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THE ROLE AND EXPERIMENTAL DETERMINATION OF EQUIVALENT MASS IN COMPLEX SEA MODELS

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The application of statistical energy analysis to vibro-acoustic systems of complex geometry has been made practicable by the introduction of the concept of equivalent mass/equivalent volume. Although previous research at the ISVR has shown that these parameters can be directly measured, it has recently been found that the published formulae for carrying out this calculation are not sufficiently accurate. This is because it has been previously incorrectly assumed that the measurement on the subsystem of interest is unaffected by the presence of other attached subsystems. The paper derives the correct expressions for equivalent mass/equivalent volume for the general case of N-connected subsystems. By utilizing these derived expressions, the paper then proceeds to show that the coupling loss factors can be obtained directly in terms of the measured input power and vibration velocity/sound pressure. The paper concludes by showing that the power balance equations can, by utilizing the above expressions, be framed in terms of the subsystem velocities/sound pressures rather than in terms of subsystem energies.

1. INTRODUCTION

Statistical energy analysis (SEA) is concerned with the steady state balance between input, dissipated and transferred powers in a vibro-acoustic system. When it was first established, the primary objective was to predict acoustic and/or structural energy levels in discrete parts (termed subsystems) of a system for known input powers. More recently, a parallel body of research has also been developed whereby an SEA model is set-up from measurements made on existing hardware. The purpose of this latter development is to diagnose noise/vibration problems and to explore various solution strategies by extrapolating the model parameters.

Since all the relevant parameters are time, frequency band and spatially averaged, the technique is not only very robust but also does not require a detailed description of the vibro-acoustic system. This is because its functionality depends on global rather than detailed properties. SEA appears, therefore, to be a very attractive technique for the analysis of complex structures (e.g., a car), particularly at high frequencies, where there are problems in applying the finite element method. Once the system under investigation has been notionally subdivided into subsystems, the setting-up of the power balance equations (which relate input powers to subsystem energies) is a relatively straightforward procedure.

Unfortunately, this underlying simplicity conceals a hidden problem. Whilst it is not difficult to predict subsystem energy levels, the conversion of these to velocities or acoustic pressures is not usually a straightforward matter. If the system considered is uniform and of

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regular geometrical shape, then the conversion factor is simply the subsystem mass or volume. However, for most actual structures and acoustic cavities, the use of such a conversion factor will generally introduce significant errors. For experimentally based models, there is a similar difficulty in converting vibration/sound pressure measurements to energy levels. This problem has been partly overcome by using the equivalent mass/equivalent volume, which can be measured experimentally, although it has been found that even this development does not entirely eliminate the errors.

Recent research at the ISVR has found that in many cases these errors have been due to the incorrect formulation of equivalent mass/equivalent volume and this paper shows the derivation of the corrected theory. However, the paper also demonstrates that the coupling loss factors can be directly obtained from measured quantities, without involving the equivalent mass and/or equivalent volume, thus significantly improving the convenience and accuracy of an experimentally based SEA model.

2. BACKGROUND THEORY

For a uniform beam or plate, the frequency band and time-averaged total energy of vibration can be expressed as

$$\overline{E_{total}} = M\langle\overline{V^2}\rangle, \tag{1}$$

where M is the the total mass of the component, and $\langle\overline{V^2}\rangle$ the frequency band, time and space-averaged square of the velocity.

Similarly, the frequency band and time-averaged total energy of a regularly shaped acoustic volume is

$$\overline{E_{total}} = \frac{V}{\rho c^2}\langle\overline{p^2}\rangle, \tag{2}$$

where V is the volume of the acoustic space, $\langle\overline{p^2}\rangle$ the frequency band, time and space-averaged square of the pressure, ρ the density of air, and c the speed of sound in air.

However, for structures or volumes having irregular geometry, it has been found [1] that the energies are not proportional to their respective masses or volumes. The reasons for this are enumerated below for the case of a complex structure, although similar arguments apply for volumes of complex shape.

Consider a stand-alone subsystem that is being excited in a particular frequency band by a force whose spectrum is constant with time. The input power $\overline{P_{in}}$ to the subsystem (which equals the power absorbed by the subsystem) is given by

$$\overline{P_{in}} = \omega\overline{E_{total}}\,.\,\eta_i = \omega M_{eq}\langle V^2\rangle\eta_i, \tag{3}$$

where ω is the band centre frequency (rad/s), η_i the band-averaged internal loss factor, and M_{eq} the band-averaged equivalent mass.

Equation (3) is, in fact, a definition of equivalent mass which, by rearranging, gives

$$M_{eq} = \frac{\overline{P_{in}}}{\omega\eta_i\langle\overline{V^2}\rangle}\,. \tag{4}$$

It should be noted that $\overline{P_{in}}$ and $\langle\overline{V^2}\rangle$ are both either frequency band totals or frequency band averages.

