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PROPORTIONALLY DAMPED SYSTEMS SUBJECTED TO DAMPING MODIFICATIONS BY SEVERAL VISCOUS DAMPERS

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1. INTRODUCTION

In the context of the derivation of the characteristic equation of a continuous structure, discretized according to the assumed-modes method to which several spring–mass systems are attached, Cha and his co-worker have introduced a new approach [1, 2].

They assumed that the discretized continuous system has n degrees of freedom (d.o.f.) and r additional spring–mass systems are to be attached to this system, where, in general $r \ll n$. The free vibration of such a combined system is governed by the solution of a generalized eigenvalue problem of order $(n \times n)$, whose stiffness and mass matrices consist of diagonal matrices modified by a total of r rank-one matrices. They manipulated the general eigenvalue problem such that the eigenfrequencies governing free vibrations can be calculated by solving a much smaller characteristic determinant of order $(r \times r)$, each element of which involves a sum of n terms, instead of finding the roots of a much larger determinant of order $(n \times n)$.

The present study deals with a quite different mechanical system. It consists of a proportionally damped mechanical system with n d.o.f. to which r additional viscous dampers are attached, for some reason. Here, making use of the above approach, the $(n \times n)$ characteristic determinant of the combined system is reduced to a determinant of order $(r \times r)$, where it is assumed that $r \ll n$, which is a more frequently encountered case in practice. As a result, an alternative formulation has been presented for the characteristic equation of the previously mentioned system. This formulation can be very convenient for numerical calculations at higher n (d.o.f.) values.

2. THEORY

The motion of a linear discrete mechanical system with n d.o.f. is governed in the physical space by the matrix differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{D}}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\tilde{\mathbf{D}}$ and $\mathbf{K}$ are the $(n \times n)$ mass, damping and stiffness matrices, respectively, and $\mathbf{q}$ is the $(n \times 1)$ vector of the generalized co-ordinates. It is assumed that the damping matrix $\tilde{\mathbf{D}}$ is a linear combination of the mass and stiffness matrices

$$\tilde{\mathbf{D}} = \alpha\mathbf{M} + \beta\mathbf{K}, \tag{2}$$

i.e., the mechanical system is proportionally damped, where α and β are some given scalars.

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Suppose that r new viscous dampers are added to the mechanical system, for some reason, such that the modified damping matrix of the system can be written as

$$\mathbf{D} = \tilde{\mathbf{D}} + \sum_{i=1}^{r} \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}} \tag{3}$$

where the vectors $\mathbf{d}_i$ include both damping constant and the orientation information in the physical space [3].

The aim of this study is to obtain the characteristic equation of the modified system and then to reduce it to a compact form as much as possible.

The transformation

$$\mathbf{q} = \mathbf{\Phi}\boldsymbol{\eta}, \tag{4}$$

where $\mathbf{\Phi}$ is the modal matrix of the undamped system, results in the following equation of motion in the modal space:

$$\ddot{\boldsymbol{\eta}} + \left(\tilde{\mathbf{D}}_0 + \sum_{i=1}^{r} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}} \right) \dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2 \boldsymbol{\eta} = \mathbf{0}. \tag{5}$$

The relations

$$\mathbf{\Phi}^{\mathrm{T}} \mathbf{M} \mathbf{\Phi} = \mathbf{I}, \quad \mathbf{\Phi}^{\mathrm{T}} \mathbf{K} \mathbf{\Phi} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2), \; i = 1, \ldots, n \tag{6}$$

are used which are due to the mass orthonormalization of the mode vectors. $\mathbf{I}$ denotes the $(n \times n)$ unit matrix and ω_i is the ith eigenfrequency of the undamped system. Additionally, the definitions

$$\tilde{\mathbf{D}}_0 = \alpha \mathbf{I} + \beta \mathbf{\Omega}^2, \qquad \mathbf{d}_i^* = \mathbf{\Phi}^{\mathrm{T}} \mathbf{d}_i \tag{7}$$

are introduced. It is worth noting that the first part of the transformed damping matrix, i.e., $\tilde{\mathbf{D}}_0$ is a diagonal matrix.

If a solution of the equation of motion in the modal space, equation (5) is assumed to be in the form of

$$\boldsymbol{\eta} = \tilde{\boldsymbol{\eta}} \mathrm{e}^{\lambda t} \tag{8}$$

where λ and $\tilde{\boldsymbol{\eta}}$ represent an eigenvalue and the corresponding eigenvector, respectively, the eigenvalue problem

$$\left[(\lambda^2 \mathbf{I} + \lambda \tilde{\mathbf{D}}_0 + \mathbf{\Omega}^2) + \lambda \sum_{i=1}^{r} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}} \right] \tilde{\boldsymbol{\eta}} = \mathbf{0} \tag{9}$$

is obtained. This means that the eigenvalues λ are obtained from the characteristic equation

$$\det \left[(\lambda^2 \mathbf{I} + \lambda \tilde{\mathbf{D}}_0 + \mathbf{\Omega}^2) + \lambda \sum_{i=1}^{r} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}} \right] = 0. \tag{10}$$

