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MODES OF CONTACT AND UNIQUENESS OF SOLUTIONS FOR SYSTEMS WITH FRICTION-AFFECTED SLIDERS

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The kinetic equations of planar multi-body systems with friction-affected sliding joints are reformulated for the computation of closed-form solutions for the kinetic parameters. The state of such systems is characterized not only by the position parameters and velocities, but in addition, the modes of contact in the sliding joints must be specified. Then the cases with one or several sets of solutions, obtained for the same position parameters, velocities, active forces and friction parameters, can be related to positions of the system with different modes of contact between sliders and guiding surfaces. They are physical unequivocal states and can be interpreted as unique solutions for the kinetic problem with specified configuration of the system. If no solutions exist, then the friction parameters considered are too large and exceed limiting values, for which friction locking occurs.

1. INTRODUCTION

Multi-body systems with determinate kinetic structure consisting of rigid parts and having constraints with friction are considered. The equations for the computation of the kinetic parameters, i.e. the generalized accelerations, the reaction forces in the constraints and the kinetic friction forces and torques, are obtained from conditions of kinetic equilibrium for the parts of the system, which are linear equations in the unknowns, in addition to the kinetic friction laws which are non-linear equations in the unknowns. These non-linear equations have significant consequences on the computation and interpretation of the solutions [1].

Whilst for systems without friction, when, for given active forces, position parameters and velocities, one set of solutions is always obtained, for systems with friction cases without, with one or with several sets of solutions can occur. This particularity for systems with friction-affected constraints has been pointed out by Painlevé [2] and is known as Painlevé's paradox (see for example, reference [3]). Related to this, if no solutions are obtained the mechanical model appear to fail, whilst, if several sets of solutions are obtained, this appears to indicate ambiguity and inconsistency in the kinetic problem. For more information on this matter, see, for example, reference [4].

For interpretation of the results for systems with friction it is necessary to know whether solutions exist or not, and, if solutions exist, then all the sets of solutions have to be determined.

The kinetic friction forces and torques are active forces. This implies a feedback connection in the block-diagram representation of systems with friction, as shown in Figure 1. As a consequence of this interaction and the non-linear friction laws, in general, no

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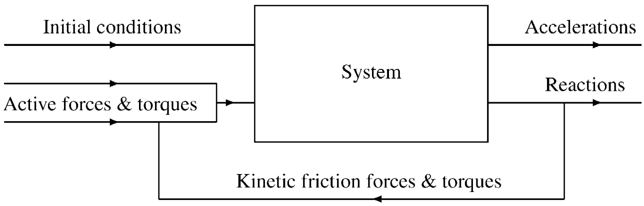

Figure 1. Block diagram representation of systems with friction-affected constraints.

closed-form solutions can be established (see, for example, reference [5]). In this paper, it is shown that for systems having only friction-affected sliding joints, closed-form solutions can be determined. Therefore, a planar system with friction-affected sliders is considered here.

The kinetic equations are reformulated, and the equations for the computation of the unknowns are put in the form of a linear algebraic system suitable for the application of a trial-and-error method. The coefficients and the absolute terms of these equations contain the position parameters and velocities, the solutions for the system without friction, friction parameters and quantities depending on the modes of contact in the sliding joints. The modes of contact define the possible orientations of the sliders within the guides, (see, for example, reference [6]). For guides without clearances, only contact along one side of the sliders can occur. In guides having small clearances contact at diagonally opposite corners can occur. The modes of contact are clearly specified by the signs of the normal reaction forces acting in the areas of contact. In general, it is *a priori* not known which modes of contact can occur. Therefore, the signs of the normal reaction forces are considered as trial assumptions in order to find the modes of contact, for which solutions exist.

The idealized concepts "rigid parts" and "sliding joints" of a multi-body system are applicable if the forces acting on the real bodies which they represent are not too large and the resulting deformations are negligible in comparison to other deformations and/or displacements relevant to the system. Therefore, the limits of applicability of the results are determined by reasonable absolute values of the accelerations and forces acting on the parts of the system.