3. REASONS FOR DIFFERENCE BETWEEN M AND M_{eq}

3.1. INSUFFICIENT SAMPLES FOR $\langle\overline{V^2}\rangle$

In order to obtain an accurate value for the mean-square velocity for a subsystem, it is necessary to take velocity samples at a sufficient number of locations. If this is not done, the measured value of $\langle\overline{V^2}\rangle$ will be subject to random variations. It can be seen from equation (4) that this will cause corresponding variations in M_{eq}.

On many structures (e.g., car door or engine cylinder block), there are parts where access for velocity sampling is not practicable. It can be seen from equation (3) that when the vibration level of such hidden components is high, increased power will be absorbed by the subsystem. However, although it is generally found that on most practical structures resonances of internal parts are detected by small response peaks on the accessible outside surfaces (due to the coupling that results from the complex three-dimensional shapes), the increase in measured $\langle\overline{V^2}\rangle$ does not adequately reflect the increase in the absorbed power. Thus, when this occurs, M_{eq} will have a high value.

Another factor related to this sampling error is due to the fact that most structures are not uniform in thickness. Hence, the velocity measurements should reflect this by being distributed such that areas of equal mass are equally sampled. Since this is usually impractical, a further error results [2].

3.2. ERRORS IN MEASURING η_i

For a single, stand-alone subsystem, the measurement of M_{eq} is essentially based on the dual definition of the internal loss factor, η_i, *viz.*

$$\eta_i = \frac{\overline{P_{diss}}}{\omega\overline{E_{total}}} = \frac{\overline{P_{diss}}}{\omega M_{eq}\langle\overline{V^2}\rangle} = \frac{13{\cdot}82}{\omega T_{60}}, \tag{5}$$

where $\overline{P_{diss}}$ is the power dissipated by the subsystem, and T_{60} the frequency band and space-averaged reverberation time (i.e., the time taken by the energy response to decay by 60 dB) of the subsystem.

Rearranging the last two expressions in equation (5) gives

$$M_{eq} = \frac{\overline{P_{diss}}T_{60}}{13{\cdot}82\langle\overline{V^2}\rangle}. \tag{6}$$

If $\langle\overline{V^2}\rangle/\overline{P_{diss}}$ is defined as v, the normalized subsystem mean-square velocity, then the expression for the equivalent mass (in terms of the internal loss factor) is also equivalent to

$$M_{eq} = \frac{1}{\omega\eta_i v}. \tag{7}$$

It is the measurement of T_{60} which can cause some problems. This is particularly the case when there are two dominant modes in the band which beat together. Furthermore, since it is the *initial* decay rate which is required, measurement is often difficult because the waveform is obscured by the initial transient behaviour of the subsystem (due to impulsive

excitation, or when steady state excitation is switched off). Care must also be exercised to ensure that the decay is controlled by the damping of the subsystem and not by the damping of the filter. These factors tend to produce an overestimate of T_{60} and hence an overestimate of M_{eq}.

It is worth mentioning at this stage that, in practice, many T_{60} measurement errors can be reduced to a minimum by using octave band filters and then interpolating the results to obtain one-third octave values. This can be justified on the basis that, although the loss factors of individual modes in a band often vary significantly, this is not generally the case with the band averages. Hence, the resulting loss factor spectrum is usually quite similar to a straight line when plotted on a log–log basis. Another reason for using octave rather than one-third octave filters is that it tends to avoid the common problem that some low-frequency one-third octave bands do not contain any modes.

3.3. SPARCE MODAL DENSITY

The fact that the total energy of each subsystem in a frequency band is calculated from measurements of velocity (or acoustic pressure) implies that the totals of the maximum kinetic and maximum strain energies are equal. However, this is often not the case, particularly when the modal density is low. Here, the equivalent mass acts as a compensator for this discrepancy [3].

3.4. EFFECT OF OTHER CONNECTED SUBSYSTEMS

Another problem in the evaluation of M_{eq} stems from the fact that the effect of coupling to other subsystems is not taken into account. It had previously been assumed that other connected subsystems would not affect the prediction of the equivalent mass since the resultant increase in input power would be balanced by a corresponding decrease in the value of T_{60}. It has recently been found that this is not the case and the following section shows how the effect of connected subsystems on equivalent mass can be taken into account.

4. REVISED FORMULATION FOR M_{eq}

It should be noted at this point that in all further analysis it will be implicitly assumed that energy, velocity and power values are averaged over space and time for a given frequency band. Therefore, in the interests of simplicity of notation, all bars above these quantities will no longer be shown.

If the response of two coupled subsystems i and j within a given frequency band is considered, then by definition

$$E_{ij} = M_{eq_i}\langle V_{ij}^2\rangle, \tag{8}$$

where E_{ij} is the energy level averaged over space and time of subsystem i when subsystem j is excited, M_{eq_i} the equivalent mass of subsystem i, and $\langle V_{ij}^2\rangle$ the measured vibration level averaged over space and time of subsystem i when subsystem j is excited.

