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ON THE FREQUENCY RESPONSE FUNCTION OF A DAMPED CANTILEVER SIMPLY SUPPORTED IN-SPAN AND CARRYING A TIP MASS

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This paper deals with the determination of the frequency response function of a cantilevered Bernoulli–Euler beam which is viscously damped by a single damper. The beam is simply supported in-span and carries a tip mass. The frequency response function is obtained through a formula that was established for the receptance matrix of discrete linear systems subjected to linear constraint equations, by considering the simple support as a linear constraint imposed on generalized co-ordinates. The comparison of the numerical results obtained via a boundary value problem formulation justifies the approach used here.

1. INTRODUCTION

The first author recently established a formula for the receptance matrix of viscously damped discrete systems subjected to several constraint equations [1]. The reliability of the formula derived was tested on an academic example of a spring–mass system with three degrees of freedom, the co-ordinates of which were assumed to be subjected to a constraint equation. In order to put forward the applicability of the method better, the formula was applied in reference [2] to a more complex but practical system. The system was made up of a cantilevered beam simply supported at a given distance from the fixed end. It was desired to determine the amplitude distribution of the beam due to harmonically varying vertical force acting at a given point.

A further study [3] dealt with the same system as in reference [2], the difference being, that viscous damping of the beam was included by introducing a single viscous damper. The present study is concerned with a more general system than in reference [3] because, here, the vibrating beam also carries a tip mass. Through the attachment of a tip mass, the system under consideration could be viewed as a more accurate and realistic model of some physical systems. The aim is to determine the amplitude distribution of the beam due to a harmonically varying vertical force acting at a given point. The problem posed is to find the frequency response function of the beam described above.

2. THEORY

The problem can best be stated referring to the cantilevered beam shown in Figure 1. The Bernoulli–Euler beam, damped by a viscous damper with damping constant c at $x = \alpha L$ is assumed to be simply supported at a distance $s^* = \eta L$ from the fixed end. The beam is

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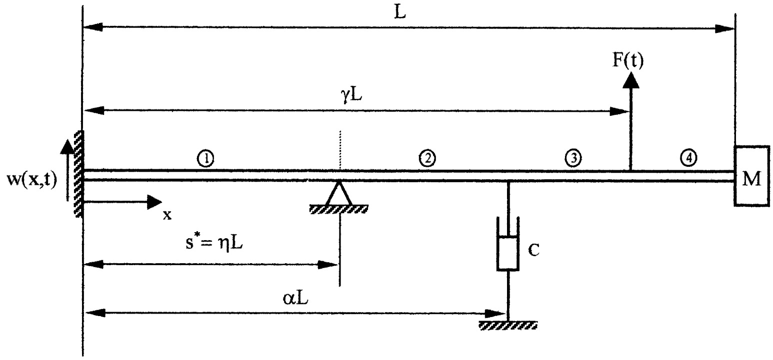

Figure 1. Viscously damped cantilevered beam simply supported in-span and carrying a tip mass, subject to a harmonically varying force.

carrying a tip mass M. At the distance $x = \gamma L$, a harmonically varying force $F(t)$ is acting on the beam. Now it is desired to determine the amplitude distribution of the beam due to this force. This problem can also be posed as finding the frequency response function of the beam.

2.1. APPLICATION OF THE FORMULA IN REFERENCE [1]

Consider with the mechanical system in Figure 1 where it is first assumed that the support does not exist. The equation of the motion of the beam is [4–6]

$$EIw^{IV}(x,t) + m\ddot{w}(x,t) + M\ddot{w}(x,t)\delta(x-L) + c\dot{w}(x,t)\delta(x-\alpha L) = F(t)\delta(x-\gamma L) \quad (1)$$

the exciting force being

$$F(t) = F_0 \mathrm{e}^{\mathrm{i}\Omega t}, \quad (2)$$

where the primes and overdots denote partial derivatives with respect to x and time t, respectively, and i is the imaginary unit. EI is the bending rigidity and m is the mass per unit length of the beam. $\delta(x)$ denotes the Dirac function, c denotes the viscous damping coefficient and M represents the tip mass.

