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LETTERS TO THE EDITOR

# COMPARISON OF FOURIER SINE AND COSINE SERIES EXPANSIONS FOR BEAMS WITH ARBITRARY BOUNDARY CONDITIONS

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## 1. INTRODUCTION

Fourier series methods have been extensively used for the dynamic analyses of beams and plates with simply supported boundary conditions. Making use of the Stokes's transformation, Chung [1] developed a Fourier series method for the free vibrations of circular cylinders with other homogeneous boundary conditions, and Lin and Wang [2] later applied it to the simply supported beams with rotational restraints at each end. Despite some desired mathematical characteristics of a Fourier series, the Fourier series methods have not been widely applied to boundary conditions other than the simply supported. This may be attributed to their difficulty in satisfying the general boundary conditions. As a remedy, simple polynomials are sometimes appended to the sinusoidal functions or Fourier series expansions to force the resulting functions or series to satisfy the specific boundary conditions under consideration [3–6]. Li [7] recently proposed a simple and unified Fourier series method for beams with arbitrary boundary conditions. The beam displacement is also expressed as the liner superposition of a Fourier series and an auxiliary polynomial. However, unlike in the previous investigations, the polynomial function is used to mathematically improve the continuity of the displacement and hence the convergence of its Fourier series expansion, regardless of boundary conditions. Accordingly, not only is the solution in the form of a Fourier series always viable for any boundary conditions, but also its accuracy and convergence is considerably improved both theoretically and numerically.

In the previous study, the solution was specifically expressed as a Fourier cosine series, and it was claimed that the cosine series expansion would converge faster than its sine counterpart for beams with arbitrary elastic restraints [7]. Although a theoretical explanation is already given there, the author believes that it would be beneficial to further the discussion and present some numerical examples to support this important conclusion.

## 2. FREE VIBRATION ANALYSIS OF BEAMS BASED ON THE FOURIER SERIES EXPANSIONS

### 2.1. BASIC EQUATIONS

Figure 1 shows a beam elastically restrained at each end. The governing differential equation for the free vibration of the beam is known as

$$D\,\mathrm{d}^4 w(x)/\mathrm{d}x^4 - \rho A \omega^2 w(x) = 0 \tag{1}$$

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Figure 1. A beam elastically restrained at both ends.

or

$$w'''' - \rho_D \omega^2 w = 0 \qquad (\rho_D = \rho A/D), \tag{2}$$

where $w$ is the flexural displacement, $D$, $\rho$ and $A$ are, respectively, the flexural rigidity, the mass density and the cross-sectional area of the beam, and $\omega$ is frequency in radians.

The boundary conditions can be expressed as

$$\hat{k}_0 w = -w''', \qquad \hat{K}_0 w' = w'' \quad (\hat{k}_0 = k_0/D, \quad \hat{k}_1 = k_1/D) \text{ at } x = 0 \tag{3, 4}$$

and

$$\hat{k}_1 w = w''', \qquad \hat{K}_1 w' = -w'' \quad (\hat{K}_0 = K_0/D, \quad \hat{K}_1 = K_1/D) \text{ at } x = L, \tag{5, 6}$$

where $k_0$ and $k_1$ are the stiffnesses of the linear springs and $K_0$ and $K_1$ are the stiffnesses of the rotational springs at $x = 0$ and $L$ respectively.

Equations (3–6) represent a set of general boundary conditions. Many classical boundary conditions can be simply considered as special cases when the stiffnesses of the springs take on some extreme values such as zero and infinity.

## 2.2. THE SOLUTION IN THE FORM OF COSINE SERIES EXPANSION

The Fourier series method used in reference [7] will be briefly reviewed here for the sake of completeness.