In the above equation, the sum of the first three matrices is a diagonal matrix. Hence, the matrix determinant which is to be equated to zero, consists of a diagonal matrix modified by r diadic products, i.e., r rank-one matrices. In the first step, equation (10) can be reformulated as

$$\det(\lambda^2\mathbf{I} + \lambda\tilde{\mathbf{D}}_0 + \boldsymbol{\Omega}^2)\det\left(\mathbf{I} + \sum_{i=1}^{r} \lambda(\lambda^2\mathbf{I} + \lambda\tilde{\mathbf{D}}_0 + \boldsymbol{\Omega}^2)^{-1}\mathbf{d}_i^*\mathbf{d}_i^{*\mathrm{T}}\right) = 0, \tag{11}$$

where the first determinant can be written in the form

$$\det(\lambda^2\mathbf{I} + \lambda\tilde{\mathbf{D}}_0 + \boldsymbol{\Omega}^2) = \prod_{i=1}^{n} (\lambda^2 + \lambda\tilde{d}_{0i} + \omega_i^2). \tag{12}$$

Now, making use of the formal similarity between equation (19) of reference [2] and the above equation (11), the latter can be expressed as

$$\prod_{i=1}^{n} (\lambda^2 + \lambda\tilde{d}_{0i} + \omega_i^2)\det\mathbf{B} = 0, \tag{13}$$

where the (i, j)th element of the matrix $\mathbf{B}$, of size $(r \times r)$ is given by

$$b_{ij} = \sum_{k=1}^{n} \frac{d_{ki}^* d_{kj}^*}{\lambda^2 + \lambda\tilde{d}_{0k} + \omega_k^2} + \frac{1}{\lambda}\delta_{ij}, \quad i, j = 1, \dots, r. \tag{14}$$

It is worth noting that each element of the matrix $\mathbf{B}$ consists of a sum of n terms. Further, d_{ki}^* denotes the kth element of the $(n \times 1)$ vector $\mathbf{d}_i^*$, defined in equation (7) and δ_{ij} represents the Kronecker delta.

Recognizing that the eigenvalues of the modified system will be different from those of the original system, the characteristic equation simplifies to

$$\det\mathbf{B} = 0. \tag{15}$$

It is quite in order to remember that eigenvalues λ can be obtained directly from the solution of the eigenvalue problem of the following matrix $\mathbf{A}$ resulting from the state-space formulation

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{array}\right]. \tag{16}$$

$\mathbf{0}$ and $\mathbf{I}$ denote the $(n \times n)$ zero matrix and unit matrix respectively. Whereas here, one has to find the eigenvalues of a matrix of size $(2n \times 2n)$, in (15) it is necessary to find the roots of a determinant of size $(r \times r)$, where in practice it is usually $r \ll n$.

In the special case of only one additional viscous damper, i.e., $r = 1$, equation (15) reduces to

$$b_{11} = \sum_{k=1}^{n} \frac{d_{k1}^{*2}}{\lambda^2 + \lambda\tilde{d}_{0k} + \omega_k^2} + \frac{1}{\lambda} = 0, \tag{17}$$

which can be reformulated as

$$1 + \lambda \sum_{k=1}^{n} \frac{d_{k1}^{*2}}{\lambda^2 + \lambda \tilde{d}_{0k} + \omega_k^2} = 0. \tag{18}$$

This equation corresponds to equation (9) in reference [4], written here in the modal space.

3. NUMERICAL EVALUATIONS

This section is devoted to the testing of the reliability of the expressions obtained. The simple system with 4 d.o.f. in Figure 1 is taken as an illustrative example. The physical parameters are chosen as $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$, $m_4 = m$ with $m = 3\,\mathrm{kg}$; $k_1 = k$, $k_2 = 2k$, $k_3 = 4k$, $k_4 = k$ and $k_5 = k$ with $k = 2\,\mathrm{N/m}$. It is assumed that the damping matrix of the original system is mass-proportional, such that $c_1 = 2m_1$, $c_2 = 2m_2$, $c_3 = 2m_3$, $c_4 = 2m_4$. In other words, in equation (2) $\alpha = 2$ and $\beta = 0$ are chosen. It is assumed further that two relative viscous dampers of constants $c_5 = 2$, $c_6 = 4\,\mathrm{N/m/s}$ are to be added between the masses m_1, m_2 and m_3, m_4, respectively, as depicted as dashed lines in Figure 1. The mass, stiffness and damping matrices of the original system are

$$\mathbf{M} = \mathbf{diag}(3, 6, 9, 3), \qquad \mathbf{K} = \begin{bmatrix} 6 & -4 & 0 & 0 \\ -4 & 12 & -8 & 0 \\ 0 & -8 & 10 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \qquad \tilde{\mathbf{D}} = \mathbf{diag}(6, 12, 18, 6).$$

