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NON-LINEAR VIBRATIONS OF SHELL-TYPE STRUCTURES: *A REVIEW WITH BIBLIOGRAPHY*

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The aim of this paper is to provide a contemporarily relevant survey of studies on non-linear vibrations of shell-type structures. The effects of geometrical non-linearity, and specific difficulties encountered in non-linear dynamic analysis of shell-type structures are presented and discussed. Studies on non-linear vibrations of shells are categorized by different shell configurations (shapes) in a chronological order. Also, the most commonly used methods of modelling and solution are reviewed and commented. Published reviews on non-linear vibrations of shell-type structures including complicating effects of anisotropy, initial stress, added mass, elastic foundation, stiffeners, open geometry (singly and doubly curved), transverse shear deformations, torsion, and interaction with fluid are also surveyed. Comments on the previous non-linear works are presented and some orientations for future research are suggested. Another purpose of this paper is to provide engineers, scientists and researchers with a list of 175 references, which should be very useful for locating relevant existing literature quickly.

1. INTRODUCTION

Problems related to the vibration of shell-type structures are encountered in many branches of industry, including aeronautical engineering, ocean engineering, and civil engineering [1, 2]. The non-linear vibration of thin circular cylindrical shells is of special interest in aerospace (design of rocket and launch vehicle structures) [3, 4], in which the structures must have a weight as low as possible and a strength as high as possible, and hence, may exhibit large amplitudes of vibration. According to the linear theory of vibration, the natural frequencies and mode shapes are independent of the amplitude of vibration. In many instances, if the amplitude of vibration is large,† such a result will not be justified, due to one or another non-linear effect. In general, the interest in the vibration of non-linear systems centres on geometrically non-linear vibration occurring at large displacement amplitudes, which leads to non-linear strain–displacement relationships.

The first available review on non-linear vibrations of shells was made by Evensen in 1974 [5]. He presented the early developments concerning isotropic circular cylindrical shells

†The large displacements mentioned here and elsewhere need only be of the order of the shell thickness for the non-linear effects to be significant. That is, they do not need to be truly large (many times the shell thickness).

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during the years from 1955 to 1971. An excellent monograph by Leissa [6] deals mostly with linear vibrations of shells, but also includes some references on large vibration amplitudes of circular cylindrical shells. This monograph provides a wealth of information on linear dynamic problems of circular cylindrical shells (approximately 500 references), but only about 25 references dealing with the problem of non-linear vibration of shells, covering the period from 1955 to 1970. Most of the monograph was devoted to the analysis of closed circular cylindrical shells having various boundary conditions, cutouts, effects of added mass, anisotropy, variable thickness, initial stress, and other complicating factors. However, a section was written on the non-linear effects resulting from large displacements. Sathyamorthy and Pandalai [7, 8] presented a review of existing literature in the area of large vibration amplitudes of plates and shells in a series of two papers. Part I of the series [7] contains a survey of vibrations of disks, membranes and rings. Also, it included information on simple non-linear systems in order to introduce the reader to non-linear dynamic problems. Non-linear vibrations of plates and shells have been surveyed in the second part [8]. These review papers, however, were mainly confined to cases with geometric-type non-linearity. In another paper authored by Leissa [9], an overview of the problem of non-linear vibrations of plates and shells reviewing the literature on the subject for the period from 1978 to 1983, was presented. The scope of this overview has been restricted to free undamped vibrations and listed 17 references. Recently, Qatu [10] reviewed the development of research into vibration of shallow shells. Specific attention was given to laminated composite shallow shells with complicating effects. More recently, a review article with bibliography documents, focussed on recent developments in the vibration analysis of thin, moderately thick, and thick shallow shells, has been presented by Liew *et al.* [11]. The studies devoted to moderately thick shells, incorporating the effects of transverse shear deformation and rotary inertia, have been reviewed in detail. However, it can be seen from the review that the studies concerned with the non-linear effects induced by large displacement amplitudes on the vibrations of thin shells have not been given sufficient attention. The author reviewed and presented in the bibliography only about 15 references dealing with non-linear free vibrations of shells. More recently, in a series of papers presented by Amabili *et al.* [12, 13], it is stated that "A full literature review of work on the non-linear dynamics of shells in vacuo and filled with fluid or surrounded by quiescent fluid has been given by Amabili *et al.*, … ." This statement concerned a previous paper by the same authors [14]. However, it may be noticed that this review was restricted only to closed circular cylindrical shells, excluding the studies of other configurations and complications which may be encountered in non-linear vibrations of shells. An addendum has been published by Amabili *et al.* [15] in order to complete the literature review of closed cylindrical shells, in which eight additional references have been mentioned. Generally, one may remark that only few references exist, in old as well as in new literature, which deal with non-linear vibrations of thin shell-type structures, compared with beams and plates, for which numerous works on the various aspects of non-linear vibrations may be found (see for example Table 1 of reference [9], and recently references [16, 17]).

The present review can be considered as a complement to the works of Evensen [5], Leissa [6], Sathyamoorthy [7, 8], Leissa [9], Qatu [10], Liew *et al.* [11], and Amabili *et al.* [14, 15]. Attention will be given to the works regarding non-linear vibrations of shell-type structures, and also to the methodological approaches used in the solutions. The papers dealing with initial imperfections, transverse shear deformation, anisotropy and orthotropy, stiffness, effects of thickness, and other complicating factors are also introduced. This review is structured as follows: In section 2, a qualitative description is given of the various effects which may be induced by the non-linearity in structural dynamic behaviour. In section 3, a discussion of the specific difficulties encountered in the analysis of shell-type structures, both linear and non-linear compared with other structures, such as beams and plates, is made. This discussion is reinforced by numerous references to the difficulties mentioned by many researchers having worked in this field. In section 4, a survey of non-linear vibrations of closed thin and thick shells is presented for different geometries. The influence of complicating factors is also introduced in this section. Studies on isotropic open shells, singly and doubly curved, are reviewed in section 5. Investigations of non-linear vibrations of closed and open, composite shells, are the subject of section 6. Also, closed and open shells, interacting with a light medium, and with a dense fluid, are presented in section 7. The experimental investigations on non-linear vibration of isotropic and composite shells, in air or in interaction with a fluid, are grouped in section 8. Section 9 is devoted to the discussion of various aspects of this complicated problem with comments. Finally, section 10 presents some concluding remarks and suggestions concerning some possible orientations for future research in the field of non-linear vibrations of shell-type structures.