Unlike in the traditional Fourier methods, the flexural displacement of the beam is here sought as the linear combination of a Fourier cosine series and an auxiliary polynomial function:

$$w(x) = \sum_{m=0}^{\infty} A_m \cos \lambda_m x + p(x), \quad 0 \leqslant x \leqslant L, \quad \lambda_m = m\pi/L, \tag{7}$$

where $p(x)$ is a polynoimial function which satisfies

$$p'''(0) = w'''(0) = \alpha_0, \qquad p'''(L) = w'''(L) = \alpha_1, \tag{8, 9}$$

$$p'(0) = w'(0) = \beta_0, \qquad p'(L) = w'(L) = \beta_1. \tag{10, 11}$$

One of such polynomials can be readily found as

$$p(x) = \boldsymbol{\zeta}(x)^{\mathrm{T}} \bar{\boldsymbol{\alpha}}, \tag{12}$$

where

$$\boldsymbol{\zeta}(x) = \begin{Bmatrix} -(15x^4 - 60Lx^3 + 60L^2x^2 - 8L^4)/360L \\ (15x^4 - 30L^2x^2 + 7L^4)/360L \\ (6Lx - 2L^2 - 3x^2)/6L \\ (3x^2 - L^2)/6L \end{Bmatrix} \tag{13}$$

and

$$\bar{\boldsymbol{\alpha}} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^{\mathrm{T}}. \tag{14}$$

It should be pointed out that the polynomial function $p(x)$ is introduced here to remove all the possible discontinuities, at $x = 0$ and $L$, from the displacement function $w(x)$ and its relevant derivatives. As a result, the Fourier series now simply represents a residual or conditioned displacement that has at least three continuous derivatives everywhere. An immediate benefit is that all the required differential operations on the Fourier series can be carried out on a term-by-term basis.

Substituting equations (7–12) into the boundary conditions (3–6), the unknown vector $\bar{\boldsymbol{\alpha}}$ can be obtained as

$$\bar{\boldsymbol{\alpha}} = \sum_{m=0}^{\infty} \mathbf{H}^{-1}\mathbf{Q}_m\mathbf{A}_m, \tag{15}$$

where

$$\mathbf{H} = \begin{bmatrix} \dfrac{\hat{k}_0 L^3}{45} + 1 & \dfrac{7\hat{k}_0 L^3}{360} & \dfrac{-\hat{k}_0 L}{3} & \dfrac{-\hat{k}_0 L}{6} \\ \dfrac{7\hat{k}_1 L^3}{360} & \dfrac{\hat{k}_1 L^3}{45} + 1 & \dfrac{-\hat{k}_1 L}{6} & \dfrac{-\hat{k}_1 L}{3} \\ \dfrac{L}{3} & \dfrac{L}{6} & \hat{K}_0 + \dfrac{1}{L} & \dfrac{-1}{L} \\ \dfrac{L}{6} & \dfrac{L}{3} & \dfrac{-1}{L} & \hat{K}_1 + \dfrac{1}{L} \end{bmatrix} \tag{16}$$

and

$$\mathbf{Q}_m = \{-\hat{k}_0\ (-1)^m\hat{k}_1\ -\lambda_m^2\ (-1)^m\lambda_m^2\}^{\mathrm{T}}. \tag{17}$$

Making use of equations (1), (7), (12) and (15), one is able to obtain

$$\lambda_m^4 A_m - \rho_D\omega^2\left(A_m + \sum_{m'=0}^{\infty} S_{mm'}A_{m'}\right) = 0, \quad m = 1, 2, 3, \ldots \tag{18}$$

and

$$\sum_{m'=0}^{\infty} S_{0m'}A_{m'} - \rho_D\omega^2 A_0 = 0, \tag{19}$$

where

$$S_{mm'} = \mathbf{P}_m^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{Q}_{m'} \tag{20}$$

with

$$\mathbf{P}_m = \frac{2}{L}\left\{\frac{1}{\lambda_m^4}\ \frac{(-1)^{m+1}}{\lambda_m^4}\ \frac{-1}{\lambda_m^2}\ \frac{(-1)^m}{\lambda_m^2}\right\}^{\mathrm{T}} \tag{21}$$

and

$$\mathbf{P}_0 = \{-1/L\ \ 1/L\ \ 0\ \ 0\}^{\mathrm{T}}. \tag{22}$$

Equations (18) and (19) represent a familiar matrix characteristic equation from which the natural frequencies and the corresponding modes shapes (actually the Fourier coefficients) can be directly obtained without any difficulty.