The power balance equation for a structure comprising N subsystems may be written as

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_N \end{Bmatrix} = \omega \begin{bmatrix} \sum_{j=1}^{N} \eta_{1j} & -\eta_{21} & \cdots & -\eta_{i1} & \cdots & -\eta_{N1} \\ -\eta_{12} & \sum_{j=1}^{N} \eta_{2j} & \cdots & -\eta_{i2} & \cdots & -\eta_{N2} \\ \vdots & & \ddots & & & \vdots \\ -\eta_{1i} & -\eta_{2i} & \cdots & \sum_{j=1}^{N} \eta_{ij} & \cdots & -\eta_{Ni} \\ \vdots & & & & \ddots & \vdots \\ -\eta_{1N} & -\eta_{2N} & \cdots & & & \sum_{j=1}^{N} \eta_{Nj} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ \vdots \\ E_N \end{Bmatrix}, \tag{9}$$

where $\eta_{ii} = \eta_i$ is the internal loss factor.

The internal loss factors can also be expressed independently [4] as

$$\omega \begin{bmatrix} E_{11} & E_{21} & \cdots & E_{N1} \\ E_{12} & \ddots & & \vdots \\ \vdots & & & \\ E_{1N} & \cdots & & E_{NN} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{Bmatrix}. \tag{10}$$

The time and space average of the vibration level of subsystem i when subsystem j is excited, normalized per unit power injected into subsystem j, is defined as

$$v_{ij} = \frac{\langle V_{ij}^2 \rangle}{P_j}. \tag{11}$$

Expressing the energies in equation (10) in terms of the equivalent masses and the averaged vibration levels normalized per unit power leads to

$$\omega \begin{bmatrix} M_{eq_1} v_{11} & M_{eq_2} v_{21} & \cdots & M_{eq_N} v_{N1} \\ M_{eq_1} v_{12} & \ddots & & \vdots \\ \vdots & & & \\ M_{eq_1} v_{1N} & \cdots & & M_{eq_N} v_{NN} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{12}$$

which may be re-written as

$$\omega \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{N1} \\ v_{12} & \ddots & & \vdots \\ \vdots & & & \\ v_{1N} & \cdots & & v_{NN} \end{bmatrix} \begin{bmatrix} M_{eq_1} & 0 & \cdots & 0 \\ 0 & M_{eq_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_{eq_N} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \tag{13}$$

or

$$\omega \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{N1} \\ v_{12} & \ddots & & \vdots \\ \vdots & & & \\ v_{1N} & \cdots & & v_{NN} \end{bmatrix} \begin{bmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \eta_N \end{bmatrix} \begin{Bmatrix} M_{eq_1} \\ M_{eq_2} \\ \vdots \\ M_{eq_N} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}. \tag{14}$$

Hence,

$$\begin{Bmatrix} M_{eq_1} \\ M_{eq_2} \\ \vdots \\ M_{eq_N} \end{Bmatrix} = \frac{1}{\omega} \begin{bmatrix} \frac{1}{\eta_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\eta_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\eta_N} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{N1} \\ v_{12} & \ddots & & \vdots \\ \vdots & & & \\ v_{1N} & \cdots & & v_{NN} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}. \tag{15}$$

In the case of two coupled subsystems, equation (15) reduces to

$$M_{eq_i} = \frac{(v_{jj} - v_{ji})}{\omega \eta_i (v_{ii} v_{jj} - v_{ij} v_{ji})} \tag{16}$$

whereas equation (7) gives

$$M_{eq_i} = \frac{1}{\omega \eta_i v_{ii}}. \tag{17}$$

Note that if the terms in equation (16) which are preceded by a minus sign are small compared with those which are preceded by a plus sign, the two expressions above are approximately equivalent. Physically, this occurs when the normalized vibration levels of a directly excited subsystem are much greater than those of a connected subsystem. This is especially true when the two subsystems are weakly coupled.

It can be seen that as the number of subsystems is increased, the expressions for the equivalent masses obtained from equation (15) become increasingly complicated due to the inversion of a larger matrix. It might be assumed that since the valid application of SEA relies on weak coupling between subsystems, any discrepancies in the equivalent masses resulting from the use of equation (17) should be minimal, which does not significantly affect the results. However, it has been noted that, in practice, even in situations where the normalized vibration levels of directly excited subsystems are much greater than those of the indirectly excited subsystems, the errors can actually be very significant. This is especially true as the number of subsystems increases.

5. ELIMINATION OF THE EQUIVALENT MASSES IN THE EXPRESSIONS FOR THE COUPLING LOSS FACTORS

In reference [5], expressions for the coupling loss factors in terms of only the normalized vibration levels and internal loss factors have been obtained in the case of two and three subsystems. The following analysis will show how these results can be generalized for N subsystems.

