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ON GROUP VELOCITY OF ELASTIC WAVES, IN AN ANISOTROPIC PLATE

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A proof is offered for the equivalence of the group velocity and the energy velocity for elastic waves in a free anisotropic homogenous plate. The proof is valid in the case where the group velocity may be negative while the wave number is positive.

1. INTRODUCTION

The equivalence of the energy velocity and the group velocity is often taught in university physics. An early proof of the equivalence of the two velocities for one-dimensional water waves was given by Rayleigh [1]. In the 1950s and 1960s many studies [2–4] were done to show the equivalence of the two velocities for more general systems. Later, in the 1990s, Awati and Howes raised a question asking for a general proof of the relationship between the group velocity and the energy velocity, and a number of people [5–8] responded to their question. However, several cases need particular considerations. There are some researchers who tried to show that the energy velocity and the group velocity are not equal for acoustic waves in piezoelectrics, but their result was proved to be incorrect later [9]. Another ambiguous case is that of the backward waves where the group velocity is negative while the wave number is positive [10–12]; in other words, the group velocity and the wave number have opposite signs. Some people believe that when the group velocity is negative, it is not equal to the energy velocity [13].

Tolstoy and Usdin [10] studying the Rayleigh–Lamb dispersion equation in 1957, numerically found that the group velocity may be negative for some modes of wave propagation in a plate. Meitzler [11] in 1965 reported an experimental result showing the existence of the mode for which the group velocity could be negative in a plate. In a recent study, photo-elastic pictures of wave modes where there may exist negative group velocity were presented [12]. People who believe that the energy and group velocities are equivalent interpret this phenomenon in such a way that the group velocity should be positive and the wave number should be negative. By the 1970s, a proof was offered by Achenbach [14] for the equivalence of the two velocities for wave in a plate under a simple assumption that the amplitude of the wave is independent of frequency ω and wave number $\mathbf{k}$ along a branch of the frequency spectrum. However, his proof is not valid for the backward waves.

In this paper, a general proof is presented to show the equivalence of the energy velocity and the group velocity for waves in an anisotropic homogenous plate and it is valid for the backward waves. The expressions for both the energy velocity and the group velocity are obtained in terms of the Lagrangian density.

2. LAGRANGIAN DENSITY

For acoustic waves in an anisotropic plate, the expression of Lagrangian density is given by

$$L = \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i - \tfrac{1}{2} C_{ijkl} u_{i,j} u_{k,l}, \tag{1}$$

where u_i is a displacement component of a particle, ρ is the mass density, and C_{ijkl} is the tensor of the elastic constants. The dot above the first term on the right side denotes the partial derivative with respect to time, the subscript comma of the second term represents the partial derivative with respect to position and any subscript takes values 1, 2, 3. The convention of a repeated alphabetic subscript in a term is used for summation. The first term and the second term on the right-hand side of equation (1) correspond to the kinetic energy and the deformation energy density respectively.

In order to show the equivalence of the energy velocity and the group velocity, the two volocities are expressed in terms of the Lagrangian density. For this purpose, we need some relations. First, the wave field must satisfy the equations of motion and the boundary conditions of free surfaces. Using Lagrangian density (1) and applying the theorem of variation calculation to Hamilton's principle [15], we obtain the equations of motion given by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_i} + \frac{\partial}{\partial x_j}\frac{\partial L}{\partial u_{i,j}} = 0. \tag{2}$$

Assuming that the plate is perpendicular to x_3 direction and that the surfaces are at $x_3 = \pm h$, we have the traction-free conditions on the surfaces of a plate given by

$$\frac{\partial L}{\partial u_{j,3}} = 0, \quad x_3 = \pm h. \tag{3}$$

When the wave in a waveguide is a combination of longitudinal waves and transverse waves which is the case of the backward waves, the amplitude of waves depends on the frequency ω or the wavenumber components k_i. Here we consider the displacement field in the general form

$$u_i = f_i(\omega, k_1, k_2, x_3, \theta), \tag{4}$$

where $\theta = \omega t - k_j x_j, j = 1, 2$ is a phase function. Furthermore, we assume that u_i and its derivatives are all continuous periodic functions in θ with a period P. The components u_i are not necessary in phase. Equation (4) gives solutions of equations (2) and (3) only if the dispersion equation is also satisfied. An important relation is the generalized dispersion equation in the form of the vanishing of the mean Lagrangian density.

