·9230	14·335	32·007	35·128	53·513	59·498	71·834	72·966
	0·50	5·2059	14·643	33·256	39·343	62·793	71·000	75·244	86·240
	0·75	5·4948	15·552	34·347	44·556	69·855	78·190	82·654	95·246
	1·00	5·7915	16·878	35·508	50·224	72·495	84·065	94·032	103·60
	1·25	6·0958	18·505	36·797	56·082	73·214	90·568	105·40	112·60
	1·50	6·4108	20·351	38·230	61·854	73·669	96·781	116·77	122·21
30°	0·00	3·8046	16·522	27·366	41·044	55·071	58·327	63·688	70·418
	0·25	4·0174	16·417	29·234	40·793	60·460	62·965	70·928	79·378
	0·50	4·2745	16·994	30·739	43·825	67·981	73·684	79·506	87·521
	0·75	4·6002	18·027	32·001	48·278	72·528	84·267	88·552	94·078
	1·00	4·9677	19·392	33·147	53·590	74·940	93·105	97·325	102·70
	1·25	5·3568	21·015	34·278	59·391	76·687	98·571	107·56	113·24
	1·50	5·7578	22·839	35·448	65·379	77·488	103·09	118·45	124·11
45°	0·00	3·3679	15·184	29·155	43·428	56·549	66·072	68·296	81·377
	0·25	3·5020	15·610	30·144	43·668	63·673	69·705	72·988	89·662
	0·50	3·7344	16·556	31·274	46·435	71·748	76·741	83·276	97·960
	0·75	4·0557	17·865	32·306	50·599	75·934	86·557	94·158	103·74
	1·00	4·4345	19·423	33·259	55·692	78·657	96·357	105·48	108·64
	1·25	4·8467	21·172	34·181	61·425	80·605	104·94	114·30	116·94
	1·50	5·2790	23·073	35·106	67·592	82·078	111·07	122·35	128·30
60°	0·00	3·1007	13·013	32·204	41·847	58·442	72·389	74·472	91·587
	0·25	3·2017	13·529	32·855	42·798	66·275	74·273	79·892	100·84
	0·50	3·4195	14·515	34·081	45·573	74·956	79·415	89·884	109·42
	0·75	3·7324	15·834	35·185	49·589	79·944	88·481	99·776	116·03
	1·00	4·1089	17·371	36·175	54·461	83·521	98·195	110·09	121·30
	1·25	4·5251	19·055	37·103	59·935	86·298	107·59	120·80	126·19
	1·50	4·9661	20·843	38·017	65·820	88·505	116·18	130·52	132·76

conditions in experiment, and the calculation errors, the errors of geometric parameters and the errors led by the assumptions in theoretical analysis. Selecting the experimental results as a reference for judging, the present results are better, the maximum and minimum discrepancies are $+4{\cdot}92\%$ and $-2{\cdot}51\%$, respectively, and the average error is only $+1{\cdot}13\%$, which are mainly caused by the geometric parameters, because the different geometric parameters are used to represent the panel and it is necessary to transform before the vibration of the panel is analyzed by the present procedure. It is not observed that the error becomes largely for the higher vibration modes, which means that the present numerical procedure has good stability.

TABLE 5

Effect of δ_θ *on frequency parameters* λ *of cylindrical panels* ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_{\bar{x}} = 1{\cdot}00$)

K	δ_θ	No. of vibration mode							
		1	2	3	4	5	6	7	8
0°	0·00	8·5470	14·776	33·755	48·133	49·470	53·298	62·149	65·218
	0·25	9·8207	13·737	34·293	54·030	56·313	58·890	70·067	83·956
	0·50	10·517	13·755	38·038	59·377	65·133	68·365	76·339	103·17
	0·75	10·932	14·470	42·140	62·783	68·643	79·730	84·989	116·26
	1·00	11·141	15·720	46·225	65·652	69·891	90·755	95·285	128·65
	1·25	11·262	17·315	50·076	68·411	70·511	101·50	106·02	139·30
	1·50	11·363	19·131	53·538	69·716	72·669	111·99	116·85	146·68
15°	0·00	6·8754	17·409	37·461	44·813	52·507	53·658	66·038	71·342
	0·25	7·6167	16·700	38·319	50·020	58·000	59·685	75·137	87·826
	0·50	8·2127	16·729	42·196	52·967	67·090	70·881	80·242	103·70
	0·75	8·7900	17·251	46·681	54·826	72·448	81·874	87·594	117·28
	1·00	9·3162	18·175	51·290	56·377	74·485	92·760	97·539	130·72
	1·25	9·7676	19·444	54·998	58·731	75·494	103·61	108·13	139·67
	1·50	10·147	20·997	56·760	62·817	76·458	114·41	118·74	144·39
30°	0·00	5·5532	19·178	36·202	46·372	54·759	58·256	72·146	75·712
	0·25	6·0379	18·797	39·495	47·608	60·234	63·637	80·131	90·664
	0·50	6·5186	19·122	42·955	49·689	69·911	73·826	85·854	105·15
	0·75	7·0493	19·864	46·095	52·132	76·830	84·027	92·638	117·67
	1·00	7·6007	20·894	48·190	55·783	80·055	94·502	101·98	128·26
	1·25	8·1417	22·163	49·433	60·464	81·571	105·17	112·35	135·45
	1·50	8·6528	23·640	50·307	65·635	82·554	115·92	122·87	140·57
45°	0·00	4·6503	18·266	35·485	48·904	57·540	63·375	75·505	82·420
	0·25	4·9924	18·575	38·313	49·040	63·720	67·878	83·060	95·187
	0·50	5·3951	19·430	40·547	51·190	73·617	76·973	89·983	109·41
	0·75	5·8731	20·608	42·234	54·446	81·373	86·228	97·200	119·91
	1·00	6·3985	21·970	43·490	58·530	85·534	96·298	106·24	128·43
	1·25	6·9425	23·465	44·461	63·234	87·731	106·71	116·34	134·88
	1·50	7·4836	25·074	45·268	68·386	89·100	117·25	126·77	139·88
60°	0·00	4·0439	15·703	37·159	47·426	60·091	69·670	80·173	89·581
	0·25	4·3013	16·446	39·053	48·218	67·195	73·409	86·427	99·953
	0·50	4·6521	17·627	40·638	50·845	77·137	80·589	93·863	113·90
	0·75	5·0892	19·085	41·811	54·498	85·596	87·937	101·82	124·25
	1·00	5·5845	20·686	42·734	58·778	90·056	97·732	110·66	132·19
	1·25	6·1111	22·362	43·513	63·521	92·954	107·90	120·17	138·49
	1·50	6·6488	24·086	44·216	68·632	94·994	118·18	130·06	143·55

In reference [12], cantilevered rectangular shallow shells with variable thickness were studied, a large amount of data about the effects of variation ratios of thickness in two directions and other geometric parameters on the vibration characteristics were proposed, the shells having $b/R_y = 0{\cdot}5$, which is the largest curvature in the reference, are analyzed by the present numerical procedure. The comparisons are shown in Table 3 where the parameters [12] are listed in order to comprehend easily, it is noted that they are different from the ones in this paper and need to transform. Good agreement can be seen between the first eight frequency parameters ($\lambda = \omega ab\sqrt{\rho h_0/D_0}$) obtained by the two methods for the panels with variable thickness whether in a lengthwise or in a chordwise direction, even in the extreme cases of $\delta_{\bar{x}} = 0{\cdot}00$ and $\delta_\theta = 0{\cdot}0$, and the maximum discrepancy is $< +3\%$.