### 2.3. THE SOLUTION IN THE FORM OF SINE SERIES EXPANSION

The beam displacement can also be expanded into a sine series

$$w(x) = \sum_{m=0}^{\infty} A_m \sin \lambda_m x + p(x), \quad 0 \leqslant x \leqslant L. \tag{23}$$

However, the polynomial function should now satisfy

$$p''(0) = w''(0) = \alpha_0, \qquad p''(L) = w''(L) = \alpha_1, \tag{24, 25}$$

$$p(0) = w(0) = \beta_0, \qquad p(L) = w(L) = \beta_1. \tag{26, 27}$$

Other modifications must also be made as follows:

$$\zeta(x)^{\mathrm{T}} = \begin{Bmatrix} -(2L^2x - 3Lx^2 + x^3)/6L \\ (x^3 - L^2x)/6L \\ (L - x)/L \\ x/L \end{Bmatrix}, \tag{28}$$

$$\mathbf{H} = \begin{bmatrix} \dfrac{\hat{K}_0 L}{3} + 1 & \dfrac{\hat{K}_0 L}{6} & \dfrac{\hat{K}_0}{L} & -\dfrac{\hat{K}_0}{L} \\ \dfrac{\hat{K}_1 L}{6} & \dfrac{\hat{K}_1 L}{3} + 1 & -\dfrac{\hat{K}_1}{L} & \dfrac{\hat{K}_1}{L} \\ -\dfrac{1}{L} & \dfrac{1}{L} & \hat{k}_0 & 0 \\ \dfrac{1}{L} & -\dfrac{1}{L} & 0 & \hat{k}_1 \end{bmatrix} \tag{29}$$

and

$$\mathbf{Q}_m = \{\hat{K}_0 \, \lambda_m \, (-1)^{m+1} \, \hat{K}_1 \lambda_m \;\; \lambda_m^3 \, (-1)^{m+1} \, \lambda_m^3\}^{\mathrm{T}}. \tag{30}$$

The final characteristic equation can still be written as

$$\lambda_m^4 A_m - \rho_D \omega^2 \left( A_m + \sum_{m'=1}^{\infty} S_{mm'} A_{m'} \right) = 0, \quad m = 1, 2, 3, \ldots \tag{31}$$

with

$$\mathbf{P}_m = \frac{2}{L} \left\{ \frac{-1}{\lambda_m^3} \;\; \frac{(-1)^m}{\lambda_m^3} \;\; \frac{1}{\lambda_m} \;\; \frac{(-1)^{m+1}}{\lambda_m} \right\}^{\mathrm{T}}. \tag{32}$$

### 2.4. THE CONVERGENCE OF THE FOURIER SERIES SOLUTIONS

It is known mathematically [8] that if a periodic continuous function $f(x)$ has $m$ derivatives, then the Fourier series of all $m$ derivatives can be obtained by term-by-term differentiation of the Fourier series of $f(x)$ and the Fourier coefficients satisfy the relations

$$\lim_{n\to\infty} a_n \lambda_n^m = \lim_{n\to\infty} b_n \lambda_n^m = 0, \tag{33}$$

where $a_n$ and $b_n$ are, respectively, the coefficients of the cosine and sine terms.

As explained in reference [7], because of the characteristics (at $x = 0$ and $L$) of the polynomial function, the Fourier cosine series in equation (7) now represents a conditioned displacement that has at least three continuous derivatives. Therefore, according to equation (33), its convergence can be estimated by

$$\lim_{n\to\infty} A_n \lambda_n^4 = 0. \tag{34}$$

Similarly, it is not difficult to show that the above equation also applies to the sine series expansion given in equation (23).

Instead of using equation (34), the convergence (or truncation error) of the Fourier series solutions can actually be estimated in a more direct manner. Multiplying equation (1) with $2/L\cos\lambda_m x$ or $2/L\sin\lambda_m x$ and integrating it from 0 to $L$, one is able to obtain

$$\lambda_m^4 A_m - \rho_D\omega^2\ (A_m + P_m) = 0 \tag{35}$$

or

$$A_m = \frac{\rho_D\omega^2 P_m}{\lambda_m^4 - \rho_D\omega^2}, \tag{36}$$

where $P_m$ are the Fourier coefficients of the polynomial function $p(x)$.