2. SOME SPECIFIC FEATURES OF THE GEOMETRICALLY NON-LINEAR BEHAVIOUR OF SHELLS

One of the most fascinating features encountered in the study of non-linear vibrations in general is the occurrence of new and totally unsuspected phenomena. New, in the sense that the phenomena are not predicted, or even hinted at, by linear theory. On the other hand, the understanding of many experimental observations cannot even be attempted if the non-linearity present in this system is not taken into account. Among these new facts, one may mention [5, 18–42]:

- The variation of the resonant frequencies with the amplitude of vibration [5, 13, 14, 18–24].
- The amplitude dependence of the mode shapes [18–24].
- The jump phenomenon, and its corrresponding multi-values region in the non-linear frequency response curve [25–27].
- The harmonic distortion of the non-linear response to harmonic excitation, and its spatial distribution [13, 22, 28–30].
- The shift to the right of the non-linear random frequency response curves [31–33].
- The internal resonance [34–38].
- The occurrence of sub- or super-harmonic response phenomena [27].
- The occurrence of chaotic vibration [13, 26, 27].
- The existence of bifurcation points [26, 27, 34].
- The coupling, due to the non-linearity, between transverse and in-plane displacements (see reference [39] for plate case, and references [40, 41] for the shell case).
- The participation of the companion mode, in addition to the driven and axisymmetric modes, in the non-linear forced response of shells [5, 13, 25, 29].
- Etc.

Determination of the modal characteristics for free and forced vibrations of shell-type structures, including or not complicating factors, is a problem of great technical interest. The majority of the analytical studies to date have been formulated within the framework of linear, small deflection, structural theories, as mentioned above. In many cases, however, the linear analysis is found to be insufficient to explain and describe the behaviour of the physical system adequately. Hence, the non-linear effects have to be taken into account in the analysis, and also, in the design process [42]. Due to high acoustic loads and severe thermal environment, the structural response is often non-linear and requires improved mathematical models for dynamic stress and fatigue life prediction [43]. For example, aircraft panels excited at high sound pressure levels exhibit a pronounced non-linear behaviour [44]. Also, recent years have witnessed an increasing use of new materials, such as composites, and an increasing demand for more appropriate design principles, satisfying the new performance requirements in future aerospace vehicles. On the other hand, the theoretical progress realized in the last decades in the development of analytical and numerical investigation tools, reinforced by the high performance of new computing systems, enables engineers to explore use of advanced materials for shell structures, via the establishment of adequate criteria design.

The modelling and simulation of the behaviour of complex aerospace structures are perhaps the more challenging shell analysis tasks to date. Following the space shuttle Challenger accident, the definition of large-scale non-linear analysis changed as a result of the analyses performed on the solid rocket boosters. New design for the space shuttle external tank and other cryogenic fuel tanks for hypersonic vehicles have also challenged the shell analyst [45].

The mode shapes are of particular interest in the dynamic behaviour of a structure since the axial and bending strains are dependent upon the first and second derivatives of the mode shapes. Therefore, accurate prediction methods are needed to determine, at large vibration amplitudes, the non-linear mode shapes and the corresponding resonance frequencies of shell-type structures. Moreover, the investigation of the geometrically non-linear vibrations of shells is intended to give not only useful information about the non-linear frequencies and mode shapes, but also to lead to interesting indications on the dangerous zones where the stresses (axial and bending) are concentrated. This is due to the fact that the distribution of these stresses at large vibration amplitudes may be completely different quantitatively as well as qualitatively from that obtained in linear theory. This important fact has been examined recently in some shell cases by Moussaoui *et al.* [40, 41, 46, 47]. On the other hand, in the view of the increasing recourse in engineering to modal testing techniques, it can be noticed that qualitative description of the non-linear behaviour can be very useful in understanding data provided by modal testing, and can open the way to the development of more appropriate modal testing models, taking into account the non-linear effects.

As a conclusion, it appears in the light of these examples, that analysis of the non-linear effects will play an important role in the coming years in design and engineering, and must be included in the mathematical models of shell vibration, in order to know how far the dynamic characteristics of real, modern, flight structures, deviate from those predicted via the linear theories.

3. DIFFICULTIES ENCOUNTERED IN GEOMETRICALLY NON-LINEAR ANALYSIS OF SHELL-TYPE STRUCTURES

The task described at the end of the above section is not an easy one. As outlined in reference [9], the subject of non-linear vibrations has always been a difficult one, since many of the solution characteristics, such as existence, uniqueness and superposition, which are guaranteed for linear vibration problems, are not guaranteed, and often not valid, in the non-linear case. The source of non-linearity may be (1) material, i.e., due to non-linear stress strain relationships; (2) geometrical, i.e., due to the stretching stresses induced by large displacements of a structure, especially if it is restrained at its ends or edges; (3) inertial, in the case of structures having a concentrated or distributed mass; (4) or, due to non-linear boundary conditions such as a non-linear spring, etc. [27]. In the present work, only the geometrically non-linear behaviour is considered (sometimes combined with other complicating effects). Specifically, the non-linear effect is introduced through the inclusion of the second order terms in the strain–displacement relationships.

One more difficulty in problems regarding geometrically non-linear vibrations appears to be that the meaning of normal modes, which are powerful tools in linear vibration problems, becomes obscure in the non-linear case, since time and space variables are not separable in most non-linear vibration problems [48–54]. It is true that the concept of a non-linear mode shape is not absolutely clear and universally accepted like its linear equivalent. However, it is becoming quite familiar in the literature (see Nayfeh and Nayfeh's paper [55] and their references [1, 2, 9, 11, 14–16, 18–20]), is very useful for the qualitative understanding of the non-linear behaviour, and is expected to play an important role in the development of a "non-linear modal analysis theory". The importance of the concept of normal modes of vibration in the non-linear case has been extensively discussed in the introduction of reference [56].