3. GENERALIZED DISPERSION EQUATION

We are interested in a linear homogeneous and conservative lossless system. For such a system, a wave field is usually of periodicity or is a superposition of periodic wave. A linear periodic dynamic system that obeys Hamilton's principle has an important property that the mean time–space Lagrangian density is zero. This property is obvious for a classical system such as a harmonic oscillator. The vanishing of the mean Lagrangian density is the key to our method so for certainty we shall show that for acoustic waves in a plate the mean Lagrangian density is zero. The averaged Lagrangian density over time and space can be expressed as

$$D = \langle L \rangle = \frac{1}{2hP}\int_0^P \mathrm{d}\theta \int_{-h}^{+h} L(\omega, k_i, x_3, \theta)\,\mathrm{d}x_3, \quad i = 1, 2. \tag{5}$$

First, it is easy to show that the first variation δD vanishes if δu_i has the same periodicity as u_i itself even if δu_i does not vanish at the boundary. The first variation δD is given by

$$\delta D = \left\langle \frac{\partial L}{\partial \dot{u}_i}\delta \dot{u}_i + \frac{\partial L}{\partial u_{i,j}}\delta u_{i,j} \right\rangle. \tag{6}$$

As usual, this first variation can be rewritten in an alternative form by the commutation property of variation and differentiation. We have

$$\delta D = -\left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}_i}\right)\delta u_i + \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{i,j}}\right)\delta u_i \right\rangle + \left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}_i}\delta u_i\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{i,j}}\delta u_i\right) \right\rangle. \tag{7}$$

The first mean integration term vanishes because of the equations of motion, and the second term is zero if δu_i has the same periodicity as u_i and if the boundary conditions are satisfied. Therefore, the first variation of the mean Lagrangian density is zero.

Second, we use the direct definition of the variation. For simplicity, we write the mean Lagrangian density as a function of u_i

$$D = D[u_i], \tag{8}$$

so that the variation can be given as

$$\delta D = D[\bar{u}_i] - D[u_i], \tag{9}$$

where $\bar{u}_i = u_i + \delta u_i$ is an arbitrarily varying function. If we assume that $\delta u_i = \alpha u_i$, where α is an arbitrary parameter, since D is homogeneous and quadratic in the derivatives of u_i, the varied mean Lagrangian density can be given by

$$D[\bar{u}_i] = (1 + \alpha)^2 D[u_i] \tag{10}$$

and equation (9) becomes

$$\delta D = \{(1 + \alpha)^2 - 1\} D[u_i]. \tag{11}$$

Since δu_i has the same properties as u_i itself, the variation is zero as we have shown. Thus for any non-zero parameter α the mean Lagrangian density must be zero:

$$D[u_i] = D(\omega, k_i) = 0. \tag{12}$$

Equation (12) gives a relationship between the frequency and the wave number components and is often called dispersion equation. Here we should emphasize that the frequency and the wave number components in the phase function θ make no contribution to the dispersion equation since the average is performed over time and space. Equation (12) determines the frequency ω as an implicit function of wave numbers k_i, $i = 1, 2$.

4. THE CONCEPTS OF ENERGY VELOCITY AND GROUP VELOCITY

The expression of the energy velocity can be given by

$$\mathbf{V}_e = \frac{\langle \mathbf{F} \rangle}{\langle E \rangle}, \tag{13}$$

where **F** is the energy flux vector and E is the energy density. It is obvious that the energy velocity is of the dimension of velocity, and that it is a vector due to the energy flux vector. For a wave system specified by the Lagrangian density (1), the component of the energy velocity can be given by

$$V_{ej} = \frac{\langle (\partial L/\partial u_{i,j})\dot{u}_i \rangle}{\langle (\partial L/\partial \dot{u}_i)\dot{u}_i \rangle}, \quad j = 1, 2, \tag{14}$$

where in the denominator we have used the result that the mean Lagrangian density is zero.