The aims of this paper are to show that for system and friction parameter values leading to reasonable absolute values of the accelerations and forces, the solutions for the kinetic parameters represent physically meaningful unequivocal states, if the system is characterized not only by position parameters and velocities but also by the modes of contact in the sliding joints. If no solutions exist, the friction parameters considered are too large and exceed the limiting values, when friction-locking occurs.

2. DESCRIPTION OF THE SYSTEM

A multi-body planar system with n-degree-of-freedom consisting of b rigid parts, with r friction-affected sliding joints and p hinge joints without friction is considered. The active forces acting on the system are situated in the plane of motion. The generalized co-ordinates are denoted by $q = \{q_1, q_2, \ldots, q_n\}$. The linear and angular momenta of the parts and their rate of change can be expressed with the help of the generalized co-ordinates q, velocities $\dot{q} = \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$ and accelerations $\ddot{q} = \{\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_n\}$.

The normal contact forces at the sliders are distributed forces. When these forces are considered as concentrated forces, denoted by F_{Nj}, $j = 1, 2, \ldots, r$, and acting in the mass centres of the sliders, respectively, then, in addition, couples with moments $C_j, j = 1, 2, \ldots, r$, must be considered. The reaction forces components in the hinges, situated in planes perpendicular to the axis of the hinges, are denoted by H_j and V_j, $j = 1,2, \dots, p$.

The relative positions of the bodies connected at the sliders are determined by the lengths $s_j, j = 1,2, \dots, r$. These parameters can be expressed as functions of the generalized co-ordinates q, and the relative velocities $\dot{s}_j, j = 1,2, \dots, r$, are functions of q and $\dot{q}$.

3. KINETIC EQUATIONS

The *differential equations of motion* for the model can be obtained from the equations for the rate of change of linear and angular momenta established for every rigid part of the system (Newton–Euler's equations). In general, they can be presented in the form

$$\sum_{j=1}^{n} a_{ij}\ddot{q}_j + \sum_{j=1}^{r} (b_{ij}F_{Nj} + c_{ij}C_j) + \sum_{j=1}^{p} (d_{ij}H_j + e_{ij}V_j) = f_i(q,\dot{q},t) + \sum_{j=1}^{r} g_{ij}F_j, \quad i = 1,2,\dots,3b. \tag{1}$$

The coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij} and g_{ij} can be functions of the generalized co-ordinates q. When $3b = n + 2r + 2p$, the multi-body system obtains a determinate kinetic structure.

If *Lagrange's equations of the second kind* are used for establishing n of the kinetic equations (1), then these equations are free of the reactions F_{Nj}, $C_j, j = 1,2,\dots,r$, and H_j, V_j, $j = 1,2,\dots,p$. The other independent equations of system (1) are conditions of kinetic equilibrium for parts of the system.

For the kinetic friction forces F_j in the sliders, the following *idealized friction laws* are considered:

$$F_j = (\mu_j|F_{Nj}| + F_{j0})\operatorname{sign}(\dot{s}_j), \quad j = 1,2,\dots,r. \tag{2}$$

Here, μ_j are the kinetic friction coefficients in the sliders. The friction terms $F_{j0} \geqslant 0$ are due to pre-stressing or result from supplementary devices used to increase friction effects.

4. KINETIC EQUATIONS WITH INFLUENCE COEFFICIENTS

For systems with determined kinetic structures, equation (1) can be solved for the accelerations and reactions. These kinetic parameters are presented as linear functions of the kinetic friction forces F_k, $k = 1,2, \dots, r$, as follows:

$$\ddot{q}_i = \ddot{q}_{i(0)} + \sum_{k=1}^{r} \kappa_{ik}F_k, \tag{3}$$

$$F_{Nj} = F_{Nj(0)} + \sum_{k=1}^{r} \eta_{jk}F_k, \qquad C_j = C_{j(0)} + \sum_{k=1}^{r} \xi_{jk}F_k, \tag{4}$$

$$H_j = H_{j(0)} + \sum_{k=1}^{r} \alpha_{jk}F_k, \qquad V_j = V_{j(0)} + \sum_{k=1}^{r} \gamma_{jk}F_k. \tag{5}$$

Here, $\ddot{q}_{i(0)}$, $i = 1,2, \dots, n$, $F_{Nj(0)}$, $C_{j(0)}, j = 1,2, \dots, r$, and $H_{j(0)}$, $V_{j(0)}, j = 1,2, \dots, p$, are the accelerations and reactions corresponding to the system without friction.