In addition to the difficulties generally encountered in the analysis of the non-linear behaviour of structures with a simple geometry, such as beams or plates, the analysis of shells involves many typical difficulties. Among these difficulties, one may mention:

1. Generally, shells have all the characteristics of plates along with an additional one: curvature; the presence of curvature implies that, typically, bending cannot be separated from stretching in the shell case, which considerably complicates the analysis of their mechanical behaviour. The deformation of a shell can vary from purely extensional to purely flexural [57].
2. As a result of (1), the bending theory of shells is governed by an eighth order system of partial differential equations of motion, while the corresponding beam- or plate-bending theories are only of the fourth order.
3. As outlined in reference [57], the generality of the shell equations permits a wide variety of mode shapes with vastly different character. For example, some of the solutions which can be obtained from the equations describing a cylindrical shell are (a) the transverse vibration of tubular beams; (b) the longitudinal vibration of tubular beams, (c) the torsional vibration of tubular beams, (d) the flexural in-plane vibration of rings, (e) the extensional in-plane vibration of rings, and (f) vibration modes unique to shells [57].
4. An added complexity enters into the problem through the boundary conditions. This observation on the boundary conditions may be illustrated as follows: in the case of a rectangular plate which is simply supported along two of its opposite edges, the number of possible problems which can arise, considering all combinations of "simple" boundary conditions which can exist on the remaining two edges, is 10; while, for a cylindrical curved panel (i.e., a shell), the corresponding number is 136! [6]. The relative complexities of plate and shell vibrations are discussed in detail elsewhere [58].
5. A large number of different theories have been developed for shells. Therefore, there is no general agreement in the literature on the linear as well as the non-linear differential equations which describe the deformations of shells. Prominent among these are the theories of Donnell [59], Mushtari [60], Love [61], Timoshenko [62], Reissener [63], Naghdi and Berry [64], Vlasov [65], Sanders [66], Byrne [67], Flügge [68], Goldenveizer [69], Lur'ye [70], and Novozhilov [71]. The difference among the theories are due to the various assumptions made about the form of small terms and the order of terms which are retained in the analysis.
6. A surprising fact encountered by the newcomer to the subject of shell vibrations is that the fundamental (i.e., lowest frequency) mode of circular cylindrical shells typically

includes many sine waves around its circumference, contrary to the usual cases of beams and plates [6, 40, 41, 46, 47].

7. In the case of large vibration amplitudes, the non-linear behaviour of the flat plates and straight bars is usually of the hardening type (i.e., frequency increasing with vibration amplitude). For shells, it may be either hardening [13, 40, 41, 46, 47, 48, 72–83], softening [13, 14, 25, 29, 74, 76, 82, 84–90], or initial softening followed by hardening [74, 82, 89, 90]. The type of non-linearity of cylindrical shells depends on the shell geometrical characteristics, the mode wave numbers, the boundary conditions and, the amplitude of vibration. A specific discussion of this point is given in section 9.
8. In non-linear forced response of circular cylindrical shells, the deflection shape has sometimes to be expanded using many degrees of freedom, in particular, two asymmetric modes (driven and companion mode), plus an axisymmetric mode [14].

It should be also noticed that the observation made in reference [25], according to which "very few experimental studies have been devoted to non-linear vibration of shells" is still true to a great extent. Also, few experimental works reported in the literature correspond to special geometrical characteristics, boundary conditions, amplitudes of vibration, and mode wave numbers. In our opinion, this is still insufficient for clarifying completely the fascinating, very rich, and often surprising subject of the non-linear behaviour of shells.

4. CLOSED ISOTROPIC SHELLS

4.1. NON-LINEAR FREE AND FORCED VIBRATIONS

Many investigators from different fields have contributed to the development of various approaches to the problem of non-linear vibrations of shells. The first investigations on the effect of geometric non-linearity on the vibration of shells were initialized by Reissner in 1955 [91]. He assumed that the non-linearity has a more pronounced effect on the arbitrary time function, which modifies the choice of the deflected shape, than on the deflected shape itself. Hence, the modified shape was used also for the non-linear vibration problem. However, he concluded that in contrast to the linear case in which the chessboard deflection pattern is a natural choice, its selection in the non-linear problem must be more carefully assessed [75]. It may be worth noticing here that the question of the choice of the deflection shape has been rediscussed recently in reference [92]. Using Donnell's shallow-shell theory, Reissner isolated a single half-wave (lobe) of the vibration mode, and analyzed it for simply supported shells, and found that the non-linearity could be either of the hardening or softening type, depending upon the geometry of the lobe. Cummings [72] has shown that the governing equations, used by Reissner, depend on the area of integration. Subsequently, Chu [48] made a similar analysis which led him to an equation of the Duffing type, and gave results indicating that the non-linearity was of the hardening type, for a circular cylindrical shell. This has been confirmed by Nowinski [73] who concluded that the non-linear effects are found to be considerably less manifest in cylinders than in the corresponding flat plates. Evensen [93] performed a series of experiments in which shells were subjected to vibration amplitudes up to three or four times the wall thickness and observed that the non-linearity was of the softening type and that the vibrations were only slightly non-linear. This observation has led him to re-examine Chu [48] and Nowinski's [73] analyses, according to which the shell non-linear behaviour is of the hardening type. Using the choice of functions proposed in reference [48] for the transverse displacement of the shell w, and for the stress function F, it appeared impossible to satisfy the constraint that the mid-plane circumferential displacement v be continuous and single curved. After some preliminary investigations, Evensen [93] showed that the deflection function should be given the form

$$w(x, y, t) = A(t)\sin(m\pi x/L)\cos(ny/R) - (n^2 A^2(t)/4R)\sin^2(m\pi x/L). \quad (1)$$

The term $(-n^2 A^2(t)/4R)\sin^2(m\pi x/L)$ was added so that the solution satisfies the periodic continuity condition on the circumferential displacement v, i.e.,

$$v(x, y, t) = v(x, y + 2\pi R, t). \quad (2)$$

Subsequently, Olson observed a softening non-linearity in a series of experiments [84]. In a later work, Evensen and Fulton [74] presented a study of the non-linear dynamic response of thin circular cylindrical shells, which was an extended version of Evensen's original investigations. They found that the non-linearity may be either hardening or softening depending upon the ratio of the number of axial waves to the number of circumferential waves, although the shear diaphragm boundary conditions were not exactly satisfied at the shell ends in their analysis. The theoretical results were in a good agreement with an experiment performed by Olson [84]. Evensen [94] analyzed the free and forced non-linear vibrations of thin circular rings by assuming two vibration modes (driven and companion modes), and then applying Galerkin's procedure to the equations of motion. The analytical and experimental results exhibited several features that are characteristics of non-linear vibrations of axisymmetric systems in general, and of circular cylindrical shells in particular.

In another work, Evensen [95] derived two coupled equations for the non-linear vibration of thin-walled circular cylindrical shells: one for the driven mode, and the other for the companion mode. An approximate solution was given for the symmetric response, i.e., when the companion mode is not ready to vibrate, and for the coupled response, i.e., when the companion mode participates in the vibration. Using a variational approach (Rayleigh–Ritz method), Mayers and Wrenn [75] used the more complicated shell theory of Sanders to surmount the restriction on the utilization of the Karman–Donnel formulation to shell problems when the number of circumferential waves is small. They arrived at the conclusion that free vibration is non-periodic and is of the hardening type. The study did suggest that the investigation of more accurate shell theories beyond that of Donnell's may be of a considerable interest. Using the harmonic balance method without considering the companion mode, Evensen [76] extended his work to infinitely long cylindrical shells vibrating in three cases: constrained, extensional, and inextensional vibrations. Dowell and Venteres [78] derived a set of modal equations for non-linear flexural vibrations of cylindrical shells employing the Donnell shallow shell theory. However, no solution was presented for their modal equations. It was concluded however that the modal equations are accurate in the limits of $L/R \to \infty$ and $L/R \to 0$, unlike the previously available results, and that the method of "averaged in-plane boundary condition" generally yields good results.