It has been shown [4] that, for the ith subsystem of a structure comprising N subsystems

$$\begin{bmatrix} \left(\frac{E_{11}}{E_{i1}} - \frac{E_{1i}}{E_{ii}}\right) & \cdots & \left(\frac{E_{r1}}{E_{i1}} - \frac{E_{ri}}{E_{ii}}\right) & \cdots & \left(\frac{E_{N1}}{E_{i1}} - \frac{E_{Ni}}{E_{ii}}\right) \\ & \ddots & & & \\ \vdots & & \left(\frac{E_{rr}}{E_{ir}} - \frac{E_{ri}}{E_{ii}}\right) & & \vdots \\ & & & \ddots & \\ \left(\frac{E_{1N}}{E_{iN}} - \frac{E_{1i}}{E_{ii}}\right) & \cdots & \left(\frac{E_{rN}}{E_{iN}} - \frac{E_{ri}}{E_{ii}}\right) & \cdots & \left(\frac{E_{NN}}{E_{iN}} - \frac{E_{Ni}}{E_{ii}}\right) \end{bmatrix}_{r \neq i} \begin{Bmatrix} \eta_{1i} \\ \vdots \\ \eta_{ri} \\ \vdots \\ \eta_{Ni} \end{Bmatrix}_{r \neq i} = \frac{P_i}{\omega E_{ii}} \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix}. \tag{18}$$

Expressing the energies in terms of the equivalent masses and the normalized vibration levels gives

$$\frac{1}{M_{eq_i}} \begin{bmatrix} M_{eq_1}\left(\frac{v_{11}}{v_{i1}} - \frac{v_{1i}}{v_{ii}}\right) & \cdots & M_{eq_r}\left(\frac{v_{r1}}{v_{i1}} - \frac{v_{ri}}{v_{ii}}\right) & \cdots & M_{eq_N}\left(\frac{v_{N1}}{v_{i1}} - \frac{v_{Ni}}{v_{ii}}\right) \\ & \ddots & & & \\ \vdots & & M_{eq_r}\left(\frac{v_{rr}}{v_{ir}} - \frac{v_{ri}}{v_{ii}}\right) & & \vdots \\ & & & \ddots & \\ M_{eq_1}\left(\frac{v_{1N}}{v_{iN}} - \frac{v_{1i}}{v_{ii}}\right) & \cdots & M_{eq_r}\left(\frac{v_{rN}}{v_{iN}} - \frac{v_{ri}}{v_{ii}}\right) & \cdots & M_{eq_N}\left(\frac{v_{NN}}{v_{iN}} - \frac{v_{Ni}}{v_{ii}}\right) \end{bmatrix}_{r \neq i} \begin{Bmatrix} \eta_{1i} \\ \vdots \\ \eta_{ri} \\ \vdots \\ \eta_{Ni} \end{Bmatrix}_{r \neq i}$$

$$= \frac{1}{\omega v_{ii} M_{eq_i}} \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{19}$$

which may be re-written as

$$[\mathbf{A}] \begin{bmatrix} \eta_{1i} & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \eta_{ri} & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & \eta_{Ni} \end{bmatrix}_{r \neq i} \begin{Bmatrix} M_{eq_1} \\ \vdots \\ M_{eq_r} \\ \vdots \\ M_{eq_N} \end{Bmatrix}_{r \neq i} = \frac{1}{\omega v_{ii}} \begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{20}$$

where

$$[\mathbf{A}] = \begin{bmatrix} \left(\frac{v_{11}}{v_{i1}} - \frac{v_{1i}}{v_{ii}}\right) & \cdots & \left(\frac{v_{r1}}{v_{i1}} - \frac{v_{ri}}{v_{ii}}\right) & \cdots & \left(\frac{v_{N1}}{v_{i1}} - \frac{v_{Ni}}{v_{ii}}\right) \\ & \ddots & & & \\ \vdots & & \left(\frac{v_{rr}}{v_{ir}} - \frac{v_{ri}}{v_{ii}}\right) & & \vdots \\ & & & \ddots & \\ \left(\frac{v_{1N}}{v_{iN}} - \frac{v_{1i}}{v_{ii}}\right) & \cdots & \left(\frac{v_{rN}}{v_{iN}} - \frac{v_{ri}}{v_{ii}}\right) & \cdots & \left(\frac{v_{NN}}{v_{iN}} - \frac{v_{Ni}}{v_{ii}}\right) \end{bmatrix}_{r \neq i}.$$