The coefficients κ_{ik}, η_{jk}, ξ_{jk}, α_{jk}, and γ_{jk} depend on the mass distribution of the system and on the values of the generalized co-ordinates. These coefficients do not depend on the method applied for establishing the kinetic equations of the system.

The coefficients occurring in equations (3)–(5) characterize the weight of every friction-affected sliding joint and the influence of the kinetic friction forces on the values of the accelerations and reactions. These coefficients can be considered as *influence coefficients*.

In particular, the first equation of system (4) shows that the normal reaction F_{Nj} is influenced not only by the friction force F_j acting in the same slider, but all the friction forces $F_k, k = 1,2, \dots, r$, that contribute to the value of F_{Nj}. The influence coefficients η_{jk}, $j, k = 1,2, \dots, r$, are measures of these interactive influences. The coefficients η_{jj}, $j = 1,2, \dots, r$, have a particular meaning when systems with only one friction-affected sliding joint are investigated.

5. COMPUTATION OF THE SOLUTIONS

The kinetic equations can be presented in a form which suggests the application of a trial-and-error method for the computation of the solutions. Substituting equation (2) in the first equations of system (4) yields

$$\sum_{k=1}^{r} [\delta_{jk}\,\mathrm{sign}(F_{Nj}) - \eta_{jk}\mu_k\,\mathrm{sign}(\dot{s}_k)]|F_{Nk}| = F_{Nj(0)} + \sum_{k=1}^{r} \eta_{jk}F_{k0}\,\mathrm{sign}\,(\dot{s}_k), \quad j = 1,2, \dots, r, \quad (6)$$

where δ_{jk} is the Kronecker delta. The solutions $F_{Nj}, j = 1,2, \dots, r$, of the kinetic problem are also the solutions of the system of equations (6).

In reference [5], the equivalent equations for the same particular case are expressed in terms of the kinetic friction forces $F_j, j = 1,2, \dots, r$. Here, these equations are expressed in terms of the unknown normal reactions $F_{Nj}, j = 1,2, \dots, r$, in order to emphasize the role of these forces in the interpretation of the solutions.

The coefficients of equations (6) contain the quantities $\mathrm{sign}(F_{Nj}), j = 1,2, \dots, r$, which can have the values $+1$ or -1. Initially, the values of these quantities for which solutions exist are unknown. Therefore, with the two types of objects $+1$ and -1, ordered sets of trial assumptions

$$\{\mathrm{sign}(F_{N1}), \mathrm{sign}(F_{N2}), \dots, \mathrm{sign}(F_{Nr})\} \quad (7)$$

of size r, numbering 2^r in total, are established. They characterize the modes of contact in the sliding joints, i.e., the configuration of the system. Equations (6) are free from the accelerations indicating that the number of degrees of freedom of the system does not determine the number of sets of solutions that exist.

For every set of trial assumptions (7), equations (6) can be solved and the corresponding solutions of $|F_{Nj}|, j = 1,2, \dots, r$, can be determined. If all the solutions are positive, they are solutions of the problem and correspond to the given position parameters and velocities and to the set of trial assumptions considered; i.e., for the modes of contact in the sliding joints considered.

The requirement

$$|F_{Nj}| \geqslant 0, \quad j = 1,2, \dots, r, \quad (8)$$

for the solutions of equations (6) can be considered as a *selection criterion* for choosing, from among the sets of trial assumptions, those corresponding to the solutions of the kinetic problem. The obtained solutions are *closed-form solutions*.

The absolute values $|F_{Nj}|, j = 1,2, \dots, r$, which are the positive solutions of equations (6), and the corresponding set of trial assumptions (7) determine the reaction forces at the sliders $F_{Nj} = |F_{Nj}|\mathrm{sign}(F_{Nj}), j = 1,2, \dots, r$. Substituting into equations (2), the kinetic friction forces

$F_j, j = 1,2, \ldots, r$, are obtained and equations (3)–(5) yield the other unknowns of the problem.