The group velocity is also a velocity vector. The component of a group velocity is defined as the rate of change of frequency with the corresponding component of wave number

$$V_{gj} = \frac{\partial \omega}{\partial k_j}, \quad j = 1, 2. \tag{15}$$

5. THE PROOF

Now we shall express the group velocity in terms of the Lagrangian density. We can express the components of the group velocity by the implicit differentiation from the generalized dispersion equation (12) as

$$V_{gj} = -\frac{\partial D/\partial k_j}{\partial D/\partial \omega}. \tag{16}$$

Let us examine the partial derivative of D with respect to explicit ω. We can change the order of the differentiation and the integration with respect to θ by the continuity of the functions since θ can be regarded as independent variables and is continuous in ω and $k_i, i = 1, 2$. Thus, the derivative of D with respect to explicit ω becomes

$$\frac{\partial D}{\partial \omega} = \left\langle \frac{\partial L}{\partial \dot{u}_i}\frac{\partial \dot{u}_i}{\partial \omega} + \frac{\partial L}{\partial u_{i,j}}\frac{\partial u_{i,j}}{\partial \omega} \right\rangle. \tag{17}$$

Since we have the expressions

$$\frac{\partial \dot{u}_i}{\partial \omega} = \frac{\partial f_i}{\partial \theta} + \omega \frac{\partial}{\partial \omega}\left(\frac{\partial f_i}{\partial \theta}\right) = \frac{\dot{u}_i}{\omega} + \frac{\partial}{\partial t}\left(\frac{\partial f_i}{\partial \omega}\right), \tag{18}$$

substituting equation (18) into equation (17) and carrying out a transform we obtain

$$\begin{aligned} \frac{\partial D}{\partial \omega} = \frac{1}{\omega}\left\langle \frac{\partial L}{\partial \dot{u}_i}\dot{u}_i \right\rangle - \left\langle \frac{\partial f_i}{\partial \omega}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}_i}\right) + \frac{\partial f_i}{\partial \omega}\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{i,j}}\right) \right\rangle \\ + \left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}_i}\frac{\partial f_i}{\partial \omega}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{i,j}}\frac{\partial f_i}{\partial \omega}\right) \right\rangle. \end{aligned} \tag{19}$$

It is easy to see that the second term vanishes by the equations of motion; in the third term, the mean integrals with respect to t and with respect to x_i, $i = 1, 2$, vanish because of the periodicity and the mean integral with respect to co-ordinate x_3 vanishes by the boundary conditions. Thus, the derivative of D with respect to explicit ω becomes

$$\frac{\partial D}{\partial \omega} = \frac{1}{\omega}\left\langle \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \right\rangle. \tag{20}$$

From similar arguments, it can be shown that the partial derivative of D with respect to explicit k_j, $j = 1, 2$, is given by

$$\frac{\partial D}{\partial k_j} = -\frac{1}{\omega}\left\langle \frac{\partial L}{\partial u_{i,j}} \dot{u}_i \right\rangle. \tag{21}$$

Therefore, the components of the group velocity in equation (16) can be expressed as

$$V_{gj} = \frac{\langle (\partial L/\partial u_{i,j})\, \dot{u}_i \rangle}{\langle (\partial L/\partial \dot{u}_i)\, \dot{u}_i \rangle}, \quad j = 1, 2. \tag{22}$$

Comparison of equation (22) and the energy velocity (14) gives the required result.

6. DISCUSSION

The backward wave phenomenon has been found not only for elastic waves but also for electromagnetic waves [16]. The method used in this paper can be easily applied to electromagnetic waves as long as the wave is linear. For a linear electromagnetic wave, the electrical field strength and the magnetic induction can be derived from a scalar potential and a vector potential. The Lagrangian density of electromagnetic field in terms of the scalar and vector potentials is homogeneous and quadratic [17] so that the mean Lagrangian density can be easily shown to be zero by using the approach given in section 3, which gives a generalized dispersion equation. The expression of the group velocity derived from the generalized dispersion equation can be shown to be the same as that of the energy velocity.

So far, the conditions for the existence of a backward wave are still not very clear in general cases and physical understanding of backward waves remains to be clarified. It is hoped that this paper will encourage further studies on the backward waves.

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