The non-linear free vibration of circular cylindrical shells has been examined by Atluri [77], using Donnell's equations given by Chu [48], and Dowell and Ventres [78]. The Galerkin technique was used to reduce the problem to a system of coupled non-linear ordinary differential equations for the modal amplitudes. These equations have been solved using the multiple-time-scaling technique. The results obtained showed a pronounced non-linear hardening effect, which was due to the fact that the structure is effectively stiffer when the axial in-plane displacements are prevented. The steady state periodic forced response of cylindrical shells to transverse excitations has been studied by Ginsberg [86].

The investigation was performed by a perturbation procedure which retained all generalized co-ordinates appearing in the first approximation of the effects of non-linearity in the strain–displacement relationships. Bending effects and tangential inertia effects were also included. Large amplitude forced vibrations of simply supported thin cylindrical shells and the stability of the response have been studied by Ginsberg [87], who employed the Flügge-Lur'e-Byrne shell theory, in conjunction with a Lagrangian description of the deformation. The solution was based upon a perturbation technique and the method of harmonic balance.

Readers interested in the historical aspect of the development of mathematical formulations for non-linear vibrations of cylindrical shells may be referred to the tutorial discussion given by Evensen in reference [5]. Attention was given to geometrically non-linear free and forced vibrations of thin circular cylindrical shells, covering the period from 1955 to 1972. A non-linear parametric vibration study, for closed circular hinge-supported shells by a pulse load in the axial direction, has been presented by Volmir [96]. Based on the differential equations of motion, the author derived an algebraic equation describing the shell behaviour under parametric oscillations. Chen and Babcock [25] analyzed the large vibration amplitudes of a simply supported thin-walled cylindrical shell, using the perturbation method to reduce the governing non-linear differential equations into a system of linear equations, for solving the steady state forced vibration problem. The results indicate that the non-linearity may be either softening or hardening depending on the mode. Also, an experimental investigation was conducted, and the results were in qualitative agreement with the theory. Accounting for geometric non-linearity, small arbitrary initial imperfections, and using the strain–displacement relations of the Senders–Koiter non-linear shell theory, Radwan and Genin [79, 97] derived a set of non-linear modal equations for thin shells of arbitrary geometry. They used the method of assumed modes (method of modal expansion), and gave a numerical result, just for the coefficients of the Duffing equation which they obtained by solving the problem. It is interesting to note that most of the results indicated a hardening-type non-linearity. An application of this model has been made to simply supported circular cylindrical shells, based on the single-mode approach. The well-known Duffing equation was obtained and numerical results have been given just for the coefficients of the terms arising in this equation.

The first general formulation, based on the finite-element theory, was presented by Raju and Rao [80], to analyze large amplitude asymmetric vibrations of shells of revolution, using a curved shell element of 12 degrees of freedom. It is noted that the trends of non-linearity in the problems considered in this paper were of the hardening type. Using the Bubnov–Galerkin method, the non-linear vibration of a thin-walled elastic cylindrical shell of elliptic cross-section has been analyzed by Kozarov and Mladenov [98]. They gave the amplitude–frequency characteristics of shells, for various eccentricities of the shell cross-section.

The non-linear characteristics of free vibrations of conical shells, including both circular cylinders and annular plates as special cases, have been investigated by Ueda [88] by imposing an axisymmetric mode, which is the square of the asymmetric mode. Donnell-type shell theory was utilized to describe the shell, and trial functions along the generatrix were constructed by the finite-element method as a cubic piecewise polynomials. Bogdanovich and Feldmane [99] have proposed a procedure for non-linear parametric vibrations of a cylindrical shell in order to select the required number and form of the functions basis used to approximate its deflection.

Szeplinska-Stupnicka [49] attempted to generalize the idea of the "single non-linear mode method" to continuous systems, and to show that the "harmonic balance principle" can be regarded as a single-term solution in a generalized Ritz method, which is the method that minimizes the time integral involved in Hamilton's principle with respect to functions of spatial variables and a set of coefficients associated with boundary conditions. The essential feature of the proposed method is that, under the assumption of a harmonic solution, it allows not only the natural frequency, but also the mode shape of vibration, to be affected by the non-linearity and, as a result, to be amplitude dependent.

The free non-linear vibrations of a long cylindrical shell resting on an elastic foundation have been considered by Victor and Pongsan [100]. Using Donnell's shallow-shell theory, Varadan *et al.* [81] have reintroduced the mode-shape expansion that has been considered by Amabili [14] as a simple generalization of Evensen [95], in order to show that the axisymmetric term of Dowell and Ventres [78], and Atluri [77], gives hardening-type results. The modal equation was obtained by the Galerkin method, and solved by the fourth order Runge–Kutta method, to obtain the amplitude–frequency relationship.

Ramachandran [101] derived the von Karman-type equations for studying the non-linear vibrations of a cantilevered helicoidal shell. A series satisfying the geometric boundary conditions has been assumed for the transverse displacement and Galerkin's procedure was employed. Emphasis has been given to the fundamental mode and results have been presented only for this mode. Using a new asymptotic method, based on Bolotin's method for the linear case, Andrianov and Kholod [102] obtained an analytical solution, describing the non-linear oscillation of a shallow cylindrical shell.

Using a C_0 continuous, QUAD-4 shear flexible shell element, based on the field consistency principle, the non-linear free flexural vibration of thin circular cylindrical shells has been studied by Ganapathi and Varadan [82], without imposing any restriction on the mode shapes. Primarily, an attempt was made to clarify the existing controversies in the prediction of the non-linear behaviour of isotropic circular cylindrical shells through a dynamic response analysis, based on a finite-element formulation. The non-linear governing equations have been solved using a Wilson-θ numerical integration scheme with $\theta = 1{\cdot}4$. For each time step, modified Newton–Raphson iterations have been employed to achieve equilibrium at the end of that time step. However, no indication was given on the form of the vibration mode shapes. Also, no computation was made of the stress distributions, when the shell vibrates at various mode shapes order.