The properties of the model concerning structure geometry, mass distribution and position parameters are characterized in the determinant $D = \det[d_{jk}]$ of the system of equations (6) by the influence coefficients $\eta_{jk}, j, k = 1,2,\ldots, r$. Together with the values of the friction coefficients $\mu_j, j = 1,2, \ldots, r$, they are determinative in the prediction, if the system with friction-affected sliders and the given modes of contact in the sliding joints has solutions for the kinetic state parameters. With the values of the friction coefficients tending to the roots of the equation $D = 0$, the solutions $|F_{Nj}| \to \infty, j = 1,2, \ldots, r$, and friction locking occurs.

For a system having only a sliding joint "j" with friction, the solutions can be determined from the equation

$$|F_{Nj}| = \frac{F_{Nj(0)} + \eta_{jj} F_{j0} \operatorname{sign}(\dot{s}_j)}{\operatorname{sign}(F_{Nj}) - \mu_j \eta_{jj} \operatorname{sign}(\dot{s}_j)}. \tag{9}$$

Only assumptions for $\operatorname{sign}(F_{Nj})$ and $\operatorname{sign}(\dot{s}_j)$ and values of μ_j yielding positive values of equation (9) correspond to the solutions of the problem. For this particular case, equation $D = 0$ yields $d_{jj} = \operatorname{sign}(F_{Nj}) - \mu_j \eta_{jj} \operatorname{sign}(\dot{s}_j) = 0$. The element d_{jj} is the denominator of equation (9). If it is of the form $\pm(1 - \mu_j |\eta_{jj}|)$, then with $\mu_j \to 1/|\eta_{jj}| = \mu_{jl}$ the solution $|F_{Nj}| \to \infty$ and kinetic friction locking occurs.

In the case of multiple frictional contacts, i.e., at least, only one of the influence coefficients η_{jk}, $j \neq k$, is different from zero, the determinant $D = \det[d_{jk}]$ has non-vanishing elements not situated on the main diagonal, i.e., $d_{jk} = -\eta_{jk} \mu_k \operatorname{sign}(\dot{s}_k) \neq 0$, $j \neq k$. In this case, the roots of equation $D = 0$ are different from the roots $\mu_j = \mu_{jl}$ of the equations $d_{jj} = 0, j = 1,2, \ldots, r$, obtained with the elements on the main diagonal and corresponding to systems with only one friction-affected slider. In this case, friction locking occurs for values $\mu_j \neq \mu_{jl}, j = 1,2, \ldots, r$.

6. EXAMPLE

For illustration, the planar system presented in Figure 2 is considered. A homogeneous bar of mass m, length l, angle of inclination α and mass centre S is hinged at the massless upper block 1 and lower block A. The blocks can slide along two horizontal slots. The bar is acted in S by the horizontal force P, by a linear elastic spring of stiffness k, undeformed in position $q = 0$, and by a viscous damper with the damping coefficient c. The scalar parameters determining the positions of the sliders are $s_1 = s_A = q$. The free body diagrams of the bar and blocks are presented in Figure 3, which also indicates the positive directions of the normal reactions, F_{N1}, F_{N2} and F_{N3}. The modes of contact of the upper block are full contact with the lower guiding surface, i.e., $F_{N1} > 0$, and full contact with the upper guiding surface, i.e., $F_{N1} < 0$. For the lower block the modes of contact are full contact with the lower guiding surface ($F_{N2} > 0$, $F_{N3} < 0$), full contact with the upper guiding surface ($F_{N2} < 0$, $F_{N3} > 0$) and, in addition, two diagonal tipping modes, $F_{N2} > 0$, $F_{N3} > 0$ and $F_{N2} < 0$, $F_{N3} < 0$. For negative values of the normal reactions F_{N1}, F_{N2} and F_{N3}, the direction lines of the friction forces F_{N1}, F_2 and F_3 change.