TABLE 6

Effect of δ_θ on frequency parameters λ of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 90°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_{\bar{x}} = 1{\cdot}00$)

		No. of vibration mode							
K	δ_θ	1	2	3	4	5	6	7	8
0°	0·00	10·689	17·024	37·038	50·775	52·598	60·120	64·915	73·360
	0·25	11·717	16·835	38·460	54·443	60·653	68·365	74·835	96·264
	0·50	12·691	16·771	42·760	61·025	70·784	76·807	83·328	113·27
	0·75	13·798	16·802	47·248	65·158	80·118	81·936	94·187	123·02
	1·00	14·853	17·124	51·783	67·336	84·467	90·959	105·25	133·45
	1·25	15·485	18·077	56·316	68·428	87·288	101·10	116·06	144·58
	1·50	15·720	19·598	60·776	68·983	89·799	111·39	126·63	156·11
15°	0·00	8·7267	19·654	40·297	50·001	54·002	58·026	70·119	78·673
	0·25	9·6058	19·294	41·843	54·832	59·187	67·430	80·119	99·568
	0·50	10·467	19·237	46·021	59·800	69·189	77·118	87·460	112·09
	0·75	11·377	19·460	50·469	62·800	78·423	83·832	96·153	123·54
	1·00	12·268	19·976	55·100	64·449	83·900	91·749	106·32	135·12
	1·25	13·058	20·813	59·865	65·323	86·485	101·59	116·90	146·82
	1·50	13·702	21·975	64·662	65·802	88·377	111·83	127·49	158·66
30°	0·00	7·1852	21·251	41·267	49·873	55·588	61·250	74·221	82·755
	0·25	7·8951	20·948	44·107	53·457	60·455	68·855	82·892	99·566
	0·50	8·6196	21·111	48·109	57·438	69·749	78·028	90·401	113·21
	0·75	9·3936	21·620	52·238	59·653	78·456	86·156	97·871	126·43
	1·00	10·181	22·385	56·224	61·283	84·816	94·052	107·08	138·87
	1·25	10·943	23·371	59·017	63·866	88·327	103·21	117·34	150·73
	1·50	11·647	24·566	60·117	68·091	90·458	113·07	127·90	162·44
45°	0·00	6·0290	20·647	40·744	53·836	57·375	63·373	76·637	87·654
	0·25	6·5934	20·861	44·241	55·231	62·474	68·961	85·446	102·68
	0·50	7·2079	21·492	47·841	57·411	71·411	78·104	93·813	117·08
	0·75	7·8821	22·436	50·842	59·436	80·047	87·105	100·75	131·25
	1·00	8·5892	23·568	52·964	62·200	87·098	95·607	108·78	143·88
	1·25	9·3014	24·833	54·232	65·910	91·640	104·59	118·24	155·16
	1·50	9·9954	26·209	54·975	70·310	94·421	114·19	128·38	165·59
60°	0·00	5·1774	18·229	42·071	54·037	58·916	66·569	82·496	92·936
	0·25	5·6279	19·016	45·031	55·034	64·473	70·847	90·291	106·12
	0·50	6·1582	20·129	47·834	56·574	73·561	79·455	98·329	121·49
	0·75	6·7585	21·487	49·730	59·108	82·302	88·006	104·98	137·41
	1·00	7·4026	22·976	50·881	62·511	89·832	96·393	112·21	149·70
	1·25	8·0672	24·531	51·616	66·523	95·158	105·31	120·62	159·18
	1·50	8·7330	26·123	52·145	70·964	98·647	114·81	129·91	167·33

Generally, it is effective and practicable to analyze the free vibrations of the cylindrical panels with variable thickness for the present numerical procedure.

4.3. EFFECTS AND DISCUSSIONS

As it is known that the study on the vibrations of twisted and curved cylindrical panels with variable thickness and the effects on it are void herein, the effects of twist, curvature in chordwise and lengthwise directions, a variation ratio of thickness also in the two directions aforementioned on the vibration characteristics are studied in detail in order to provide data for applications and researches.

TABLE 7

Effect of $\delta_{\bar{x}}$ on frequency parameters λ of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 30°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_\theta = 1{\cdot}00$)

		No. of vibration mode							
K	$\delta_{\bar{x}}$	1	2	3	4	5	6	7	8
0°	0·00	11·668	13·357	24·258	29·071	40·159	43·366	49·776	57·304
	0·25	8·9876	12·627	27·283	39·096	43·757	53·186	72·459	82·590
	0·50	7·7210	13·577	30·639	43·064	59·229	62·963	84·880	88·773
	0·75	6·9875	14·424	32·677	47·429	69·688	74·444	80·429	98·069
	1·00	6·5046	15·150	33·950	52·501	71·358	76·559	93·667	107·43
	1·25	6·1599	15·793	34·949	57·739	67·364	82·501	108·46	117·20
	1·50	5·9003	16·377	35·862	62·929	64·154	88·038	122·22	127·40
15°	0·00	9·5042	15·518	26·421	30·042	44·643	48·288	48·845	59·565
	0·25	7·3755	14·671	31·951	34·547	47·997	57·934	68·944	85·694
	0·50	6·5753	15·505	34·950	38·814	62·516	66·891	83·584	89·484
	0·75	6·1067	16·251	35·298	44·560	73·168	78·297	82·196	94·655
	1·00	5·7915	16·878	35·508	50·224	72·495	84·065	94·032	103·60
	1·25	5·5621	17·428	35·808	55·553	69·532	88·965	108·29	113·68
	1·50	5·3864	17·928	36·193	60·170	67·461	93·858	120·45	125·40
30°	0·00	7·6240	16·852	27·849	34·248	42·638	53·737	57·694	66·319
	0·25	5·9621	16·535	32·751	35·926	52·385	63·329	71·995	88·612
	0·50	5·4379	17·699	33·390	42·141	66·182	71·451	83·221	97·195
	0·75	5·1527	18·649	33·284	48·034	75·050	81·504	90·015	96·341
	1·00	4·9677	19·392	33·147	53·590	74·940	93·105	97·326	102·70
	1·25	4·8362	19·996	33·095	58·830	73·255	97·459	109·67	113·15
	1·50	4·7371	20·509	33·130	63·729	71·604	101·28	120·15	126·01
45°	0·00	6·6711	16·032	29·433	35·070	45·736	57·176	67·372	74·972
	0·25	5·2213	16·059	33·575	37·476	58·077	66·101	78·716	92·444
	0·50	4·7924	17·364	33·959	44·302	69·497	74·718	88·496	104·87
	0·75	4·5715	18·483	33·718	50·207	77·118	84·552	96·260	106·38
	1·00	4·4345	19·423	33·259	55·692	78·657	96·357	105·48	108·64
	1·25	4·3408	20·228	32·810	60·839	78·265	105·33	113·43	116·82
	1·50	4·2729	20·930	32·434	65·687	77·433	109·94	122·27	128·77
60°	0·00	6·1531	14·761	28·061	38·248	46·489	61·596	76·438	82·857
	0·25	4·8171	14·558	34·124	38·485	63·040	68·259	85·318	96·642
	0·50	4·4263	15·603	36·101	43·997	73·697	76·264	95·385	111·12
	0·75	4·2291	16·543	36·447	49·399	80·214	86·952	102·37	117·83
	1·00	4·1089	17·371	36·175	54·461	83·521	98·195	110·09	121·30
	1·25	4·0285	18·106	35·717	59·197	84·630	109·02	118·18	125·99
	1·50	3·9715	18·769	35·244	63·607	84·891	117·45	125·67	134·09