It will be noted that although equation (15) expresses all the M_{eq}'s from 1 to N, equation (20) does not include the ith term. Therefore, before the M_{eq}'s can be eliminated, the terms in $M_{eq_i}\eta_i$ must be removed from equation (15). This is achieved as follows: firstly equation (14) is re-written as

$$\begin{aligned}
&M_{eq_1}\eta_1 v_{11} + M_{eq_2}\eta_2 v_{21} + \cdots + M_{eq_i}\eta_i v_{i1} + \cdots + M_{eq_N}\eta_N v_{N1} = \frac{1}{\omega}\\
&M_{eq_1}\eta_1 v_{12} + M_{eq_2}\eta_2 v_{22} + \cdots + M_{eq_i}\eta_i v_{i2} + \cdots + M_{eq_N}\eta_N v_{N2} = \frac{1}{\omega}\\
&\vdots\\
&M_{eq_1}\eta_1 v_{1i} + M_{eq_2}\eta_2 v_{2i} + \cdots + M_{eq_i}\eta_i v_{ii} + \cdots + M_{eq_N}\eta_N v_{Ni} = \frac{1}{\omega}\\
&\vdots\\
&M_{eq_1}\eta_1 v_{1N} + M_{eq_2}\eta_2 v_{2N} + \cdots + M_{eq_i}\eta_i v_{iN} + \cdots + M_{eq_N}\eta_N v_{NN} = \frac{1}{\omega}
\end{aligned} \tag{21}$$

In order to eliminate M_{eq_i}, the rth equation is multiplied by v_{ii} and the ith equation by v_{ir}, and then the rth equation is subtracted from the ith one.

By carrying out this procedure, the following equation is obtained:

$$[\mathbf{B}]\begin{bmatrix} \eta_1 & 0 & \cdots & & 0\\ 0 & \ddots & & & \vdots\\ \vdots & & \eta_r & & \\ & & & \ddots & 0\\ 0 & \cdots & & 0 & \eta_N \end{bmatrix}_{r\neq i}\begin{Bmatrix} M_{eq_1}\\ \vdots\\ M_{eq_r}\\ \vdots\\ M_{eq_N}\end{Bmatrix}_{r\neq i} = \frac{1}{\omega}\begin{Bmatrix}1\\ \vdots\\ 1\\ \vdots\\ 1\end{Bmatrix} \tag{22}$$

or

$$\begin{Bmatrix} M_{eq_1}\\ \vdots\\ M_{eq_r}\\ \vdots\\ M_{eq_N}\end{Bmatrix}_{r\neq 1} = \frac{1}{\omega}\begin{bmatrix} \dfrac{1}{\eta_1} & 0 & \cdots & & 0\\ 0 & \ddots & & & \vdots\\ \vdots & & \dfrac{1}{\eta_r} & & \\ & & & \ddots & 0\\ 0 & \cdots & & 0 & \dfrac{1}{\eta_N} \end{bmatrix}_{r\neq i}[\mathbf{B}]^{-1}\begin{Bmatrix}1\\ \vdots\\ 1\\ \vdots\\ 1\end{Bmatrix}, \tag{23}$$

where

$$[\mathbf{B}] = \begin{bmatrix} \dfrac{(v_{11}v_{ii} - v_{1i}v_{i1})}{(v_{ii} - v_{i1})} & \cdots & \dfrac{(v_{r1}v_{ii} - v_{ri}v_{i1})}{(v_{ii} - v_{i1})} & \cdots & \dfrac{(v_{N1}v_{ii} - v_{Ni}v_{i1})}{(v_{ii} - v_{i1})}\\ & \ddots & & & \\ \vdots & & \dfrac{(v_{rr}v_{ii} - v_{ri}v_{ir})}{(v_{ii} - v_{ir})} & & \vdots\\ & & & \ddots & \\ \dfrac{(v_{1N}v_{ii} - v_{1i}v_{iN})}{(v_{ii} - v_{iN})} & \cdots & \dfrac{(v_{rN}v_{ii} - v_{ri}v_{iN})}{(v_{ii} - v_{iN})} & \cdots & \dfrac{(v_{NN}v_{ii} - v_{Ni}v_{iN})}{(v_{ii} - v_{iN})} \end{bmatrix}_{r\neq i}.$$

Substituting equation (23) into equation (20) to eliminate the M_{eq}'s gives

$$[\mathbf{A}]\begin{bmatrix} \eta_{1i} & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \eta_{ri} & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & \eta_{Ni} \end{bmatrix}_{r\neq i} \begin{bmatrix} \dfrac{1}{\eta_1} & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \dfrac{1}{\eta_r} & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & \dfrac{1}{\eta_N} \end{bmatrix}_{r\neq i} [\mathbf{B}]^{-1}\begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \frac{1}{v_{ii}}\begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \tag{24}$$

or

$$[\mathbf{A}]\begin{bmatrix} \dfrac{\eta_{1i}}{\eta_1} & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \dfrac{\eta_{ri}}{\eta_r} & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & \dfrac{\eta_{Ni}}{\eta_N} \end{bmatrix}_{r\neq i} \{\mathbf{C}\} = \frac{1}{v_{ii}}\begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{25}$$

where

$$\{\mathbf{C}\} = [\mathbf{B}]^{-1}\begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}. \tag{26}$$