The kinetic parameters, obtained from conditions of kinetic equilibrium of the forces in Figure 3, are presented with the notations in equations (3) (without index "i") and the first of equations (4) (with $r = 3$) as follows: $\ddot{q}_{(0)} = (P - kq - c\dot{q})/m$, and $\kappa_1 = \kappa_2 = \kappa_3 = -1/m$. For the normal reactions the values $F_{N1(0)} = \frac{1}{2}mg$, $F_{N2(0)} = -F_{N3(0)} = \frac{1}{4}mg$, $\eta_{11} = -\eta_{12} = -\eta_{13} = -\frac{1}{2}\tan(\alpha)$, $\eta_{21} = \frac{1}{4}\tan(\alpha)$, $\eta_{22} = (h/2b)\operatorname{sign}(F_{N2}) - \frac{1}{4}\tan(\alpha)$,

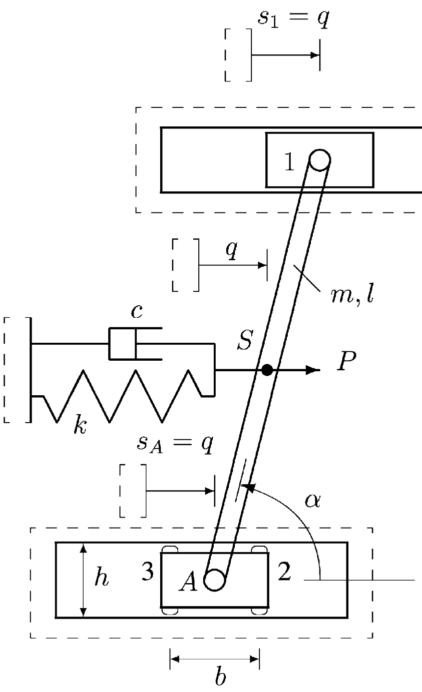

Figure 2. A planar system with two friction-affected sliding joints.

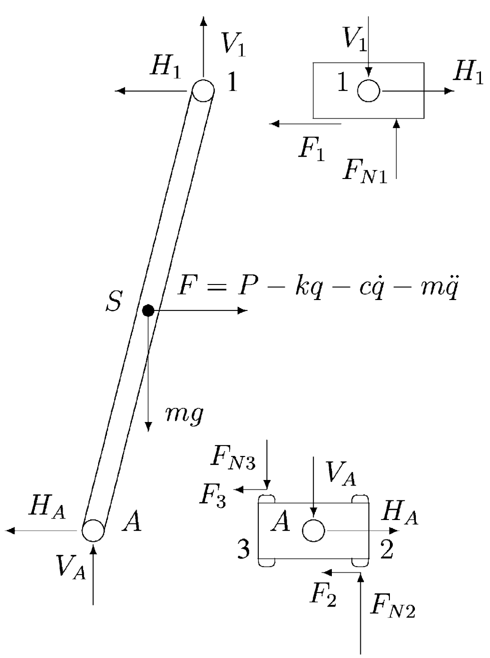

Figure 3. The free body diagrams of the system in Figure 2 indicating the positive directions of the normal reactions.

$\eta_{23} = -(h/2b)\,\mathrm{sign}(F_{N3}) - \frac{1}{4}\tan(\alpha)$, $\eta_{31} = -\frac{1}{4}\tan(\alpha)$, $\eta_{32} = (h/2b)\,\mathrm{sign}(F_{N2}) + \frac{1}{4}\tan(\alpha)$, and $\eta_{33} = -(h/2b)\,\mathrm{sign}(F_{N3}) + \frac{1}{4}\tan(\alpha)$ are obtained. For a given position parameter q, velocity $\dot{q}$ and active force P, the force $F = F_1 + F_2 + F_3 = P - kq - c\dot{q} - m\ddot{q}$ yields the acceleration $\ddot{q}$.