Tables 4–6 and 7–9 correspond to the cases of $\delta_{\bar{x}} = 1{\cdot}00$ and $\delta_\theta = 1{\cdot}00$, respectively, where the central angle ($\Omega_1 = 30, 60°, 90°$), the twist angle at the free end ($K = 0°–60°$ step $15°$), the variation ratio of thickness ($\delta_{\bar{x}}$ or $\delta_\theta = 0{\cdot}00–1{\cdot}50$ step $0{\cdot}25$) in a direction and the other given geometric parameters are considered and their effects on vibration characteristics are studied.

In the case of $\delta_{\bar{x}} = 1{\cdot}00$, or the panels having variable thickness in the chordwise direction, almost the frequency parameters λ corresponding to vibration modes decrease with decreasing parameter δ_θ and the variations of the higher λ are greater than those of the lower λ, which leads the region of the first eight frequency parameters λ distribution to

TABLE 8

Effect of $\delta_{\bar{x}}$ *on frequency parameters* λ *of cylindrical panels* ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_\theta = 1{\cdot}00$)

		No. of vibration mode							
K	$\delta_{\bar{x}}$	1	2	3	4	5	6	7	8
0°	0·00	14·444	19·651	30·326	33·508	47·650	50·206	65·953	67·812
	0·25	13·246	15·578	33·647	41·290	58·554	65·632	85·357	92·257
	0·50	13·291	14·139	37·685	57·949	62·390	75·056	84·350	102·96
	0·75	12·032	14·951	42·151	64·034	71·134	79·376	86·066	116·03
	1·00	11·141	15·720	46·225	65·652	69·891	90·755	95·285	128·65
	1·25	10·486	16·401	49·532	66·050	68·270	100·63	108·05	139·44
	1·50	9·9820	17·013	51·976	63·306	71·016	108·48	121·82	148·52
15°	0·00	13·067	20·610	29·008	36·521	48·339	56·111	66·528	70·863
	0·25	11·001	17·390	35·702	42·597	60·138	68·630	78·088	94·570
	0·50	10·314	17·181	41·148	54·852	63·156	76·555	90·485	96·531
	0·75	9·7603	17·640	46·337	56·740	74·415	80·741	90·142	114·23
	1·00	9·3162	18·175	51·290	56·377	74·485	92·760	97·539	130·72
	1·25	8·9608	18·684	54·073	57·686	72·201	105·71	107·30	140·72
	1·50	8·6716	19·159	53·740	61·369	71·071	113·79	121·12	146·54
30°	0·00	10·776	21·712	28·002	42·647	46·532	62·934	69·770	73·158
	0·25	8·7901	19·104	36·011	46·542	59·455	68·910	78·548	97·766
	0·50	8·2085	19·490	43·274	50·881	66·279	77·714	91·488	103·27
	0·75	7·8537	20·232	47·519	52·114	78·271	83·191	94·847	115·31
	1·00	7·6007	20·894	48·190	55·783	80·055	94·502	101·98	128·26
	1·25	7·4090	21·445	47·552	60·563	78·212	107·73	111·23	136·06
	1·50	7·2576	21·913	46·781	65·375	76·543	118·14	122·83	141·24
45°	0·00	9·1230	19·613	30·377	43·914	48·179	65·325	71·798	78·236
	0·25	7·3245	18·842	37·522	46·773	58·070	74·488	83·250	97·457
	0·50	6·8306	19·918	43·468	48·948	69·603	80·401	93·630	113·56
	0·75	6·5681	21·032	44·171	53·510	82·056	85·722	99·023	120·37
	1·00	6·3985	21·970	43·490	58·530	85·534	96·298	106·24	128·43
	1·25	6·2793	22·734	42·648	63·440	84·569	108·83	115·75	134·24
	1·50	6·1912	23·364	41·861	68·182	83·309	119·76	126·91	138·49
60°	0·00	8·0064	17·352	30·608	44·013	49·871	66·818	77·056	85·857
	0·25	6·3709	17·241	39·187	43·428	61·719	80·126	86·733	97·006
	0·50	5·9316	18·520	43·671	47·451	73·199	83·495	96·906	118·47
	0·75	5·7151	19·690	43·439	53·524	85·118	88·040	103·57	126·37
	1·00	5·5845	20·686	42·734	58·778	90·056	97·732	110·66	132·19
	1·25	5·4984	21·526	41·899	63·644	90·281	109·95	119·30	137·36
	1·50	5·4390	22·242	41·094	68·203	89·848	120·97	129·32	141·64

decrease greatly. It means that the stiffness of the twisted and curved panel decreases with decreasing parameter δ_θ or the thickness of the panel thinning in the chordwise direction. With increasing the central angle Ω_1 the frequency parameters λ increase, which is greater for the lower λ than for the higher one, or the λ increases with decreasing radius r of the reference surface because the Ω_1 is inversely proportional to the r when the length of arc b is constant. The first λ decreases monotonically with increasing twist angle and the second λ tends to decrease with decreasing twist angle, which does not change for different Ω_1 and δ_θ. The effects of the K on the other λ are complicated. Figure 2 shows the vibration mode shapes of the panels whose geometric parameters are $\Omega_1 = 60°$, $\Omega_2 = 30°$, $K = 0°$ and 30°

TABLE 9

Effect of $\delta_{\bar{x}}$ on frequency parameters λ of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 90°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_\theta = 1{\cdot}00$)