Making use of equations (21) and (32), equation (36) can be expressed as

$$A_m = \frac{\rho_D\omega^2 P_m}{\lambda_m^4 - \rho_D\omega^2} = \frac{2\rho_D\omega^2/\lambda_m}{(\lambda_m^4 - \rho_D\omega^2)L}\left(\frac{\alpha_1(-1)^m - \alpha_0}{\lambda_m^2} + \beta_1(-1)^{m+1} + \beta_0\right) \tag{37}$$

for sine series expansion, and

$$A_m = \frac{\rho_D\omega^2 P_m}{\lambda_m^4 - \rho_D\omega^2} = \frac{-2\rho_D\omega^2/\lambda_m^2}{(\lambda_m^4 - \rho_D\omega^2)L}\left(\frac{\alpha_1(-1)^m - \alpha_0}{\lambda_m^2} + \beta_1(-1)^{m+1} + \beta_0\right) \tag{38}$$

for cosine series expansion.

It should be noted that although the same symbols are used in equations (37) and (38), the boundary constants, $\alpha_i$ and $\beta_i$ ($i = 0$, 1), actually have different meanings, referring to equations (8–11) and (24–27).

Suppose that the boundary constants are somehow known *a priori*, then equation (37) and (38) can be directly used to assess the convergence rates of the sine and cosine series expansions. For instance, if a beam is simply supported at each end with only rotational restraint, the constants $\beta_0$ and $\beta_1$ (representing the displacements at $x = 0$ and $L$) in the sine series expansion are then both equal to zero. However, this is not generally true in the cosine series representation because there these constants denote the boundary values of the first derivatives. Thus, from equations (37) and (38) one see that $A_m \sim O(\lambda_m^{-7})$ for the sine series and $A_m \sim O(\lambda_m^{-6})$ for the cosine series. However, the cosine series will be converging much faster if the first derivative vanishes at $x = 0$ and $L$ as in the cases when a beam is guided at each end with only translational restraint. In this situation, it is clear from equations (37) and (38) that $A_m \sim O(\lambda_m^{-5})$ for the sine series and $A_m \sim O(\lambda_m^{-8})$ for the cosine series. For beams with general elastic restraints, the constants $\beta_0$ and $\beta_1$ will not normally equal to zero. Therefore, the sine and cosine series will, respectively, converge according to $A_m \sim O(\lambda_m^{-5})$ and $A_m \sim O(\lambda_m^{-6})$.

In the real calculations, however, the boundary constants have to be determined from equation (15) in terms of the Fourier coefficients. Each time the Fourier series is differentiated, the convergence of the corresponding series will be accordingly slowed by a factor of $\lambda_m$. For a generally supported beam, since the second derivative of the Fourier series of the displacement is involved in the cosine series representation, the resulting cosine series solution is then expected to converge according to $A_m \sim O(\lambda_m^{-4})$. In comparison, because the third derivative is used in the sine series representation, one can see that the sine series solution is converging at a much slower speed of $A_m \sim O(\lambda_m^{-2})$.

The above convergence estimate is only based on the continuity characteristic of the displacement function. In essence, however, the convergence or accuracy of the final solution or results will also be affected by truncation errors associated with the discretization procedure that is used to convert the original governing differential equation into a set of linear algebraic equations about the unknown Fourier coefficients.