7. NUMERICAL RESULTS

For numerical simulation the values $\alpha = 80°$, $b/l = 0{\cdot}2$, $h/l = 0{\cdot}1$, $\mu_1 \in [0,\ 0{\cdot}8]$, and $\mu_2 = \mu_3 = 0{\cdot}1$ are considered. The determinant of the system of equations (6) corresponding to this case has the root $\mu_1 = \mu_1^*$ between 0·4526 and 0·4527. The solutions obtained for the normal reaction forces $F_{Nj}, j = 1, 2, 3$, with positive velocities are denoted by $F_{Nj}^{(+)}$ and with negative velocities by $F_{Nj}^{(-)}$.

The numerical results can be verified with the help of kinetic equilibrium conditions; for example, with the force equations: $F - F_1 - F_2 - F_3 = 0$, $F_{N1} + F_{N2} - F_{N3} - mg = 0$, and the moment equations about point A: $(F_1 - \frac{1}{2}F)\,l\sin(\alpha) + (F_{N1} - \frac{1}{2}mg)\,l\cos(\alpha) = 0$ and $\frac{1}{2}(F_{N2} + F_{N3})b - \frac{1}{2}[F_2\ \mathrm{sign}(F_{N2}) - F_3\ \mathrm{sign}(F_{N3})]h = 0$.

Figure 4 presents the results obtained with zero friction terms, i.e. $F_{10} = F_{20} = F_{30} = 0$, and with $\dot{q} > 0$. For $\mu_1 \leqslant 0{\cdot}4526$ unique solutions and for $\mu_1 \geqslant 0{\cdot}4527$, two sets of solutions are obtained. Reasonable solutions exist for all the values of μ_1 considered for the modes of contact characterized by $\mathrm{sign}(F_{N1}) = +1$, $\mathrm{sign}(F_{N2}) = +1$ and $\mathrm{sign}(F_{N3}) = -1$, i.e., the upper and the lower sliders are in contact with the corresponding lower guiding surfaces. These solutions are denoted by $F_{Nj(1)}^{(+)}$, $j = 1,2,3$.

In addition, for $\mu_1 \geqslant 0{\cdot}4527$ solutions with decreasing absolute values exist, for which the upper slider is in contact with the upper guiding surface and the lower slider touches the lower guiding surface. These solutions are denoted by $F_{Nj(2)}^{(+)}, j = 1, 2, 3$. If the upper limits for the absolute values of the reaction forces are established due to rigidity considerations, then the domain of μ_1-values which yield excessive absolute values, $[\mu_1^*, \mu_{1a}]$, must be

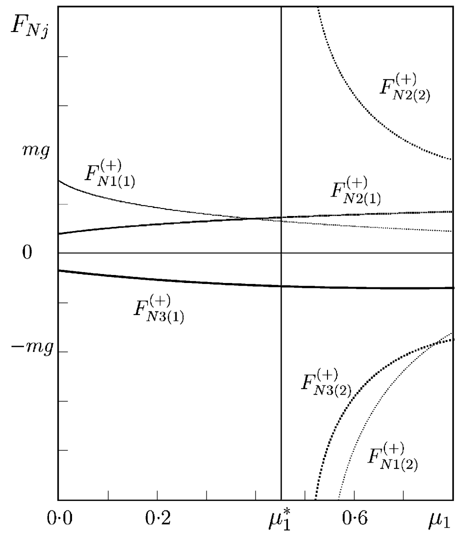

Figure 4. The solutions $F_{Nj}^{(+)}, j = 1, 2, 3$, of the kinetic problem for increasing values of the friction coefficient μ_1, with $\mu_2 = \mu_3 = 0{\cdot}1$, $F_{j0} = 0, j = 1, 2, 3$, and positive velocities.