The corresponding boundary conditions are

$$w(0,t) = w'(0,t) = w''(L,t) = w'''(L,t) = 0. \quad (3)$$

An approximate series solution of the differential equation (1) can be taken in the form

$$w(x,t) \approx \sum_{r=1}^{n} w_r(x)\eta_r(t), \quad (4)$$

where $w_r(x)$ are the orthogonal eigenfunctions of the bare clamped-free beam, normalized with respect to the mass density and $\eta_r(t)$ are the generalized co-ordinates. Through application of the Galerkin procedure, (After substitution of expression (4) into the differential equation (1), both sides of the equation are multiplied by the sth eigenfunctions $w_s(x)$ and integrated over the beam length L. Then, the orthogonality property of the eigenfunctions is used.) the system of modal equations, i.e., the system of differential equations for the $\eta_i(t)$, is obtained as [6]

$$\ddot{\eta}_i(t) + Mw_i(L)\sum_{j=1}^{n} w_j(L)\ddot{\eta}_j(t) + cw_i(\alpha L)\sum_{j=1}^{n} w_j(\alpha L)\dot{\eta}_j(t) + \omega_i^2\eta_i(t) = N_i(t)$$

$$(i = 1, \ldots, n), \tag{5}$$

where [7]

$$\omega_i^2 = (\beta_i L)^4 \frac{EI}{mL^4}, \quad \bar{\beta}_1 = \beta_1 L = 1{\cdot}875104068712, \quad \bar{\beta}_2 = \beta_2 L = 4{\cdot}694091132974,$$

$$N_i(t) = F(t)w_i(\gamma L). \tag{6}$$

The system of differential equations in equation (5) can be written in matrix notation as

$$\mathbf{M}\ddot{\boldsymbol{\eta}}(t) + \mathbf{D}\dot{\boldsymbol{\eta}}(t) + \boldsymbol{\omega}^2\boldsymbol{\eta}(t) = \mathbf{N}(t), \tag{7}$$

where

$$\boldsymbol{\eta}(t) = [\eta_1(t)\ldots\eta_n(t)]^{\mathrm{T}}, \quad \boldsymbol{\omega}^2 = \mathbf{diag}(\omega_i^2), \quad \mathbf{N}(t) = \bar{\mathbf{N}}\mathrm{e}^{\mathrm{i}\Omega t}, \quad \bar{\mathbf{N}} = F_0\mathbf{w}(\gamma L),$$

$$\mathbf{M} = \mathbf{I} + M\mathbf{w}(\mathrm{L})\mathbf{w}^{\mathrm{T}}(\mathrm{L}), \quad \mathbf{D} = c\mathbf{w}(\alpha L)\mathbf{w}^{\mathrm{T}}(\alpha L), \quad \mathbf{w}(x) = [w_1(x)\ldots w_n(x)]^{\mathrm{T}}, \tag{8}$$

ω_i $(i = 1, \ldots, n)$ are the eigenfrequencies of the bare cantilever beam.

Substitution of

$$\boldsymbol{\eta}(t) = \bar{\boldsymbol{\eta}}\mathrm{e}^{\mathrm{i}\Omega t} \tag{9}$$

into the matrix differential equation (7) yields

$$\bar{\boldsymbol{\eta}} = \mathbf{H}(\Omega)\bar{\mathbf{N}}, \tag{10}$$

where the receptance matrix is in the form

$$\mathbf{H}(\Omega) = \{-\Omega^2[\mathbf{I} + M\mathbf{w}(L)\mathbf{w}^{\mathrm{T}}(L)] + \mathrm{i}\Omega\mathbf{D} + \boldsymbol{\omega}^2\}^{-1}. \tag{11}$$