Denote

$$\begin{aligned} R_M &= D\partial^4 w_M/\partial x^4 - \rho A\omega^2 (w_M + p_M) \\ &= -[D\partial^4 \varepsilon_M^w/\partial x^4 - \rho A\omega^2 (\varepsilon_M^w + \varepsilon_M^p)], \end{aligned} \tag{39}$$

where

$$w(x) = \sum_{m=0}^{M} A_m \cos \lambda_m x + \varepsilon_M^w + p(x) = w_M + \varepsilon_M^w + p(x) \tag{40}$$

and

$$p(x) = \sum_{m=0}^{M} P_m \cos \lambda_m x + \varepsilon_M^p = p_M + \varepsilon_M^p. \tag{41}$$

Then equations (18) and (19) can be alternatively written as

$$\int R_M \cos \lambda_m x \, \mathrm{d}x = 0, \quad m = 0, 1, 2, \ldots, M. \tag{42}$$

Clearly, the solution of equation (42) bears an error that results from the neglecting of the following equations.

$$\int R_M \cos \lambda_m x \, \mathrm{d}x = 0, \quad m = M+1, M+2, \ldots . \tag{43}$$

Making use of equations (21), (34) and (39), the discretization errors given by equation (43) can be expressed as

$$\int [D\partial^4 \varepsilon_M^w/\partial x^4 - \rho A\omega^2 (\varepsilon_M^w + \varepsilon_M^p)] \cos \lambda_m x \, \mathrm{d}x \sim O(\lambda_M^{-4}) + O(\lambda_M^{-2}) \sim O(\lambda_M^{-2}). \tag{44}$$

Similarly, the discretization errors corresponding to the sine series representation can be directly written as

$$\int [D\partial^4 \varepsilon_M^w/\partial x^4 - \rho A\omega^2 (\varepsilon_M^w + \varepsilon_M^p)] \sin \lambda_m x \, \mathrm{d}x \sim O(\lambda_M^{-4}) + O(\lambda_M^{-1}) \sim O(\lambda_M^{-1}). \tag{45}$$

It is seen from equations (44) and (45) that in both cases the discretization-related errors are more serious than their counterparts resulting from the truncation of the Fourier series of the displacement function and will be ultimately responsible for the accuracy and convergence of the final solutions. This conclusion agrees with the previous observation that when a different discretization scheme, the Galerkin's method, is used to solve equation (1), the modal results can be meaningfully improved with respect to the convergence and accuracy [9].

The discretization error, and hence the convergence speed of the final solution, is also dependent upon the boundary conditions. As shown in equations (44) and (45), for beams with a generally supported end the cosine and sine series solutions will, respectively,

TABLE 1

*Frequency parameters, $\mu = L/\pi(\mathrm{u}\sqrt{\rho A/D})^{1/2}$, for a clamped–pinned beam*

| Mode | $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$ | | | | |
|---|---|---|---|---|---|
| | Cosine series $M = 10$ | Sine series $M = 10$ | Cosine series $M = 20$ | Sine series $M = 20$ | Exact [10] |
| 1 | 1·24994 | 1·24988 | 1·24988 | 1·24988 | 1·24988 |
| 2 | 2·25036 | 2·25002 | 2·25005 | 2·25 | 2·25 |
| 3 | 3·25108 | 3·25011 | 3·25014 | 3·25 | 3·25 |
| 4 | 4·25293 | 4·25044 | 4·25032 | 4·25002 | 4·25 |

TABLE 2

*Frequency parameters, $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$, for a clamped–clamped beam*

| Mode | $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$ | | | | |
|---|---|---|---|---|---|
| | Cosine series $M = 10$ | Sine series $M = 10$ | Cosine series $M = 20$ | Sine series $M = 20$ | Exact [10] |
| 1 | 1·50562 | 1·50562 | 1·50562 | 1·50562 | 1·50562 |
| 2 | 2·49976 | 2·4998 | 2·49975 | 2·49975 | 2·49975 |
| 3 | 3·50003 | 3·50042 | 3·50001 | 3·50002 | 3·50001 |
| 4 | 4·5002 | 4·50091 | 4·5 | 4·50004 | 4·5 |

converge according to $\lambda_M^{-2}$ and $\lambda_M^{-1}$. However, when a beam is simply supported with only rotational restraints at each end (i.e., $\beta_0 = \beta_1 \equiv 0$), the sine series expansion will be converging at a much faster speed of $\lambda_M^{-3}$. Similarly, for the boundary conditions that do not allow a beam to rotate at any end, the convergence speed of the cosine series solution will be increased from $\lambda_M^{-2}$ to $\lambda_M^{-4}$. In the following section, several numerical examples will be given to check these remarks.