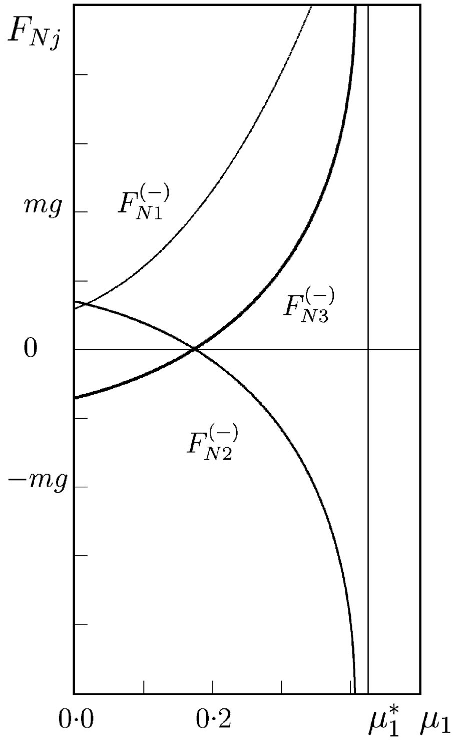

Figure 5. The solutions $F_{Nj}^{(-)}, j = 1, 2, 3$, of the kinetic problem for increasing values of the friction coefficient μ_1, with $\mu_2 = \mu_3 = 0{\cdot}1$, $F_{j0} = 0, j = 1, 2, 3$, and negative velocities.

excluded. Thus, for this configuration of the system only solutions corresponding to $\mu_1 > \mu_{1a}$ are meaningful.

The two sets of solutions occurring for $\mu_1 \geqslant 0{\cdot}4527$ and accepted for $\mu_1 > \mu_{1a}$ do not indicate ambiguity or inconsistency of the mechanical model, which takes into account friction in the sliding joints, but they correspond to two different modes of contact of the sliders, i.e., for two different configurations of the system.

For example, with $\mu_1 = 0{\cdot}6$, the first set of solutions is $F_{N1(1)}^{(+)} = 0{\cdot}2625mg$, $F_1^{(+)} = 0{\cdot}1575mg$, $F_{N2(1)}^{(+)} = 0{\cdot}3872mg$, $F_2^{(+)} = 0{\cdot}0387mg$, $F_{N3(1)}^{(+)} = -0{\cdot}3503mg$, $F_3^{(+)} = 0{\cdot}0350mg$, and $F^{(+)} = 0{\cdot}2313mg$. The second set of solutions is $F_{N1(2)}^{(+)} = -1{\cdot}8754mg$, $F_1^{(+)} = 1{\cdot}1252mg$, $F_{N2(2)}^{(+)} = 1{\cdot}5096mg$, $F_2^{(+)} = 0{\cdot}1510mg$, $F_{N3(2)}^{(+)} = -1{\cdot}3658mg$, $F_3^{(+)} = 0{\cdot}1366mg$, and $F^{(+)} = 1{\cdot}4128mg$. These numerical results can be easily proved.

Figure 5 shows the results obtained with zero friction terms and $\dot{q} < 0$. In the domain $\mu_1 \in [0, 0{\cdot}1763]$, solutions exist for the modes of contact sign$(F_{N1}) = +1$, sign$(F_{N2}) = +1$, sign$(F_{N3}) = -1$, with increasing values of $F_{N1}^{(-)}$ and with $F_{N2}^{(-)}$ and $F_{N3}^{(-)}$ tending to zero. In the domain $\mu_1 \in [0{\cdot}1764, 0{\cdot}4526]$, solutions exist for the modes of contact sign$(F_{N1}) = +1$, sign$(F_{N2}) = -1$, sign$(F_{N3}) = +1$, with increasing absolute values of $F_{Nj}^{(-)}, j = 1, 2, 3$. From rigidity considerations, a limit value of $\mu_1 < \mu_1^*$ denoted by μ_{1b}, ensuring reasonable absolute values for the reaction forces, can be established and values of $\mu_1 \in (\mu_{1b}, \mu_1^*)$ must be excluded. In the domain $\mu_1 > \mu_1^*$, no solutions are obtained for any of the possible modes of contact. In this case, the system is jammed and kinetic friction locking occurs.

With $\mu_1 < \mu_{1b} < \mu_1^*$ the system passes position q in the positive direction in the configuration $\text{sign}(F_{N1}) = +1$, $\text{sign}(F_{N2}) = +1$, $\text{sign}(F_{N3}) = -1$. In the negative direction and for $\mu_1 \leqslant 0{\cdot}1763$, the system passes position q in the same configuration, and for $\mu_1 \in [0{\cdot}1764, \mu_{1b}]$ in the configuration $\text{sign}(F_{N1}) = +1$, $\text{sign}(F_{N2}) = -1$, $\text{sign}(F_{N3}) = +1$. With $\mu_1 > \mu_{1a} > \mu_1^*$, only motion in the positive direction is possible. For motion in the negative direction the system is jammed. For an oscillating system, motions in both directions are expected. Therefore, for the system considered in Figure 3, only values of $\mu_1 < \mu_{1b}$ allow oscillatory motions.