Considering equation (8), it can be arranged as

$$\mathbf{H}(\Omega) = (\mathbf{K}' + \mathbf{u}'\mathbf{v}'^{\mathrm{T}})^{-1}, \tag{12}$$

where

$$\mathbf{K}' = \boldsymbol{\omega}^2 - \Omega^2[\mathbf{I} + M\mathbf{w}(L)\mathbf{w}^{\mathrm{T}}(L)], \quad \mathbf{u}' = \mathrm{i}\Omega c\mathbf{w}(\alpha L), \quad \mathbf{v}'^{\mathrm{T}} = \mathbf{w}^{\mathrm{T}}(\alpha L). \tag{13}$$

Using the Sherman–Morrison formula which gives the inverse of the sum of a regular matrix and a dyadic product [8]

$$(\mathbf{K} + \mathbf{u}\mathbf{v}^{\mathrm{T}})^{-1} = \mathbf{K}^{-1} - \mathbf{K}^{-1}\mathbf{u}(1 + \mathbf{v}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{u})^{-1}\mathbf{v}^{\mathrm{T}}\mathbf{K}^{-1}, \tag{14}$$

the receptance matrix can be written as follows:

$$\mathbf{H}(\Omega) = \mathbf{K}'^{-1} - \frac{\mathbf{K}'^{-1}\mathrm{i}\Omega c\,\mathbf{w}(\alpha L)\,\mathbf{w}^{\mathrm{T}}(\alpha L)\,\mathbf{K}'^{-1}}{[1 + \mathbf{w}^{\mathrm{T}}(\alpha L)\mathbf{K}'^{-1}\mathrm{i}\Omega c\,\mathbf{w}(\alpha L)]}. \tag{15}$$

Matrix $\mathbf{K}'$ defined in equation (13) can be expressed as

$$\mathbf{K}' = \mathbf{K} + \mathbf{u}\mathbf{v}^{\mathrm{T}} \tag{16}$$

with

$$\mathbf{K} = \boldsymbol{\omega}^2 - \Omega^2\mathbf{I}, \quad \mathbf{u} = -\Omega^2 M\mathbf{w}(L), \quad \mathbf{v}^{\mathrm{T}} = \mathbf{w}^{\mathrm{T}}(L). \tag{17}$$

Making use of formula (14) and noting that

$$\mathbf{K}^{-1} = \mathbf{diag}\left(\frac{1}{\omega_i^2 - \Omega^2}\right), \tag{18}$$

$$\mathbf{K}'^{-1} = \mathbf{diag}\left(\frac{1}{\omega_i^2 - \Omega^2}\right) + \frac{\mathbf{diag}\,(1/(\omega_i^2 - \Omega^2))\,\Omega^2 M\,\mathbf{w}(L)\,\mathbf{w}^{\mathrm{T}}(L)\,\mathbf{diag}(1/(\omega_i^2 - \Omega^2))}{1 - \mathbf{w}^{\mathrm{T}}(L)\,\mathbf{diag}\,(1/(\omega_i^2 - \Omega^2))\,\Omega^2 M\,\mathbf{w}(L)} \tag{19}$$

is obtained. This can further be arranged as

$$\mathbf{K}'^{-1} = \mathbf{diag}\left(\frac{1}{\omega_i^2 - \Omega^2}\right)\left[\mathbf{I} + \frac{\Omega^2 M\mathbf{w}(L)\,\mathbf{w}^{\mathrm{T}}(L)\,\mathbf{diag}(1/(\omega_i^2 - \Omega^2))}{1 - \mathbf{w}^{\mathrm{T}}(L)\,\mathbf{diag}(1/(\omega_i^2 - \Omega^2))\,\Omega^2 M\,\mathbf{w}(L)}\right]. \tag{20}$$