		No. of vibration mode							
K	$\delta_{\bar{x}}$	1	2	3	4	5	6	7	8
0°	0·00	16·252	24·227	36·641	37·464	50·444	58·148	72·532	76·161
	0·25	14·159	21·434	38·880	41·863	64·520	75·447	91·678	100·55
	0·50	14·472	18·888	42·922	56·556	72·106	83·508	91·650	114·75
	0·75	14·962	17·447	47·260	67·452	78·044	83·807	97·235	123·33
	1·00	14·853	17·124	51·783	67·336	84·467	90·959	105·25	133·45
	1·25	14·285	17·496	56·197	64·788	85·530	105·17	113·21	144·56
	1·50	13·700	18·011	60·301	62·305	86·657	118·71	121·31	156·07
15°	0·00	15·213	25·278	32·800	41·990	49·890	64·518	72·108	77·006
	0·25	12·961	21·879	37·832	46·483	64·957	77·661	90·783	96·746
	0·50	12·724	20·230	45·143	58·974	68·061	85·312	95·705	108·14
	0·75	12·545	19·846	50·197	66·843	73·867	87·384	99·126	121·42
	1·00	12·268	19·976	55·100	64·449	83·900	91·749	106·32	135·12
	1·25	11·960	20·283	59·951	61·971	84·565	104·90	114·45	147·99
	1·50	11·666	20·642	59·720	64·667	84·398	118·21	123·03	160·11
30°	0·00	13·215	27·756	28·913	47·339	48·771	69·882	72·612	82·148
	0·25	11·070	22·775	36·766	51·706	68·058	75·428	84·250	98·220
	0·50	10·651	21·808	46·167	60·903	67·939	82·704	94·629	112·64
	0·75	10·401	21·962	52·061	62·813	75·719	88·512	99·724	124·65
	1·00	10·181	22·385	56·224	61·283	84·816	94·052	107·08	138·87
	1·25	9·9864	22·830	56·582	63·568	86·274	105·63	115·97	152·44
	1·50	9·8164	23·241	55·063	67·963	85·998	117·73	125·66	165·05
45°	0·00	11·429	23·834	32·205	45·868	53·726	70·271	79·215	87·516
	0·25	9·4293	22·044	37·577	56·950	65·001	73·316	87·475	98·891
	0·50	8·9849	22·135	46·741	60·676	70·180	78·776	97·756	118·75
	0·75	8·7545	22·813	51·998	60·251	79·175	87·409	102·56	130·70
	1·00	8·5892	23·568	52·964	62·200	87·098	95·607	108·78	143·88
	1·25	8·4603	24·240	51·860	66·086	89·336	106·65	117·60	156·44
	1·50	8·3572	24·808	50·506	70·334	89·634	117·81	127·82	168·05
60°	0·00	10·025	20·484	33·461	45·116	59·175	66·996	86·780	89·867
	0·25	8·1677	20·022	39·431	56·244	62·360	78·559	91·749	98·459
	0·50	7·7346	20·960	48·053	56·454	73·154	80·245	101·87	122·12
	0·75	7·5307	22·001	51·638	58·188	82·604	87·450	106·67	137·58
	1·00	7·4026	22·976	50·881	62·511	89·832	96·393	112·21	149·70
	1·25	7·3129	23·816	49·509	66·949	92·603	107·35	120·22	159·32
	1·50	7·2475	24·521	48·207	71·193	93·541	118·06	130·01	167·12

and $\delta_\theta = 0{\cdot}00$–$1{\cdot}00$. It is seen that the distribution of contour lines on the panels is uniform for the first and second vibration modes even in the case of $\delta_\theta = 0{\cdot}00$, but the contour lines of the higher ones scatter on the local zone as $\delta_\theta = 0{\cdot}00$, which means that there are abrupt changes in the transverse displacements on the panels. It is found that modes do not exchange with each other as the δ_θ changes.

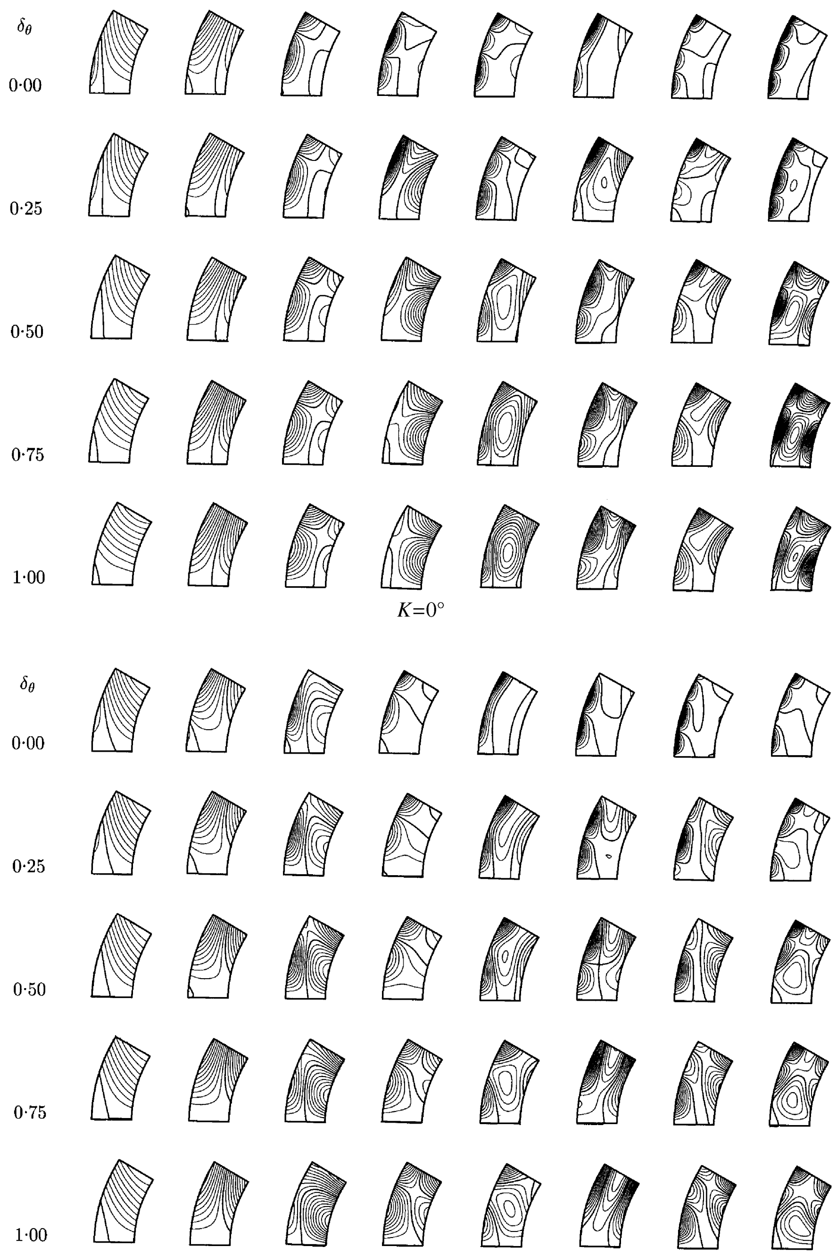

Figure 2. Effect of δ_θ on vibration mode shapes of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_{\bar{x}} = 1{\cdot}00$).