The large-amplitude forced vibrations of empty steel thin-walled silos have been investigated by Fernando and Godoy [103]. The basic geometry configuration modelled was a cylinder clamped at the bottom with a top conical roof. The instability was identified from finite-element computations of the time response of the shell using a criterion due to Budianski and Roth. Popov *et al.* [104] and McRobie *et al.* [105] presented two studies; one on the vibrations of cylindrical shells, parametrically exited by axial forcing, and another on the internal auto-parametric instabilities in the non-linear free vibrations of a cylindrical shell, using geometric averaging. A Rayleigh–Ritz discretization of the von Karman–Donnel equations was used, and led to the Hamiltonian and transformation into action-angle co-ordinates, followed by averaging, which provided readily a geometric description of the modal interaction. The dynamics of shallow shells, taking into account physical non-linearity, has been considered by Chinh [106]. More recently, Fu and Chen [107] presented an analysis of non-linear vibration of a truncated conical shell in large overall motion. The shell considered was elastic and moderately thick.

4.2. NON-LINEAR VIBRATIONS WITH COMPLICATING FACTORS

Although many publications deal with geometric-type non-linearity (see section 4.1), only a few papers are available which take into account, in addition to this type of non-linearity, other complicating effects.

Kildiberkov [108] investigated the non-linear vibrations of a thin circular cylindrical shell with initial imperfections. He studied also the influence of static forces and acoustic pressure on the observed non-linear vibrations. The effect of initial camber on the non-linear vibrations of cylindrical shells has been investigated analytically by Kubenko *et al.* [109], within the framework of geometrically non-linear theory. It was shown that initial camber results in splitting of the frequency spectrum and natural vibrations of the shell. Liu and Arbocz [110] presented a theoretical investigation of the influence of different boundary conditions on the non-linear vibration of thin-walled cylinders, employing the non-linear Donnell-type equations, and the method of averaging. The effect of initial geometric imperfections on large-amplitude vibrations of truncated conical shells subjected to pressure load has been investigated by Reseka and Helmy [111]. A mathematical model has been performed by means of the finite-element method, and the method of generalized co-ordinates, by Kovtunov [112], for studying the dynamic stability and non-linear vibration parameters of ideal cylindrical shells, and cylindrical shells with added mass with regard to their geometrical non-linearity. The numerical methods suggested by the author have been realized in the software package STADYS (stability and dynamic of structures).

Sivadas [113] analyzed the vibration characteristics of pre-stressed circular conical shells, using the moderately thick shell theory with shear deformation and rotary inertia. A semi-analytical isoparametric finite element with three nodes per element and five degrees of freedom per nodes was used. The non-linear response of thin cylindrical shells with longitudinal cracks, subjected to internal pressure and axial compression load has been studied analytically by Starnes and Rose [114]. More recently, a model of three-dimensional vibrations of thick spherical shell segments with variable thickness was presented by Kang and Leissa [115]. Also, the non-linear supersonic flutter of circular cylindrical shells has been investigated by Amabili and Pellicano [116], using the Donnell non-linear shallow-shell theory. The effect of viscous structural damping has been taken into account and results show that the system loses stability by standing-wave flutter through specific bifurcation. A very good agreement between theoretical and experimental data has been found for flutter amplitudes.

5. OPEN ISOTROPIC SHELLS

Non-linear problems concerning open shells, which may be either shallow or deep, of various geometries have not received much attention in the literature as can be seen, recently, in a review paper by Liew *et al.* [11]. This section introduces some references on the non-linear vibrations of isotropic shallow shells which have not been documented in reference [11]. To complete this review, studies on non-linear vibrations of composite shallow shells will be presented in section 6.

The paper by Leissa and Kadi [90] was the first published study of large-amplitude vibrations of doubly curved shallow shells having arbitrary curvatures. It was also shown that all these curvatures yielded first a soft spring, followed by a hard spring, type of frequency–amplitude response, except for the hyperbolic paraboloidal shallow shell, which was entirely hard spring. Kulterbaev [117] analyzed the non-linear transverse vibrations of a flexible cylindrical panel subjected to a time-random longitudinal force. The effects of large vibration amplitudes on flexural vibrations of an orthotropic shallow cylindrical panel, on an elastic Winkler foundation, have been examined by Ramachandran and Murthy [118], using the dynamic von Karman field equations and assuming the edges of the panel to be simply supported. Lau and Cheung [119] derived an amplitude incremental variational principle for non-linear vibrations of elastic systems, adopting the

Rayleigh–Ritz method, and especially the finite-element method. The linear solution for the system was used as the starting point of the solution procedure for the non-linear case, and the amplitude was then increased incrementally. The panel was treated as a system with one degree of freedom, and the presence of a phenomena called “vibration break-off”, and resonance type was revealed.

The stability of non-linear forced vibrations of shallow rectangular-planform cylindrical shells has been investigated by Grigorinko *et al.* [120], using a shell modelled as a system with one degree of freedom, and the Bubnov–Galerkin method. Sinharay and Banergee [121] investigated the large-amplitude free vibrations of shallow spherical and cylindrical shells following a new approach based on Berger’s hypothesis. In this study, it was assumed that the differential equations are uncoupled and thus simplified. Excellent results were obtained for both movable and immovable edge conditions in a single-mode approach.

Sathyamoorthy [122] presented a shallow-shell theory for the geometrically non-linear analysis of moderately thick isotropic spherical shells, including the effects of transverse shear deformation and rotary inertia. Solution to the system of thick shell equations has been obtained by means of Galerkin’s method and the Runge–Kutta procedure. He concluded that the transverse strain and rotary inertia effects are important in the linear as well as in the non-linear response of shallow spherical shells. Kobayashi and Leissa [89] examined the effect of thickness and curvature upon the large-amplitude vibration of shallow shells supported by shear diaphragms. The governing equations for non-linear vibration of doubly curved thick shallow shells were derived, based upon FSDF. Applying Galerkin’s procedure and eliminating all of the variables except the transverse displacement, the governing equations were reduced to an elliptic ordinary differential equation in time.

Lasiecka and Valente [123] studied a coupled non-linear system of equations describing the vibrations of shallow thin spherical shells. Ye [124] used the Marguerre-type dynamic equations to investigate the non-linear vibration and dynamic instability of thin shallow spherical and conical shells subjected to periodic transverse and in-plane loads. The solution of the differential equations was made by the Galerkin’s method, which led to Duffing and Mathieu equations. A new shell-type dynamic vibration absorber was presented by Aida *et al.* [125], for suppressing several modes of vibration of a shallow shell under harmonic load. Using the Den Hartog method, numerical results were obtained which demonstrated the usefulness of shell-type dynamic vibration absorbers. Based on the investigation made by Popov *et al.* [126] on the vibrations of cylindrical shells parametrically excited by axial forcing, Foale *et al.* [127] investigated a number of methods for obtaining a low-dimensional dynamical system from a set of partial differential equations describing the non-linear vibrations of a shallow cylindrical panel under axial forcing. The non-linear free vibration of a cylindrical thin panel with curvature and twist has been treated by Hu and Tsuji [128], by means of the Rayleigh–Ritz method, assuming two-dimensional polynomial functions as displacement functions.