## 3. RESULTS AND DISCUSSIONS

Let us start with a beam with the clamped–pinned boundary condition. This boundary condition can be easily generated by setting the rotational stiffness to zero at $x = L$ and all the others to infinity (which is actually represented by a very large number, 1.0E + 10, in the following calculations). Table 1 shows the four lowest frequency parameters, $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$, obtained using both the sine and cosine expansions. In this example, while the displacement is identically equal to zero at each end, the first derivative is normally not zero at $x = L$. As mentioned earlier, this is the case that the sine series representation is better suited for. Now, if the stiffness of the rotational spring at $x = L$ also becomes infinite (i.e., the clamped–clamped boundary condition), then both the displacement and its derivative will be equal to zero at each end. Although this is a favorable scenario for both solutions, the cosine series should now outperform the sine series ($\lambda_M^{-4}$ versus $\lambda_M^{-3}$), which is evident from the results in Table 2.

TABLE 3

*Frequency parameters, $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$, for a simply supported beam with rotational restraints, $\hat{K}_1 L = \hat{K}_0 L = 1$, at both ends*

| | $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$ | | | | |
|---|---|---|---|---|---|
| Mode | Cosine series $M = 10$ | Cosine series $M = 20$ | Cosine series $M = 40$ | Sine series $M = 5$ | Sine series $M = 10, 20$ |
| 1 | 1·08192 | 1·08188 | 1·08187 | 1·08187 | 1·08187 |
| 2 | 2·04636 | 2·04592 | 2·04587 | 2·04587 | 2·04586 |
| 3 | 3·0331 | 3·03191 | 3·03175 | 3·03174 | 3·03173 |
| 4 | 4·02908 | 4·02474 | 4·02427 | 4·02429 | 4·02421 |

TABLE 4

*Frequency parameters, $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$, for a guided–guided beam with translational restraints, $\hat{k}_0 L^3 = \hat{k}_1 L^3 = 1$, at both ends*

| | $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$ | | | | |
|---|---|---|---|---|---|
| Mode | Sine series $M = 10$ | Sine series $M = 20$ | Sine series $M = 40$ | Cosine series $M = 3$ | Cosine series $M = 5, 10$ |
| 1 | 0·38217 | 0·380202 | 0·379232 | 0·378274 | 0·378274 |
| 2 | 1·02962 | 1·02008 | 1·01515 | 1·01011 | 1·01011 |
| 3 | 2·04667 | 2·02262 | 2·01168 | 2·00128 | 2·00128 |
| 4 | 3·06496 | 3·03142 | 3·0157 | 3·00038 | 3·00038 |

Next, consider a few examples involving elastic restraints at the ends. The first one concerns a simply supported beam with only rotational restraints, $\hat{K}_1 L = \hat{K}_0 L = 1$, at both ends. The frequency parameters $\mu = L/\pi(\omega\sqrt{\rho A/D})^{1/2}$ calculated using the sine and cosine series are compared in Table 3. Like in the first example, because of the zero displacements at both ends, the sine series is clearly converging at a faster speed than the cosine series.

To demonstrate that the cosine series solution is better suited for the other (kinds of) boundary conditions, let us consider a guided–guided beam with the translational restraints, $\hat{k}_0 L^3 = \hat{k}_1 L^3 = 1$. This is an ideal case for the cosine series solution because the first derivative is always zero at each end. As shown in Table 4, the cosine series solution converges so quickly that all the four natural frequencies can be accurately calculated with only four terms ($M = 3$). In comparison, the results even obtained from the 40-term sine series are not as nearly accurate. This should not come as a surprise because for this kind of boundary conditions the cosine and sine series solutions will, respectively, converge according to $\lambda_M^{-4}$ and $\lambda_M^{-1}$.