In the case of $\delta_\theta = 1{\cdot}00$, or when the panels have variable thickness in the lengthwise direction, although the same effect of the parameter $\delta_{\bar{x}}$ on the region of the first eight frequency parameters λ distribution as that of the parameter δ_θ is found, there are different phenomena for λ observed from those aforementioned. Only the first frequency parameter λ increases as the $\delta_{\bar{x}}$ decreases and the largest first λ occurs when the thickness at the free end is equal to zero, which can be explained from the mode shapes shown in Figure 3, the

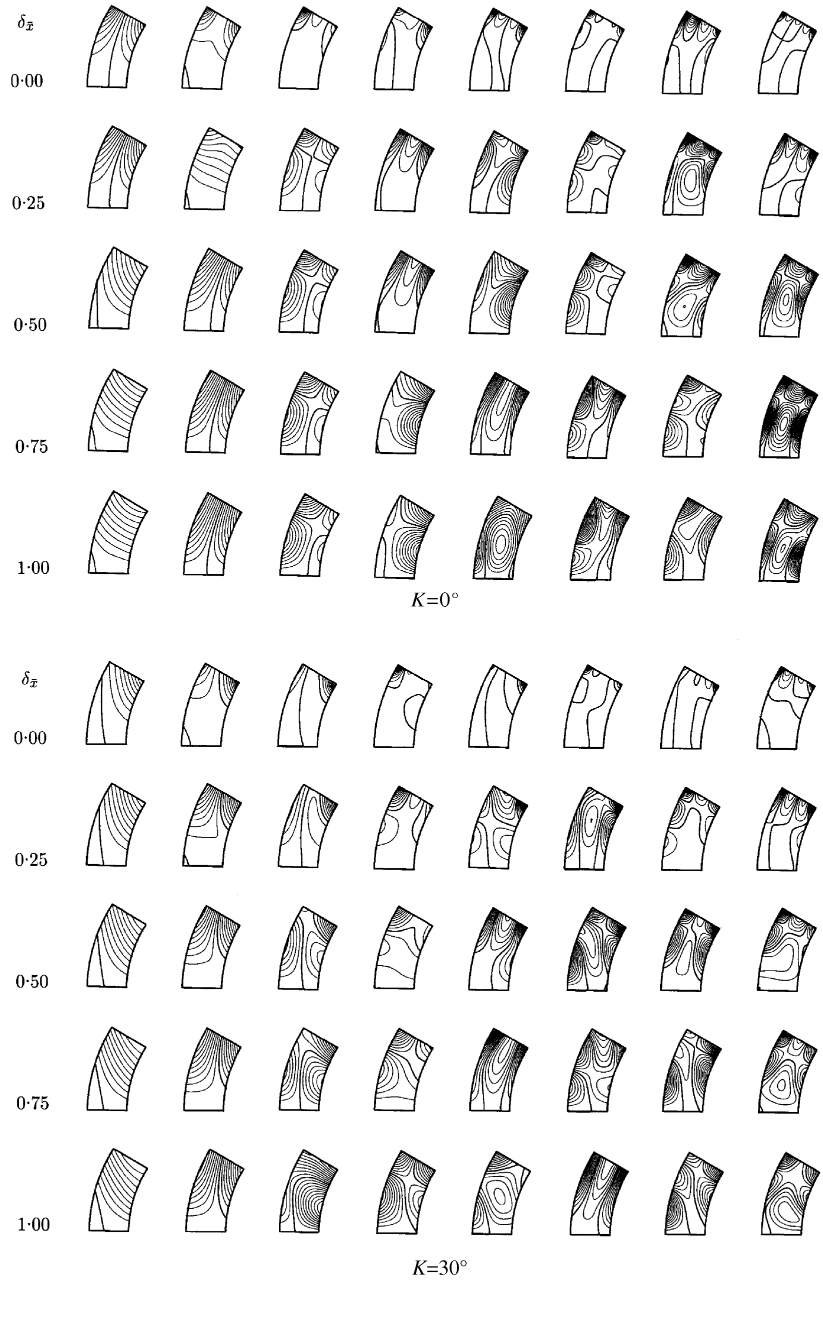

Figure 3. Effect of $\delta_{\bar{x}}$ on vibration mode shapes of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $\Omega_2 = 30°$, $\phi = 0°$, $\delta_\theta = 1{\cdot}00$).

first vibration mode is a bending mode for twisted and curved cylindrical panels with $\delta_{\bar{x}} = 1{\cdot}00$, but it changes to a vibration mode which is similar to a torsion mode with decreasing $\delta_{\bar{x}}$. The higher frequency parameters λ decrease with the thickness thinning. The frequency parameters λ tend towards an increase of the central angle Ω_1 or a decrease of the radius r. The same effects of the twist angle occur on the first frequency parameter λ aforementioned. From the given mode shapes, it can also be seen that the adjacent modes exchange with each other as the $\delta_{\bar{x}}$ varies in the case of $K = 0°$, for instance, the first and second modes in $\delta_{\bar{x}} = 0{\cdot}25$ become the second and first modes in $\delta_{\bar{x}} = 0{\cdot}50$, but it is not obvious for the higher modes and for the modes of the twisted panels. In the extreme case of $\delta_{\bar{x}} = 0{\cdot}00$, the contour lines concentrate on the free ends except the first mode.

The effect of the curvature $1/R$ in the lengthwise direction on the frequency parameters λ of the panels with $\Omega_1 = 60°$, $K = 0°$, $\phi = 0°$ and the different variation ratios of thickness $\delta_{\bar{x}}$ and δ_θ are shown in Table 10. As it is known there is a linear relation between the

TABLE 10

Effect of Ω_2 on frequency parameters λ of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $K = 0°$, $\phi = 0°$)