6. CLOSED AND OPEN COMPOSITE SHELLS

Analysis of shell structures composed of composite materials has been of a considerable research interest, particularly, in the development of space technology. For advanced information on this subject, readers can refer the monograph by Leissa [6] for early works, or Oevy’s paper [129] for directions for the design of orthotropic cylindrical shells. More recently, a literature review on the vibration of composite structures has been presented by Bert [130].

Using the Galerkin method and assumed modes, Pandalai and Sathyamoorthy [131] analyzed the non-linear flexural vibrations of thin elastic orthotropic oval cylindrical shells. Kvasha [132] studied the forced vibrations of three-layer plates and shells made from physically non-linear materials. An orthotropic model has been applied by Volmir and Ponomarev [133] to the analysis of the dynamic behaviour of closed cylindrical shells, prepared from a composite material. The solution of the problem was obtained on the basis of geometrically non-linear dynamic equations in the theory of shallow shells, derived from the Kirchhoff–Love hypothesis. El-Zaouk and Dym [134] employed the same approximation, for the transverse displacement assumed by Evensen and Fulton [74], in the analysis of isotropic cylinders, to study orthotropic doubly curved shells. Using Bubnov's method to reduce the non-linear equations and a strain energy method, the problem of non-linear vibrations excited by a transverse harmonic load in three-layer plates, and shallow shells of rectangular plan-form and asymmetric thickness, has been examined by Goloskokov and Dmitrenko [135]. Bogdanovich and Feldman [136] analyzed the non-linear parametric vibrations of closed, viscoelastic, orthotropic cylindrical shells. Solution of the problem was obtained by the Bubnov–Galerkin method, which led to a non-linear ordinary integrodifferential equation. A finite-element analysis of geometrically non-linear transient response of laminated anisotropic shells has been presented by Chao and Reddy [137].

The transverse shear effects on the non-linear instability behaviour of composite cylindrical shells under axial compression, hydrostatic pressure, and torsion, has been investigated by Chengti and Chienbin [138]. It has been shown that in many cases the transverse shear effect is important, and should be taken into account in the non-linear stability analysis of composite shells. Reddy and Chandrashekhara [139] studied non-linear transient response of doubly curved shells by the finite-element method using first order shear deformation theory. Chia [140] investigated the non-linear vibration and post-buckling of imperfect panels by a perturbation procedure. Introducing the effect of initial imperfection, the laminated cross-ply circular cylindrical shells with clamped and simply supported ends have been studied by Iu and Chia [141]. The transverse equation of motion was fulfilled by the Galerkin procedure, and the solution has been obtained by the harmonic balance method.

Non-linear free vibrations of circular cylindrical shells composed of composite materials have been examined analytically by Vinson [142]. Galerkin's method and a fourth order Runge–Kutta method were employed to determine the amplitude–frequency relationships for isotropic and composite (orthotropic) cylindrical shells.

Fu and Chia [143, 144] analyzed multi-mode vibrations of thick panels by the harmonic balance method. An extension of Donnell's equations for isotropic cylindrical shells has been made by Kumar [145], via the anisotropy concept of Lekhnitsky, to analyze the large-scale oscillation and buckling behaviour of anisotropic cylindrical shells, subjected to biaxial in-plane normal stresses. Raouf and Palazotto [146, 147] attempted to solve non-linear dynamic problems of laminated shell panels using numerical methods. Kapania and Byum [148] studied imperfect laminated panels by the finite-element method.

Dong [149] proposed a new and more direct approach to the von Karman–Donnell-type governing equations of non-linear vibrations of shallow shells of revolution, and applied it to non-linear axisymmetric free vibrations of orthotropic, shallow, thin conical and spherical shells. Asymptotic expressions for spacial modes, and the amplitude–frequency response of the shells were derived.

The effects of large vibration amplitudes on flexural vibrations of moderately thick orthotropic spherical shells, taking into account the transverse shear deformation and rotary inertia effects, have been studied by Sathyamoorthy [150]. He obtained a coupled set of non-linear equations which have been solved using Galerkin's method and employing the numerical Runge–Kutta integration procedure. A geometrically non-linear theory and numerical results have been presented by Andrianov *et al.* [151], for free vibrations of thin, simply supported circular cylindrical, stringer-stiffened shells. An asymptotic procedure was followed, which separated the solution of the non-linear equations of motion into two parts: an inner part which applies to the boundary layer, and an outer part. Employing Sanders' shell theory, and a specialized finite-element method, Selman and Lakis [83] studied the non-linear dynamics of open and closed orthotropic cylindrical shells. A non-linearity of the hardening type was obtained for the same closed circular cylindrical shell simply supported at the ends investigated by Nowinsky [73], and Raju and Rao [80].

Abe *et al.* [152] examined the non-linear free vibration characteristics of the first and second vibration modes of laminated shallow shells, with rigidly clamped edges. Using Galerkin's procedure, simultaneous non-linear ordinary differential equations were derived in terms of amplitudes of the first and second vibration modes. The Gauss–Legendre integration method, and the shooting method, were used for the solution.

7. SHELLS IN INTERACTION WITH FLUID

Non-linear vibrations of a cylindrical shell, filled with fluid, have been investigated by Obraztsova [153], and a numerical example of a steel cylindrical shell, filled with water, has been treated at assigned parameter values. The non-linear equations were obtained by the theory of shallow cylindrical shells, and solved by application of the Bubnov–Galerkin and harmonic balance methods. In another paper by Obraztsova [154], a study on non-linear axisymmetric vibrations of shallow spherical shell, containing an ideal incompressible fluid, was presented.

Using geometrically non-linear equations for shallow shells, the interactions between the conjugate flexural modes, the flexural modes corresponding to the nearest natural frequencies, and the flexural modes of shells in gas flow have been analyzed by Kubenko *et al.* [155]. Particular attention has been given to the determination and analysis of the possible flexural modes of shells under conditions of various resonance.