With friction terms $F_{10} = 0$, $F_{20} = mg$, and $F_{30} = 0$ and $\mu_2 = \mu_3 = 0.1$, in the domain $\mu_1 \in [0, 0{\cdot}4526]$, as well for positive as for negative velocities, only one set of solutions exists. In the domain $\mu_1 \in [0{\cdot}4527, 0{\cdot}4674]$, two sets and with $\mu_1 \geqslant 0{\cdot}4675$, three sets of solutions are obtained for positive velocities. For negative velocities and $\mu_1 \geqslant 0{\cdot}4527$, two sets of solutions exist.

In particular, with $\mu_1 = 0{\cdot}5$ and $\dot{q} > 0$ the following three sets of solutions are obtained, which correspond to three different configurations of the system. The first set of solutions is $F_{N1}^{(+)} = 1{\cdot}4357mg$, $F_1^{(+)} = 0{\cdot}7178mg$, $F_{N2}^{(+)} = 0{\cdot}0213mg$, $F_2^{(+)} = 1{\cdot}0021mg$, $F_{N3}^{(+)} = 0{\cdot}4570mg$, $F_3^{(+)} = 0{\cdot}0457mg$, and $F^{(+)} = 1{\cdot}7657mg$. The upper slider touches the corresponding lower guiding surface and the lower slider is in a diagonal contact mode, right below and left above. The second set of solutions is $F_{N1}^{(+)} = 1{\cdot}4408mg$, $F_1^{(+)} = 0{\cdot}7204mg$, $F_{N2}^{(+)} = -0{\cdot}4814mg$, $F_2^{(+)} = 1{\cdot}0481mg$, $F_{N3}^{(+)} = -0{\cdot}0406mg$, $F_3^{(+)} = 0{\cdot}0041mg$, and $F^{(+)} = 1{\cdot}7726mg$. The upper slider touches the corresponding lower guiding surface and the lower slider is in a diagonal contact mode, right above and left below. The third set of solutions is $F_{N1}^{(+)} = -26{\cdot}9574mg$, $F_1^{(+)} = 13{\cdot}4787mg$, $F_{N2}^{(+)} = 14{\cdot}9276mg$, $F_2^{(+)} = 2{\cdot}4928mg$, $F_{N3}^{(+)} = -13{\cdot}0298mg$, $F_3^{(+)} = 1{\cdot}3030mg$, and $F^{(+)} = 17{\cdot}2745mg$. The upper slider touches the corresponding upper guiding surface and the lower slider is in contact with the lower guiding surface. The kinetic equilibrium of these systems of forces can also be easily proved.

8. CONCLUSIONS

From the investigation of the kinetic problem of a multi-body planar system with determinate kinetic structure and friction-affected sliders, the following facts emerge:

(1) The equations for the computation of the absolute values of the normal reactions in the sliders can be presented in the form of a linear system of algebraic equations.
(2) Closed-form solutions for the kinetic parameters can be determined with the help of a trial-and-error method.
(3) The possible sets of solutions correspond to the given position parameters, velocities, active forces, friction parameters and to the specified modes of contact in the sliding joints. Every set of solutions can be assigned to a physically meaningful unequivocal state of the system.
(4) For particular values of the kinetic friction coefficients and particular modes of contact in the sliding joints, solutions with infinite absolute values of the kinetic parameters occur (kinetic friction locking), whereas, for the same friction coefficients and other modes of contact, reasonable values for the kinetic parameters are obtained.
(5) Only solutions with reasonable absolute values of the kinetic parameters are meaningful.
(6) If no solutions of the problem exist, then the kinetic friction coefficients considered are extremely large and exceed limiting values, for which kinetic friction locking occurs.

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