			No. of vibration mode							
Ω_2	$\delta_{\bar{x}}$	δ_θ	1	2	3	4	5	6	7	8
30°	1·00	0·00	8·5470	14·776	33·755	48·133	49·470	53·298	62·149	65·218
		0·25	9·8207	13·737	34·293	54·030	56·313	58·890	70·067	83·956
		0·50	10·517	13·755	38·038	59·377	65·133	68·365	76·339	103·17
		0·75	10·932	14·470	42·140	62·783	68·643	79·730	84·989	116·26
		1·00	11·141	15·720	46·225	65·652	69·891	90·755	95·285	128·65
60°	1·00	0·00	8·7343	16·246	30·873	43·335	51·245	57·510	60·389	63·787
		0·25	9·6869	14·985	31·076	48·519	54·985	66·007	71·346	74·249
		0·50	10·248	15·060	34·181	57·153	61·971	71·121	80·439	93·404
		0·75	10·613	15·673	37·541	62·578	69·345	75·668	89·489	112·13
		1·00	10·848	16·693	40·889	65·099	75·538	82·052	99·150	129·58
90°	1·00	0·00	9·2713	17·913	29·041	39·843	49·833	57·126	62·190	69·941
		0·25	9·7743	16·480	28·938	43·308	53·022	66·882	71·650	78·317
		0·50	10·151	16·657	31·743	50·236	58·801	80·284	82·394	86·222
		0·75	10·422	17·236	34·696	56·150	64·103	85·884	94·987	98·551
		1·00	10·630	18·083	37·660	59·939	70·177	90·458	104·88	113·71
30°	0·00	1·00	14·444	19·651	30·326	33·508	47·650	50·206	65·953	67·812
	0·25		13·246	15·578	33·647	41·290	58·554	65·632	85·357	92·257
	0·50		13·291	14·139	37·685	57·949	62·390	75·056	84·350	102·96
	0·75		12·032	14·951	42·151	64·034	71·134	79·376	86·066	116·03
	1·00		11·141	15·720	46·225	65·652	69·891	90·755	95·285	128·65
60°	0·00	1·00	14·973	19·430	27·978	32·471	44·514	49·594	64·876	69·353
	0·25		14·022	14·981	29·793	42·351	54·992	68·371	80·616	87·463
	0·50		12·859	14·957	33·533	58·035	63·885	72·557	81·674	104·45
	0·75		11·658	15·884	37·383	65·933	71·021	75·800	86·954	117·69
	1·00		10·848	16·693	40·889	65·099	75·538	82·052	99·150	129·58
90°	0·00	1·00	15·415	19·319	25·627	32·921	39·299	50·975	60·753	69·109
	0·25		14·228	15·360	27·095	43·891	47·505	73·712	76·130	79·209
	0·50		12·439	16·242	30·663	54·702	58·028	79·387	84·567	90·966
	0·75		11·351	17·243	34·290	59·342	64·767	87·344	90·741	103·38
	1·00		10·630	18·083	37·660	59·939	70·177	90·458	104·88	113·71

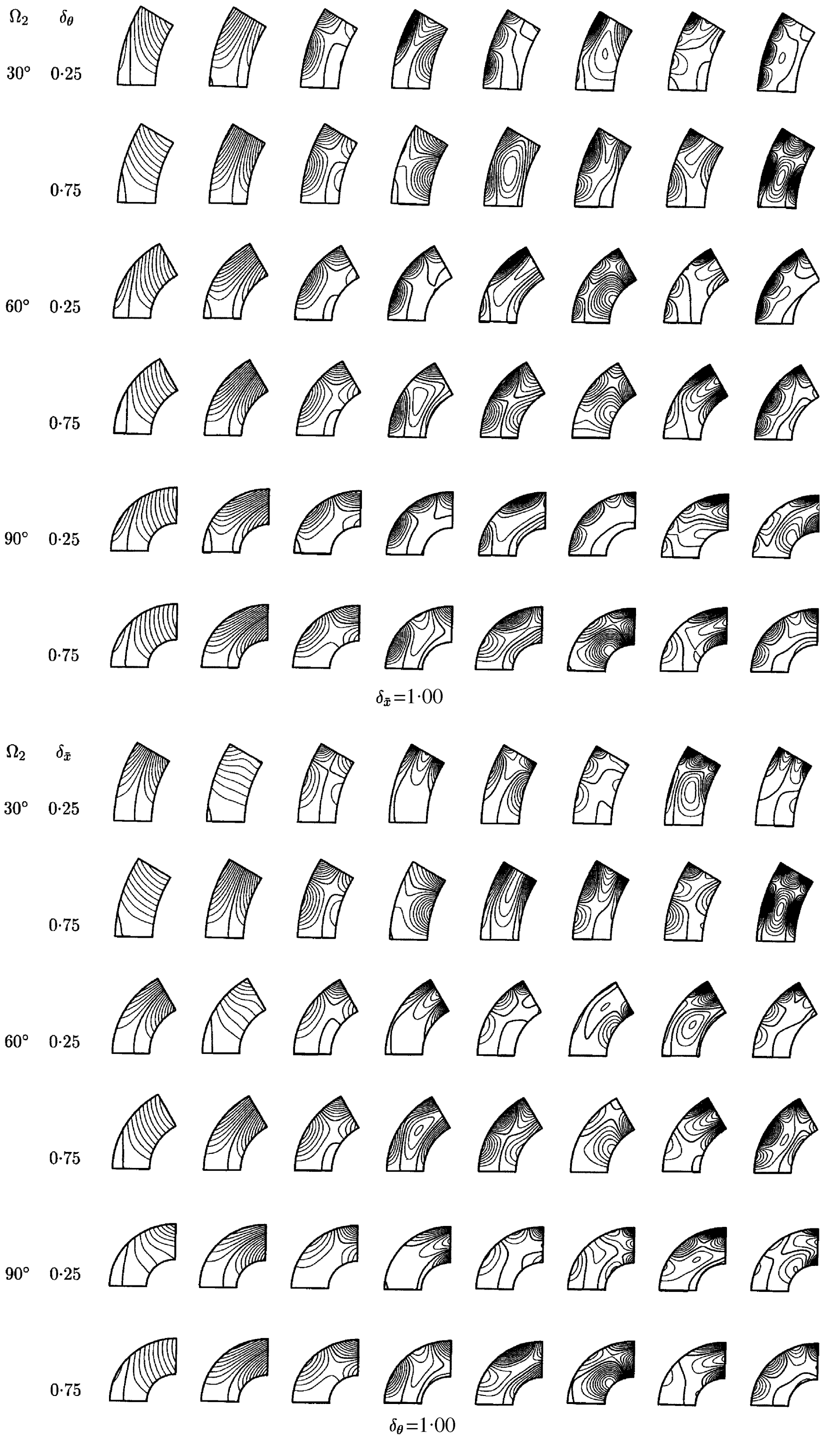

Figure 4. Effect of Ω_2 on vibration mode shapes of cylindrical panels ($l/b = 2$, $b/h_0 = 25$, $\Omega_1 = 60°$, $K = 0°$, $\phi = 0°$).

curvature $1/R$ and the Ω_2, or $\Omega_2 = (1/R)\,l$, so the change in the Ω_2 is used to represent the variation of the curvature. Under the different Ω_2, the first frequency parameter λ always increases with increasing parameter δ_θ and decreases with increasing parameter $\delta_{\bar{x}}$, the other frequency parameters λ tend to increase with parameters $\delta_{\bar{x}}$ and δ_θ increasing except the second λ in the extreme conditions of zero thickness at the free end. The effect of the Ω_2 on the frequency parameters λ can be observed from the table. In general, the variations of the λ are not great and have no simple relations with the Ω_2. In the case of $\delta_{\bar{x}} = 1{\cdot}00$, with increasing Ω_2, the first λ tends towards an increase at $\delta_\theta = 0{\cdot}00$ which changes with the δ_θ and shows a decrease at $\delta_\theta = 1{\cdot}00$, and the second λ shows an increase. In the case of $\delta_\theta = 1{\cdot}00$, the same variation of the first λ with the Ω_2 as mentioned before can be seen and the variation of second λ is contrary to it. It is complicated for the higher frequency parameters and those aforementioned become complex for the twisted panels. The part of mode shapes corresponding to Table 10 are given in Figure 4 where the effects of Ω_2 on mode shapes can be seen. It is known that coupled vibrations occur due to the Ω_2, there is an exchange of the adjacent lower modes with the Ω_2 varying only in the case of $\delta_\theta = 1{\cdot}00$ and the effects on the higher modes are complicated.