Boiarshina [156] analyzed the non-linear vibrations induced by periodic external forces in an elastic cylindrical shell partially filled with fluid. For resonance conditions, first integrals were determined, indicating a coupling between the antisymmetric vibrational modes of the free surface of the fluid and the flexural vibrations of the shell. The non-linear parametric vibrations of liquid-filled cylindrical shells with an initial imperfection have been studied by Pavlovskii and Filin [157] using basic functions in the form of coupled bending vibration modes. The equations of motion were solved by an asymptotic method. Gonçalave and Batista [158] have presented a model for non-linear dynamic interaction between a fluid and thin elastic shells, in order to examine the effect of shell and fluid parameters on the non-linear structural vibration response in terms of frequencies. These studies led, using the Galerkin procedure, to a system of coupled non-linear algebraic equations for the modal amplitudes u_i, v_i, and w_i, which were solved by the Newton–Raphson method. The results obtained have shown how variations in the geometrical parameters (L/R, h/R), and wave numbers (m, n), affect the degree and type of non-linearity. In another paper, Gonçalave and Batista [159] considered non-linear vibrations of simply supported circular cylindrical shells filled with inviscid and incompressible fluid, using Sanders' non-linear theory. They excluded the companion mode and found that the presence of a dense fluid gives more strongly softening results, compared to those for the same shell *in vacuo*.

The linear and non-linear natural frequencies versus shell amplitudes have been examined by Selmane and Lakis [160] for thin, orthotropic and non-uniform open cylindrical shells, submerged and subjected simultaneously to an internal and external fluid. They developed a method including thin shell theory, fluid theory, and the finite-element method, and obtained a non-linear equation of motion, which was solved by the fourth-order Runge–Kutta numerical method. Both softening- and hardening-type non-linearity were found depending on the geometry. The non-linear free and forced vibrations of a simply supported, circular cylindrical shell, in contact with an incompressible and inviscid, quiescent and dense fluid have been investigated by Amabili *et al.* [14]. The problem was reduced to a system of ordinary differential equations by means of Galerkin's method, and the effects of both internal and external dense fluid have been examined.

In a series of paper presented recently by Amabili *et al.* [12, 13, 29, 30], the non-linear dynamics and stability of simply supported circular cylindrical shells containing inviscid incompressible fluid flow was investigated in detail. The non-linearity due to large-amplitude shell motion has been considered using the non-linear Donnell's shallow shell theory, taking into account the effect of viscous structural damping. Part I deals with a detailed study of the stability of the structure [12], Part II deals with large vibration amplitudes without flow [13], Part III deals with truncation effects without flow and experiments [29], and Part IV deals with large vibration amplitudes with flow [30].

8. EXPERIMENTAL STUDIES

The development of non-linear models cannot be made adequately, unless they are made in the light of the results of experimental investigations. These investigations should play an important role before and after the theoretical work. The experimental data guide the choice of the basic assumptions, and indicate the form of the expected solutions before starting the modelling. In the second stage, they permit one to validate the numerical results, and indicate the range of validity and the degree of accuracy of each type of solution. It is believed that the first experimental investigations on the effect of geometric non-linearity on the vibrations of shells were made by Evensen [93], for closed circular cylindrical shells. He noticed that the experiments suggest that the non-linearity is of the softening type, and for the shells that were tested, the vibrations were only slightly non-linear. These observations have led him to a re-examination of the non-linear analysis of circular cylindrical shell, as has been mentioned in section 3.

In preparing for flutter experiments [161], Olson [84] performed vibration tests on several cylindrical shells made of copper, including results on large vibration amplitudes. The shell tested had a thickness-to-radius ratio of 0·00055 (a very thin shell). He concluded that large vibration amplitudes of cylindrical shells exhibit a slight non-linearity of the softening type, as has been observed by Evensen [93]. The experimental result of Olson is frequently used by researchers to validate their models [13, 14, 82, 88, 95]. Evensen [93, 94] extended his work, and examined both theoretically and experimentally non-linear flexural vibrations of thin circular rings. A ring is a particular case of a shell when the length tends to zero. He obtained a slight non-linearity of the softening type and a good agreement between theory and tests for a shell having a thickness-to-radius ratio of 0·00127. Matsuzaki and Kobayashi [85] analyzed theoretically and experimentally a clamped circular cylindrical thin shell having a thickness-to-radius ratio of 0·00092, and observed the participation of a companion mode over a range of frequency and amplitude. He also modified the mode expansion in order to satisfy the different boundary conditions, and retained both the particular and the homogeneous solutions for the stress function. A softening-type non-linearity was found in this case.

Careful non-linear vibration experiments have been performed by Chen and Babcock [25] on a circular cylindrical shell, simply supported, having a thickness-to-radius ratio of 0·00243. They pointed out that it is impossible to simulate this boundary condition in the laboratory, and used a shell with rings at both ends to approximate the simply supported condition. Various non-linear phenomena have been observed, and the response–frequency relation of the companion mode has been measured. It has been concluded that the corresponding theoretical results show a satisfactory agreement with the experimental results, and the resulting single-mode response is either hardening or softening, depending upon the mode of vibration.

Sivak and Telalov [162] studied experimentally a vertical circular cylindrical shell, made of titanium alloy, in contact with water having a free surface. Experiments were performed with the shell partially filled and partially submerged in water. Many experiments indicated a softening-type non-linearity, while the completely filled shell showed a hardening non-linearity, an effect which was increasing with the number of nodal diameters. Large-amplitude vibrations of two cantilevered composite circular cylindrical shells have been examined experimentally by Fu and Chia [163]. They found that almost all responses have a softening non-linearity. More recently, an important paper presented by Amabili *et al.* [29] deals with theoretical and experimental studies on the non-linear dynamics and stability of circular cylindrical shells containing flowing fluid. This work has been presented in Part III of a series of papers. A series of experiments were carried out on a water-filled circular cylindrical shell, made of steel. The results obtained were in very good agreement with the theory presented. It was observed, also, that the liquid (water) contained in the shell generated a much stronger softening behaviour of the system.

9. COMMENTS AND DISCUSSION

The problem of large vibration amplitudes of shell-type structures has given rise to a number of theoretical studies, as has been reviewed above. It appears from the literature survey that in spite of much research work on non-linear vibrations of shells, no exact solution for the problem of non-linear vibrations of shell-type structures is known. It can be seen also that, generally, the dynamic behaviour of shell-type structures is governed by three non-linear coupled partial differential equations, involving three dependent variables u, v, and w, which are, respectively, two tangential and one normal displacement components of a point of the shell middle surface. These equations can be reduced to two simultaneous partial differential equations involving only the Airy stress function F and the transverse displacement when the tangential inertia terms are neglected. This approach is consequently limited to shallow-shell analysis, or deep shells having large numbers of waves in their mode shapes. Using three equations of motion, in terms of u, v, and w, is more general. However, exact solutions of the non-linear partial differential equations in closed form do not exist in the literature owing to the difficulty of the mathematical treatment. Approximate analytical solutions exist, but under very special conditions.