The damping matrix of the modified system reads as

$$\mathbf{D} = \tilde{\mathbf{D}} + \mathbf{d}_1 \mathbf{d}_1^{\mathrm{T}} + \mathbf{d}_2 \mathbf{d}_2^{\mathrm{T}}$$

$$= \mathbf{diag}(6, 12, 18, 6) + \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \\ 0 \end{bmatrix} [\sqrt{2}\ -\sqrt{2}\ 0\ 0] + \begin{bmatrix} 0 \\ 0 \\ \sqrt{4} \\ -\sqrt{4} \end{bmatrix} [0\ 0\ \sqrt{4}\ -\sqrt{4}]$$

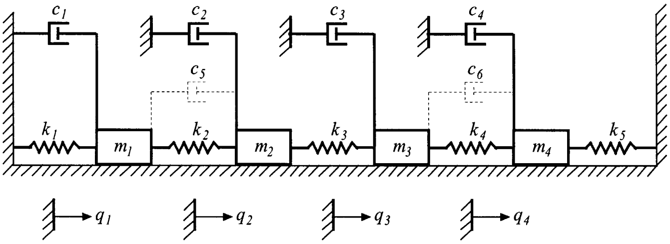

Figure 1. Sample system with four degrees of freedom.

TABLE 1

Eigenvalues λ of the modified system in Figure 1

Eigenvalues of **A** given in equation (16)	Roots of equation (15)
$-3{\cdot}30822599$	$-3{\cdot}30822599$
$-1{\cdot}92345123$	$-1{\cdot}92345123$
$-1{\cdot}27002504 \pm 0{\cdot}48534308i$	$-1{\cdot}27002504 \pm 0{\cdot}48534308i$
$-1{\cdot}23681637 \pm 1{\cdot}07151904i$	$-1{\cdot}23681637 \pm 1{\cdot}07151904i$
$-0{\cdot}46498404$	$-0{\cdot}46498404$
$-0{\cdot}06743369$	$-0{\cdot}06743369$

$$= \begin{bmatrix} 8 & -2 & 0 & 0 \\ -2 & 14 & 0 & 0 \\ 0 & 0 & 22 & -4 \\ 0 & 0 & -4 & 10 \end{bmatrix}.$$

The solution of the eigenvalue problem of the undamped system yields

$$\mathbf{\Phi} = \begin{bmatrix} 0{\cdot}16702739 & 0{\cdot}23699340 & -0{\cdot}38616296 & 0{\cdot}31646087 \\ 0{\cdot}23370715 & 0{\cdot}12999193 & -0{\cdot}06605818 & -0{\cdot}30130723 \\ 0{\cdot}24349274 & -0{\cdot}04719554 & 0{\cdot}18178137 & 0{\cdot}12864878 \\ 0{\cdot}13539191 & -0{\cdot}48650735 & -0{\cdot}27631637 & -0{\cdot}04429703 \end{bmatrix},$$

$$\mathbf{\Omega}^2 = \mathbf{diag}(0{\cdot}13438064,\ 1{\cdot}26866073,\ 1{\cdot}77191603,\ 3{\cdot}26948704)$$

for the modal matrix and the matrix of the squares of the eigenfrequencies respectively. Further results are

$$\tilde{\mathbf{D}}_0 = \mathbf{diag}(\tilde{\mathrm{d}}_{0i}) = \mathbf{diag}(2, 2, 2, 2),$$

$$\mathbf{d}_1^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{d}_1 = [-0{\cdot}09429942 \quad 0{\cdot}15132292 \quad -0{\cdot}45269651 \quad 0{\cdot}87365603]^{\mathrm{T}},$$

$$\mathbf{d}_2^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{d}_2 = [0{\cdot}21620167 \quad 0{\cdot}87862361 \quad 0{\cdot}91619548 \quad 0{\cdot}34589162]^{\mathrm{T}}.$$

The eigenvalues λ of the modified system in Figure 1 are given in Table 1. The real and complex numbers in the first column are the eigenvalues obtained directly by solving the eigenvalue problem of matrix **A** defined in (16). The numbers in the second column are obtained by solving equation (15), where **B** is a (2×2) matrix each element of which consists of a sum of four terms. Both numerical operations are carried out with MATLAB. The agreement of the numbers in both columns is excellent. This in turn justifies clearly the validity of the characteristic equation obtained via the new approach.

4. CONCLUSION

This study is concerned with a proportionally damped linear mechanical system with n d.o.f. to which r additional viscous dampers are attached, for some reason. Making use of a recently developed approach, the $(n \times n)$ characteristic determinant of the above described combined system is reduced to a much smaller determinant of order $(r \times r)$ where $r \ll n$ is a frequently encountered case in practice.

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