Equation (25) is also equivalent to

$$[\mathbf{A}]\begin{bmatrix} C(1) & 0 & \cdots & & 0 \\ 0 & C(2) & & & \vdots \\ \vdots & & \ddots & & \\ & & & & 0 \\ 0 & \cdots & & 0 & C(N-1) \end{bmatrix}\begin{Bmatrix} \dfrac{\eta_{1i}}{\eta_1} \\ \vdots \\ \dfrac{\eta_{ri}}{\eta_r} \\ \vdots \\ \dfrac{\eta_{Ni}}{\eta_N} \end{Bmatrix}_{r\neq i} = \frac{1}{v_{ii}}\begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}, \tag{27}$$

where $C(r)$ corresponds to the element in the rth row of $\{\mathbf{C}\}$.

Rearranging equation (27) gives

$$\begin{Bmatrix} \frac{\eta_{1i}}{\eta_1} \\ \vdots \\ \frac{\eta_{ri}}{\eta_r} \\ \vdots \\ \frac{\eta_{Ni}}{\eta_N} \end{Bmatrix}_{r \neq i} = \frac{1}{v_{ii}} \begin{bmatrix} \frac{1}{C(1)} & 0 & \cdots & & 0 \\ 0 & \frac{1}{C(2)} & & & \vdots \\ \vdots & & \ddots & & \\ & & & & 0 \\ 0 & \cdots & & 0 & \frac{1}{C(N-1)} \end{bmatrix} [\mathbf{A}]^{-1} \begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}. \tag{28}$$

6. VERIFICATION

Although it is not practicable to verify the preceding theory independently, a comparison can be made with the previously published work. In reference [5], expressions for the ratios of the coupling loss factors to the internal loss factors in terms of the averaged vibration levels have been obtained in the cases of two- and three-coupled subsystems. It will now be shown that the same expressions may be obtained using equation (28).

6.2. TWO SUBSYSTEM CASE ($N = 2$)

For $i = 2$, equation (24) may be re-written as

$$\left(\frac{v_{11}v_{22} - v_{12}v_{21}}{v_{12}v_{22}}\right)\left(\frac{\eta_{12}}{\eta_1}\right)\left(\frac{v_{22} - v_{21}}{v_{11}v_{22} - v_{12}v_{21}}\right) = \frac{1}{v_{22}} \tag{29}$$

and hence

$$\frac{\eta_{12}}{\eta_1} = \frac{v_{22}}{(v_{22} - v_{21})}. \tag{30}$$

Similarly, if $i = 1$,

$$\frac{\eta_{21}}{\eta_2} = \frac{v_{11}}{(v_{11} - v_{12})}. \tag{31}$$

These results are the same as those obtained in reference [5].

6.2. THREE SUBSYSTEM CASE ($N = 3$)

Assume, for example, that $i = 2$. The following expressions are obtained for $[\mathbf{A}]$ and $[\mathbf{B}]$:

$$[\mathbf{A}] = \begin{bmatrix} \left(\frac{v_{11}}{v_{21}} - \frac{v_{12}}{v_{22}}\right) & \left(\frac{v_{31}}{v_{21}} - \frac{v_{32}}{v_{22}}\right) \\ \left(\frac{v_{13}}{v_{23}} - \frac{v_{12}}{v_{22}}\right) & \left(\frac{v_{33}}{v_{23}} - \frac{v_{32}}{v_{22}}\right) \end{bmatrix} \text{ and } [\mathbf{B}] = \begin{bmatrix} \frac{(v_{11}v_{22} - v_{12}v_{21})}{(v_{22} - v_{21})} & \frac{(v_{31}v_{22} - v_{32}v_{21})}{(v_{22} - v_{21})} \\ \frac{(v_{13}v_{22} - v_{12}v_{23})}{(v_{22} - v_{23})} & \frac{(v_{33}v_{22} - v_{32}v_{23})}{(v_{22} - v_{23})} \end{bmatrix}.$$