The basic approach used in majority of these studies is to assume the shape of the vibration modes, sometimes referred to as generalized co-ordinates, and then to derive a set of non-linear ordinary differential equations in time using Galerkin's approximation procedure (averaging method). This method is a very powerful approximation method that reduces a system of non-linear partial differential equations into a system of non-linear ordinary differential equations. Also, the Galerkin method provides an insight into the non-linear coupling of various vibration modes during the solution procedure. However, the results are highly dependent on the assumed deflxtion shapes. Completely different results can be obtained by a small difference in the assumed deflexion shapes as can be seen from the previous investigations on the subject of non-linear vibrations of shells. In the analytical methods, the spatial and temporal problems are solved analytically. The temporal problem is usually solved by the harmonic balance method, the perturbation method, the multiple scales method or the averaging method. These methods are generally used for simple structures and simple boundary conditions. The perturbation method can only be applied to a system with weak non-linearity and the computation procedures become quite cumbersome if many terms in the perturbation series are required to achieve a desired degree of accuracy. The finite-element approach has also been used in the problem of non-linear vibration of shells. Application of the finite-element method has the disadvantages of the very long time for the solution and the amount of manual labour involved in preparing the input data. But, it can be considered as a very important tool, especially for studying geometrically complex shell-type structures.

The single-mode method was effectively used in investigations of large vibration amplitudes of a wide class of continuous systems [49]; in particular, in most of the papers concerned with the non-linear dynamic response of beams, the single-mode assumption was made as a tool for investigating the effect of the geometric non-linearities on resonant frequencies [13, 14, 18, 164]. This is due to the simplification it introduces in the theory on the one hand, and on the other hand, because the error it introduces in the estimation of the non-linear frequency remains small. In the case of non-linear vibration of shells, it has never been clarified how many terms in the expression of the flexural displacement are necessary to obtain a good accuracy in the model. This question, which is still open, has been examined recently by Amabili *et al.* [13], and it was concluded that not only the first but also the third axisymmetric mode is fundamental for the adequate description of the non-linear response of closed circular cylindrical shells. In another study on constrained vibrations of infinitely long circular cylindrical shells, a spectral expansion of the transverse displacement has been assumed with 12 basic functions (multi-mode approach) [40, 41, 46, 47]. It has been shown that, for the first mode, calculations can be made with only six basic functions of w and the results exhibited no significant change in both the values of the non-linear frequencies, and the basic function contributions.

One problematic point, which is not encountered in non-linear vibration problems of plates and beams, is the question of the behaviour type, softening or hardening, of cylindrical shells. Some of the analyses have led to the conclusion that the non-linear behaviour was of the hardening type (i.e., the increase in resonant frequencies as the amplitude increases), others have concluded that it was of the softening type (i.e., the decrease in resonant frequencies as the amplitude increases), while other analyses yielded softening first, followed by hardening, depending on the mode of vibration and the structural parameters. The experiments on cylindrical shells that yielded only softening response may have been for small amplitudes, too small to obtain subsequent hardening. From the review on the non-linear works exposed in sections 3 (point 7) and 4.1, it can be seen that experiments on forced vibrations, however, have shown that the non-linear behaviour is of the softening type on certain cylindrical shells. This discrepancy, between the analyses, seems to have stemmed from the choice of the trial functions that were assumed in solving the equations, rather than from differences in shell theories or other assumptions [88]. However, Evensen [5, 74], and Ganapathy and Varadan [82] assume that the type of non-linearity of the shells depends on the values of shell aspect ratio ξ and the thickness parameter ε. A reliable discussion and interesting comment have been made about this subject between Raju and Rao [165, 166], and Evensen [167], and between Prathab [168] and Evensen [169]. It is interesting to note that these comments caused Ganapathy and Varadan [82] to reinvestigate recently the large-amplitude free flexural vibrations of isotropic circular cylindrical shells. Other discussions between leading authors have recently been published in the *Journal of Fluids and Structures* [170–172]. More recently, another discussion has been made between Amabili *et al.* [173] and Moussaoui *et al.* [92] on the hardening strong non-linearity observed in reference [41] for moderately thin, infinite circular cylindrical shells. This set of discussions testify that the topic of circular cylindrical shells behaviour still presents open questions, which should be clarified through experiments on non-linear vibrations of moderately thin and thick shells. To our knowledge, the experiences reported in the available literature, concerning large vibration amplitudes, have been carried out on very thin shells (see section 8), for which the softening behaviour has been obtained.

10. CONCLUDING REMARKS

On the basis of the references reviewed in this paper, and the comments presented above, the following remarks can be made:

- Although many publications deal with geometric-type non-linearity, including or not complicating effects, few papers have appeared on material-type non-linearity or combinations of the two types. Also, very little information can be found in the literature about the effects of geometrical non-linearity on viscoelastic shells.
- The number of references reporting experimental investigations is low. This area needs renewed attention in order to clarify more the non-linear behaviour of shell-type structures.
- The non-linear vibrations of hybrid shell-type structures, and combined structures, remain a field to develop.
- Most of the studies on non-linear vibrations of shells (except those of Evensen [94], Chen and Babcock [25], Olson [84] and Amabili *et al.* [12, 116], Moussaoui and Benamar [40] and Moussaoui *et al.* [41, 46, 47]), do not give any indication of the physical form of the vibration modes at large vibration amplitudes. In other words, if the form of the circular cylindrical shell vibrating freely at small amplitudes (linear case) is known (see for example references [174, 175]), what does it become when the shell vibrates at large amplitudes?
- One may observe also that most of the works on non-linear vibrations of shells (including complicating factors or not) restrict the investigations to the amplitude–frequency relationship and stability. However, the stress distributions on the surface of the structure are not analyzed. This non-linear effect is of crucial importance in engineering design, and very significant, in comparison with the linear case, as has been shown by Moussaoui and Benamar [40], and Moussaoui *et al.* [41, 46, 47].
- In many applications (aircraft fuselage, missile bodies, etc.), stiffeners are used to strengthen thin shells [4]. However, although the studies on linear vibrations of stiffened shells have been extensive, the studies on non-linear vibrations of stiffened shells are very little.

These remarks show that many aspects of the complicating problem of non-linear vibrations of shell-type structures need to be clarified and that much research work has still to be done.

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