5. CONCLUSIONS

The method for analyzing the free vibrations of shells based on general shell theory and using the principle of virtual work and the Rayleigh–Ritz method with two-dimensional algebraic polynomial functions as the displacements has been extended to accommodate twisted and curved cylindrical panels with variable thickness, and it has been verified by comparing the present results with the previous experimental and theoretical results to several shallow cylindrical panels with variable thickness in both chordwise and lengthwise directions, the high accuracy of the solutions applied by the present numerical procedure is also demonstrated.

The effects of the geometric parameters on the vibration characteristics are studied, the first frequency parameter changes monotonically with both the variation ratios of thickness, or the first λ increases with the decrease of $\delta_{\bar{x}}$, in contrast, it decreases with decreasing δ_θ. The higher frequency parameters tend towards an increase with increasing parameter $\delta_{\bar{x}}$ and δ_θ, the variations are greater than those of the lower ones, which leads the region of the frequency parameters distribution to increase greatly. As the twist angle increases the lower frequency parameters decrease, and the effect of the curvature on the frequency parameters can also be seen. Whether or not the twist angle and curvature are large, the trend of the frequency parameters caused by the variation ratio of thickness are the same. A phenomenon in which modes exchange with a variation in the ratio of thickness is observed.

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APPENDIX A

The quantities in this paper are given as follows:

$$A = 1 - \frac{r}{R}\sin(kx + \phi - \theta) + \frac{e}{R}\sin(kx + \phi), \qquad B = \sqrt{e^2k^2\sin^2\theta + A^2},$$

$$p = -r\cos(kx + \phi - \theta) + e\cos(kx + \phi), \qquad q = \frac{Ar}{R}\cos(kx + \phi - \theta) + e^2k^2\sin\theta\cos\theta,$$

$$c_{11} = -\frac{A}{BR}\sin(kx + \phi - \theta) + \frac{Aek^2}{B^3R}p\sin\theta,$$

$$c_{12} = \frac{ek}{BR}\sin\theta\cos(kx + \phi) - \frac{Ak}{B}\cos\theta + \frac{e^2k^3}{B^3R}p\sin^3\theta,$$

$$c_{13} = \frac{k}{B}\sin\theta\left[-\frac{e}{R}\sin(kx + \phi) + A + \frac{e^2k^2}{B^2R}p\sin\theta\cos\theta\right],$$

$$c_{21} = \frac{ek}{Br}\left(-\cos\theta + \frac{1}{B^2}q\sin\theta\right),$$

$$c_{22} = \frac{1}{Br}\left[\frac{r}{R}\cos(kx + \theta - \phi)\sin\theta + A\cos\theta - \frac{A}{B^2}q\sin\theta\right],$$

$$c_{23} = \frac{1}{Br}\left[\frac{r}{R}\cos(kx + \theta - \phi)\cos\theta - A\sin\theta - \frac{A}{B^2}q\cos\theta\right],$$

$$F = B(1 + zp_1), \quad e_1 = e - r\cos\theta, \quad e_2 = e\cos\theta - r, \quad h_1 = A - \frac{e}{R}\sin\theta\cos(kx + \phi - \theta),$$

$$h_2 = \frac{A}{R}\cos(kx + \phi - \theta) + ek^2\sin\theta, \qquad h_3 = A\cos\theta - \frac{r}{R}\sin\theta\cos(kx + \phi - \theta),$$

$$h_4 = \frac{ek^2}{B^2R}p\sin\theta - \frac{1}{R}\sin(kx + \phi - \theta), \qquad p_1 = \frac{1}{B}\left(\frac{A}{r} + h_4 + \frac{ek^2}{B^2r}h_3e_2\right).$$

APPENDIX B

The non-zero elements in three matrices $\mathbf{G}_{x1}$, $\mathbf{G}_{\theta 1}$ and $\mathbf{G}_{x\theta}$ are as follows:

$$_{x1}G_{1,1} = B, \qquad _{x1}G_{1,2} = -\frac{Bk}{r}e_2, \qquad _{x1}G_{1,3} = \frac{Ak}{BR}p, \qquad _{x1}G_{1,6} = \frac{1}{Br}q,$$

$$_{x1}G_{1,12} = h_4 + \frac{ek^2}{B^2r}h_3e_2, \qquad _{x1}G_{2,1} = \frac{A}{r} + h_4, \qquad _{x1}G_{2,2} = \frac{k}{r}(h_1 - h_4e_2),$$

$$ {}_{x1}G_{2,3} = \frac{k}{R}\left[\frac{2ek^2}{B^4 r} pqe_2 \sin\theta - \frac{ek^2}{B^2 r} pe_2 \cos\theta - \frac{2Aek^2}{B^4 R} p^2 \sin\theta - \cos(kx+\phi-\theta) + \frac{A^2}{B^2 r} p - \frac{e^2k^2}{B^2}\sin^2\theta\cos(kx+\phi-\theta)\right], \quad {}_{x1}G_{2,4} = -\frac{ek}{B^2 r}h_3, \quad {}_{x1}G_{2,5} = \frac{ek^2}{B^2 r^2}h_3 e_2, $$

$$ {}_{x1}G_{2,6} = \frac{1}{r}\left[\frac{A}{B^2}h_2 - \frac{ek^2}{B^2 R}e_2\cos\theta\cos(kx+\phi-\theta) + \frac{2Aek^2}{B^4 R}ph_3 - \frac{2ek^2}{B^4 r}qe_2 h_3 + \frac{1}{R}\cos(kx+\phi-\theta)\right], \quad {}_{x1}G_{2,7} = -\frac{1}{B}, \quad {}_{x1}G_{2,8} = -\frac{k^2}{Br^2}e_2^2, \quad {}_{x1}G_{2,9} = \frac{2k}{Br}e_2, $$

$$ {}_{x1}G_{2,10} = \frac{k}{B^3}\left(\frac{A}{R}p - \frac{1}{r}qe_2\right), \quad {}_{x1}G_{2,11} = -\frac{1}{Br}\left[-\frac{ek^2e_2}{r}\sin\theta + \frac{Ak^2e_2}{B^2R}p + \frac{1}{B^2 r}q(B^2 - k^2e_2^2)\right], $$

$$ {}_{x1}G_{2,12} = \frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1h_3\right), \quad {}_{x1}G_{3,4} = -\frac{Aek}{B^3 r^2}h_3, $$