Inverting both these matrices gives

$$[\mathbf{A}]^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} \left(\frac{v_{33}}{v_{23}} - \frac{v_{32}}{v_{22}}\right) & \left(-\frac{v_{31}}{v_{21}} + \frac{v_{32}}{v_{22}}\right) \\ \left(-\frac{v_{13}}{v_{23}} + \frac{v_{12}}{v_{22}}\right) & \left(\frac{v_{11}}{v_{21}} - \frac{v_{12}}{v_{22}}\right) \end{bmatrix} \tag{32}$$

and

$$[\mathbf{B}]^{-1} = \frac{1}{|\mathbf{B}|}\begin{bmatrix} \frac{(v_{33}v_{22} - v_{32}v_{23})}{(v_{22} - v_{23})} & \frac{(-v_{31}v_{22} + v_{32}v_{21})}{(v_{22} - v_{21})} \\ \frac{(-v_{13}v_{22} + v_{12}v_{23})}{(v_{22} - v_{23})} & \frac{(v_{11}v_{22} - v_{12}v_{21})}{(v_{22} - v_{21})} \end{bmatrix}, \tag{33}$$

where

$$|\mathbf{A}| = \frac{(v_{11}v_{22}^2v_{33} - v_{11}v_{22}v_{23}v_{32} - v_{12}v_{21}v_{22}v_{33} - v_{13}v_{22}^2v_{31} + v_{13}v_{21}v_{22}v_{32} + v_{12}v_{22}v_{23}v_{31})}{v_{21}v_{22}^2v_{23}}$$

and

$$|\mathbf{B}| = \frac{v_{21}v_{22}^2v_{23}}{(v_{22} - v_{21})\ (v_{22} - v_{23})}\,|\mathbf{A}|\,.$$

By definition (see equation (26))

$$\{\mathbf{C}\} = [\mathbf{B}]^{-1}\begin{Bmatrix}1\\1\end{Bmatrix}, \tag{34}$$

which gives

$$\{\mathbf{C}\} = \frac{v_{22}}{|\mathbf{B}|\,(v_{22} - v_{21})\ (v_{22} - v_{23})} \times \begin{Bmatrix} v_{22}v_{33} - v_{21}v_{33} - v_{23}v_{32} + v_{21}v_{32} - v_{22}v_{31} + v_{23}v_{31} \\ v_{12}v_{23} - v_{13}v_{22} - v_{12}v_{21} + v_{13}v_{21} - v_{11}v_{23} + v_{11}v_{22} \end{Bmatrix} \tag{35}$$

or

$$\{\mathbf{C}\} = \frac{v_{22}}{|\mathbf{B}|(v_{22} - v_{21})\ (v_{22} - v_{23})}\begin{Bmatrix}D_1\\D_2\end{Bmatrix}, \tag{36}$$

where $D_1 = (v_{22}v_{33} - v_{21}v_{33} - v_{23}v_{32} + v_{21}v_{32} - v_{22}v_{31} + v_{23}v_{31})$ and

$$D_2 = (v_{12}v_{23} - v_{13}v_{22} - v_{12}v_{21} + v_{13}v_{21} - v_{11}v_{23} + v_{11}v_{22}).$$

Substituting into equation (28) gives

$$\begin{Bmatrix} \frac{\eta_{12}}{\eta_1} \\ \frac{\eta_{32}}{\eta_3} \end{Bmatrix} = v_{21}v_{23}\begin{bmatrix} \frac{1}{D_1} & 0 \\ 0 & \frac{1}{D_2} \end{bmatrix}\begin{bmatrix} \left(\frac{v_{33}}{v_{23}} - \frac{v_{32}}{v_{22}}\right) & \left(-\frac{v_{31}}{v_{21}} + \frac{v_{32}}{v_{22}}\right) \\ \left(-\frac{v_{13}}{v_{23}} + \frac{v_{12}}{v_{22}}\right) & \left(\frac{v_{11}}{v_{21}} - \frac{v_{12}}{v_{22}}\right) \end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}. \tag{37}$$

Hence,

$$\begin{Bmatrix} \eta_{12}/\eta_1 \\ \eta_{32}/\eta_3 \end{Bmatrix} = \begin{Bmatrix} (v_{21}v_{33} - v_{23}v_{31}) \\ \hline (v_{22}v_{33} - v_{21}v_{33} - v_{23}v_{32} + v_{21}v_{32} - v_{22}v_{31} + v_{23}v_{31}) \\ (v_{11}v_{23} - v_{13}v_{21}) \\ \hline (v_{12}v_{23} - v_{13}v_{22} - v_{12}v_{21} + v_{13}v_{21} - v_{11}v_{23} + v_{11}v_{22}) \end{Bmatrix}. \tag{38}$$

For $i = 1$ and 3, similar expressions may be obtained. Again, the same results may be found in reference [5]. In addition, Bharj *et al.* [6] have experimentally verified equations (30), (31) and (38) for two and three subsystem models. They found that the coupling loss factor values calculated using these equations were indistinguishable from those obtained by the normal method.

7. ELIMINATION OF THE EQUIVALENT MASS IN THE BACK CALCULATION OF SUBSYSTEM VELOCITY

A very useful check on the accumulated error due to the various approximations used in estimating the SEA parameters is to compare the back calculation of a subsystem velocity with the directly measured transfer function. Of course, this is only of value when the subsystem in question is remote from the excited subsystem—otherwise exact agreement will result because the measured transfer function has already been used in the parameter computation. This checking procedure has been described in detail by Hermans *et al.* [7].