The last example deals with a general case in which a beam is supported by both translational and rotational springs at each end. Assuming $\hat{k}_0 L^3 = \hat{k}_1 L^3 = 1$ and $\hat{K}_1 L = \hat{K}_0 L = 100$, Table 5 lists the five lowest frequency parameters $\mu = (L^2\omega\sqrt{\rho A/D})^{1/2}$. It is seen that the cosine series expansion has clearly outperformed the sine series for this

TABLE 5

*Frequency parameters, $\mu = (L^2\omega\sqrt{\rho A/D})^{1/2}$, for a beam with general elastic restraints, $\hat{K}_1 L = \hat{K}_0 L = 100$ and $\hat{k}_1 L^3 = \hat{k}_0 L^3 = 1$*

| | $\mu = (L^2\omega\sqrt{\rho A/D})^{1/2}$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Mode | Sine $M = 10$ | Cosine $M = 10$ | Sine $M = 20$ | Cosine $M = 20$ | Sine $M = 40$ | Cosine $M = 40$ | [11] |
| 1 | 1·20053 | 1·188301 | 1·19436 | 1·188301 | 1·19131 | 1·188301 | 1·188301 |
| 2 | 3·20606 | 3·14418 | 3·17581 | 3·14418 | 3·16018 | 3·14418 | 3·144179 |
| 3 | 6·37557 | 6·227224 | 6·29699 | 6·227221 | 6·26121 | 6·22722 | 6·22722 |
| 4 | 9·5534 | 9·337013 | 9·44123 | 9·336975 | 9·38844 | 9·336971 | 9·336969 |
| 5 | 12·8067 | 12·45001 | 12·6042 | 12·44990 | 12·5229 | 12·44988 | 12·44988 |

general boundary condition. For example, the frequency parameters calculated from a 11-term cosine series are actually much more accurate than those obtained using 40 terms in the sine series expansion.

## 4. CONCLUDING REMARKS

A Fourier series method was previously proposed in reference [7] for the free vibration analysis of beams with arbitrary support at each end. This method can be presented in two versions corresponding to the sine and cosine series expansions of the displacement function. In this study, the sine and cosine series solutions are compared with respect to their convergence and accuracy. It is concluded that for a generally supported beam the cosine and sine series solutions are, respectively, converging according to $\lambda_M^{-2}$ and $\lambda_M^{-1}$. However, for the cases when a beam is simply supported with only rotational restraints, the convergence speed of the sine series solution can be greatly increased to $\lambda_M^{-3}$. It is also shown that for beams with zero rotations at both ends, the cosine series solution will be converging at the speed of $\lambda_M^{-4}$. Several numerical examples have been presented to verify these conclusions.

## REFERENCES


1. H. CHUNG 1981 *Journal of Sound and Vibration* **196**, 285–293. Free vibration analysis of circular cylindrical shells.
2. J. T.-S. WANG and C.-C. LIN 1996 *Journal of Sound and Vibration* **196**, 285–293. Dynamic analysis of generally supported beams using Fourier series.
3. A. W. LEISSA 1993 *Vibration of Plates*, 118–119. Acoustical Society of America.
4. W. K. TSO 1996 *Journal of the American Institute of Aeronautics and Astronautics* **4**, 733–735. On the fundamental frequency of a four point-supported square elastic plate.
5. D. ZHOU 1995 *Computers and Structures* **57**, 731–735. Natural frequencies of elastically restrained rectangular plates using a set of static beam functions in the Rayleigh–Ritz method.
6. C. P. FILIPICH and M. B. ROSALES 2000 *Journal of Sound and Vibration* **230**, 521–539. Arbitrary precision frequencies of a free rectangular thin plate.
7. W. L. LI 2000 *Journal of Sound and Vibration* **237**, 709–725. Free vibrations of beams with general boundary conditions.
8. G. P. TOLSTOV 1965 *Fourier Series*. Englewood Cliffs, NJ: Prentice-Hall.

9. W. L. Li 2001 *Journal of Sound and Vibration* **246**, 751–756. Dynamic analysis of beams with arbitrary elastic supports at both ends.
10. R. D. Blevins 1979 *Formulas for Natural Frequency and Mode Shape*, Vol. 108. New York: Van Nostrand Reinhold Company.
11. C. Kameswara Rao and S. Mirza 1989 *Journal of Sound and Vibration* **130**, 453–465. A note on vibration of generally restrained beams.