$$ {}_{x1}G_{3,5} = -\frac{ek^2}{B^3 r^2}h_1h_3, \quad {}_{x1}G_{3,7} = -\frac{A}{B^2 r}, $$

$$ {}_{x1}G_{3,8} = \frac{k^2}{B^2 r^2}e_2h_1, \quad {}_{x1}G_{3,9} = \frac{k}{B^2r^2}(Ae_2 - rh_1), \quad {}_{x1}G_{3,10} = \frac{k}{B^4 r}\left(\frac{A^2}{R}p + h_1 q\right), $$

$$ {}_{x1}G_{3,11} = -\frac{1}{r^2}\left[\frac{A^2k^2}{B^4R}pe_2 + \frac{1}{R}\cos(kx+\phi-\theta) + \frac{ek^2}{B^2}h_1\sin\theta + \frac{k^2}{B^4}qe_2h_1\right], \quad {}_{\theta 1}G_{1,2} = \frac{Bk}{r}e_2, $$

$$ {}_{\theta 1}G_{1,5} = \frac{B}{r}, \quad {}_{\theta 1}G_{1,12} = \frac{A}{r}, \quad {}_{\theta 1}G_{2,1} = \frac{ek^2}{B^2 r}e_2h_3, \quad {}_{\theta 1}G_{2,2} = \frac{k}{r}(h_4e_2 - h_1), $$

$$ {}_{\theta 1}G_{2,3} = \frac{e^2k^3}{B^2Rr}p\sin^2\theta, \quad {}_{\theta 1}G_{2,4} = \frac{ek}{B^2 r}h_3, \quad {}_{\theta 1}G_{2,5} = \frac{1}{r}\left(h_4 + \frac{A}{r}\right), \quad {}_{\theta 1}G_{2,6} = -\frac{2e^2k^2}{B^2r^2}h_3\sin\theta, $$

$$ {}_{\theta 1}G_{2,8} = -\frac{B}{r^2}, \quad {}_{\theta 1}G_{2,10} = -\frac{ek}{Br}\sin\theta, \quad {}_{\theta 1}G_{2,11} = \frac{ek^2}{Br^2}e_2\sin\theta, \quad {}_{\theta 1}G_{2,12} = \frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1h_3\right), $$

$$ {}_{\theta 1}G_{3,1} = -\frac{ek^2}{B^3 r}h_1h_3, \quad {}_{\theta 1}G_{3,3} = \frac{ek^3}{B^3Rr}\left[eph_4\sin^2\theta - e_2h_3\cos(kx+\phi-\theta) + \frac{A}{B^2}ph_1h_3\right], $$

$$ {}_{\theta 1}G_{3,4} = \frac{Aek}{B^3r^2}h_3, \quad {}_{\theta 1}G_{3,6} = \frac{e^2k^2}{B^3r^2}h_3\left[\frac{1}{B^2}h_2h_3 - h_4\sin\theta + \frac{1}{R}\sin\theta\sin(kx+\phi-\theta)\right], $$

$$ {}_{\theta 1}G_{3,8} = -\frac{1}{r^2}h_4, \quad {}_{\theta 1}G_{3,9} = -\frac{ek}{B^2r^2}h_3, \quad {}_{\theta 1}G_{3,10} = \frac{ek}{B^4 r}(h_2h_3 - B^2h_4\sin\theta), $$

$$ {}_{\theta 1}G_{3,11} = \frac{ek^2}{B^4r^2}e_2(B^2h_4\sin\theta - h_2h_3), \quad {}_{x\theta}G_{1,1} = ke_2, \quad {}_{x\theta}G_{1,2} = \frac{1}{r}(B^2 - k^2e_2^2), $$

$$ {}_{x\theta}G_{1,4} = 1,\ {}_{x\theta}G_{1,5} = -\frac{k}{r}e_2,\ {}_{x\theta}G_{1,6} = -\frac{ek}{r}\sin\theta,\ {}_{x\theta}G_{1,12} = -\frac{2ek}{Br}h_3,\ {}_{x\theta}G_{2,1} = \frac{2Ak}{Br}e_2, $$

$$ {}_{x\theta}G_{2,2} = \frac{2}{Br}(B^2h_4 + k^2e_2h_1), \quad {}_{x\theta}G_{2,3} = \frac{2ek^2}{BRr}\left[\frac{2A}{B^2}ph_3 - e_2\cos(kx+\phi)\right], \quad {}_{x\theta}G_{2,4} = \frac{2A}{Br}, $$

$$ {}_{x\theta}G_{2,5} = -\frac{2Ak}{Br^2}e_2, \quad {}_{x\theta}G_{2,6} = \frac{2ek}{Br}\left[\frac{1}{R}\sin\theta\sin(kx+\phi-\theta) + \frac{2}{B^2r}h_3q\right], \quad {}_{x\theta}G_{2,8} = \frac{2k}{r^2}e_2, $$

$$ {}_{x\theta}G_{2,9} = -\frac{2}{r}, \qquad {}_{x\theta}G_{2,10} = \frac{2}{B^2r}q, \qquad {}_{x\theta}G_{2,11} = -\frac{2k}{B^2r^2}qe_2, $$

$$ {}_{x\theta}G_{3,3} = \frac{k^2}{B^2}\left[\frac{e^2k^2}{B^2r}(1-A)h_3\sin\theta - \frac{e}{R^2r}e_2\sin\theta\cos^2(kx+\phi-\theta) - \frac{2Ae^2k^2}{B^4R^2r}p^2h_3\sin\theta \right.$$
$$ + \frac{1}{Rr}ph_1 - \frac{e}{Rr}h_3\cos(kx+\phi-\theta) + \frac{e}{Rr}ph_4\cos\theta - \frac{e}{R}h_4\sin\theta\sin(kx+\phi-\theta) $$
$$ \left. - \frac{2e}{B^2Rr}pqh_4\sin\theta - \frac{1}{Rr}h_1e_2\cos(kx+\phi-\theta)\right], $$

$$ {}_{x\theta}G_{3,7} = -\frac{ek}{B^3r}h_3, \qquad {}_{x\theta}G_{3,8} = \frac{k}{Br^2}(e_2h_4 - h_1), \qquad {}_{x\theta}G_{3,9} = \frac{1}{Br}\left(\frac{ek^2}{B^2r}e_2h_3 - \frac{A}{r} - h_4\right), $$

$$ {}_{x\theta}G_{3,10} = \frac{1}{Br}\left[\frac{1}{R}\cos(kx+\phi-\theta) + \frac{1}{B^2}h_4q + \frac{Aek^2}{B^4R}ph_3\right], $$

$$ {}_{x\theta}G_{3,11} = -\frac{k}{Br}\left[\frac{1}{Rr}e_2\cos(kx+\phi-\theta) + \frac{e}{B^2r}h_2h_3 \right.$$
$$ \left. + \frac{ek^2}{B^2Rr}pe_2\cos\theta - \frac{1}{B^2Rr}qe_2\sin(kx\phi-\theta)\right]. $$