If power P_i is simultaneously input into each subsystem i, then from equation (9)

$$\begin{Bmatrix} E_1 \\ \vdots \\ \vdots \\ E_N \end{Bmatrix} = \frac{1}{\omega} \begin{bmatrix} \sum_{j=1}^{N} \eta_{1j} & \cdots & -\eta_{N1} \\ \vdots & \ddots & \vdots \\ -\eta_{1N} & \cdots & \sum_{j=1}^{N} \eta_{Nj} \end{bmatrix}^{-1} \begin{Bmatrix} P_1 \\ \vdots \\ \vdots \\ P_N \end{Bmatrix} \tag{39}$$

or

$$\begin{Bmatrix} E_1 \\ \vdots \\ \vdots \\ E_N \end{Bmatrix} = \frac{1}{\omega} [\mathbf{L}]^{-1} \begin{Bmatrix} P_1 \\ \vdots \\ \vdots \\ P_N \end{Bmatrix}. \tag{40}$$

Expressing the resultant subsystem energies in terms of their velocities $\langle V_i^2 \rangle$ gives

$$\begin{bmatrix} M_{eq_1} & & \cdots & 0 \\ & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & M_{eq_N} \end{bmatrix} \begin{Bmatrix} \langle V_1^2 \rangle \\ \vdots \\ \vdots \\ \langle V_N^2 \rangle \end{Bmatrix} = \frac{1}{\omega} [\mathbf{L}]^{-1} \begin{Bmatrix} P_1 \\ \vdots \\ \vdots \\ P_N \end{Bmatrix}. \tag{41}$$

It can also be easily shown [8] that, by injecting power sequentially into each subsystem in turn

$$[\mathbf{L}]\begin{bmatrix} E_{11} & \cdots & \cdots & E_{1N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ E_{N1} & \cdots & \cdots & E_{NN} \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} P_1 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & P_N \end{bmatrix}. \tag{42}$$

Equation (42) can be rearranged to give

$$[\mathbf{L}]^{-1} = \omega\begin{bmatrix} \dfrac{E_{11}}{P_1} & \cdots & \cdots & \dfrac{E_{1N}}{P_N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \dfrac{E_{N1}}{P_1} & \cdots & \cdots & \dfrac{E_{NN}}{P_N} \end{bmatrix} \tag{43}$$

or

$$[\mathbf{L}]^{-1} = \omega\begin{bmatrix} M_{eq_1} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & M_{eq_N} \end{bmatrix}\begin{bmatrix} v_{11} & \cdots & \cdots & v_{1N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ v_{N1} & \cdots & \cdots & v_{NN} \end{bmatrix}. \tag{44}$$

Eliminating $[\mathbf{L}]^{-1}$ by substituting equation (44) into equation (41) gives

$$\begin{bmatrix} M_{eq_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{eq_N} \end{bmatrix}\begin{Bmatrix} \langle V_1^2 \rangle \\ \vdots \\ \langle V_N^2 \rangle \end{Bmatrix} = \begin{bmatrix} M_{eq_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{eq_N} \end{bmatrix}\begin{bmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{bmatrix}\begin{Bmatrix} P_1 \\ \vdots \\ P_N \end{Bmatrix} \tag{45}$$

or

$$\begin{Bmatrix} \langle V_1^2 \rangle \\ \vdots \\ \langle V_N^2 \rangle \end{Bmatrix} = \begin{bmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{bmatrix}\begin{Bmatrix} P_1 \\ \vdots \\ P_N \end{Bmatrix}. \tag{46}$$

Equation (46) is essentially the same as the power balance equation (9) but without the equivalent masses. It will be recalled that the coupling loss factors can also be expressed in terms of the normalized mean-square velocities v_{ij} and the internal loss factors, using equation (28).

Equations (45) and (46) are particularly interesting because they show that the energy of any subsystem is the linear sum of the energies due to each input acting on its own, as has previously been reported by Hermans [9]. It should also be noted that since equation (46) does not contain any coupling loss factors, it is always valid, regardless of the validity of the SEA model. The equation is, in fact, a space and frequency band-averaged version of the well-known transfer function matrix equation.

8. CONCLUDING REMARKS

The paper demonstrates that the expression currently used for the equivalent mass is not consistent with the power balance equations. A correct expression has been obtained for the general N subsystem case. This relationship has then been used to derive an expression for the coupling loss factors solely in terms of measured velocities, internal loss factors and input powers.

Since the calculation of equivalent mass has been much prone to error, it is to be expected that its elimination will significantly improve the experimental determination of the coupling loss factors. The paper also shows that, by using the above-derived relationships, the power balance equations can be framed solely in terms of directly measured quantities.

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