Finally,

$$\mathbf{K}'^{-1} = \frac{1}{\omega_0^2}\,\mathrm{diag}\left(\frac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{G}^* \tag{21}$$

is obtained where the following definitions are introduced:

$$\Omega^* = \frac{\Omega}{\omega_0}, \qquad \omega_0^2 = \frac{EI}{mL^4}, \qquad \beta_M = \frac{M}{mL}, \qquad \omega_i^2 = \omega_0^2\bar{\beta}_i^4$$

$$\mathbf{w}^{\mathrm{T}}(x) = \frac{1}{\sqrt{mL}}\mathbf{a}^{\mathrm{T}}(x) = \frac{1}{\sqrt{mL}}[a_1(x) \ldots a_n(x)],$$

$$a_i(x) = \cosh\bar{\beta}_i\frac{x}{L} - \cos\bar{\beta}_i\frac{x}{L} - \bar{\eta}_i^*\left(\sinh\bar{\beta}_i\frac{x}{L} - \sin\bar{\beta}_i\frac{x}{L}\right),$$

$$\bar{\eta}_{\mathrm{i}}^* = \frac{(\cosh\bar{\beta}_i + \cos\bar{\beta}_i)}{(\sinh\bar{\beta}_i + \sin\bar{\beta}_i)}, \qquad \bar{c} = \frac{c}{mL\omega_0},$$

$$\mathbf{G}^* = \mathbf{I} + \frac{\beta_M\Omega^{*2}\mathbf{a}(L)\mathbf{a}^{\mathrm{T}}(L)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))}{1 - \beta_M\Omega^{*2}\,\mathbf{a}^{\mathrm{T}}(L)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{a}\,(L)}. \tag{22}$$

Substituting equation (21) and considering equation (22), the receptance matrix in equation (15) can be expressed as

$$\mathbf{H}(\Omega^*) = \frac{1}{\omega_0^2}\,\mathbf{diag}\left(\frac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{G}^*\mathbf{R}^*, \tag{23}$$

where

$$\mathbf{R}^* = \mathbf{I} - \frac{\mathrm{i}\bar{c}\Omega^*\mathbf{a}(\alpha L)\mathbf{a}^{\mathrm{T}}(\alpha L)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{G}^*}{1 + \mathrm{i}\bar{c}\Omega^*\mathbf{a}^{\mathrm{T}}(\alpha L)\,\mathbf{diag}\,(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{a}(\alpha L)} \tag{24}$$

is introduced.

Now return to the actual system with the support at $x = \eta L$. The introduction of the support leads to the constraint equation

$$\sum_{r=1}^{n} w_r(s^*)\eta_r(t) = 0, \tag{25}$$

which can be written compactly as

$$\mathbf{a}_1^{\mathrm{T}}\boldsymbol{\eta} = 0, \tag{26}$$

where

$$\mathbf{a}_1^{\mathrm{T}} = \mathbf{w}^{\mathrm{T}}(s^*) = [w_1(s^*) \ldots w_n(s^*)]^{\mathrm{T}}, \quad s^* = \eta L. \tag{27}$$

The amplitude vector $\bar{\boldsymbol{\eta}}$ in the constrained case can be written from equation (10) analogously as

$$\bar{\boldsymbol{\eta}} = \mathbf{H}_{cons}(\Omega^*)\bar{\mathbf{N}}, \tag{28}$$

where from reference [1] the receptance matrix of the constrained system reads as

$$\mathbf{H}_{cons}(\Omega^*) = \mathbf{H}(\Omega^*)\left[\mathbf{I} - \frac{\mathbf{w}(s^*)\mathbf{w}^{\mathrm{T}}(s^*)\mathbf{H}(\Omega^*)}{\mathbf{w}^{\mathrm{T}}(s^*)\mathbf{H}(\Omega^*)\mathbf{w}(s^*)}\right], \tag{29}$$

$\mathbf{I}$ being the $n \times n$ unit matrix.

Therefore, the displacements of the constrained (i.e., supported) beam can be written by using equation (9) as

$$w_{cons}(x, t) = \bar{w}_{cons}(x)\mathrm{e}^{\mathrm{i}\Omega t}, \tag{30}$$

where

$$\bar{w}_{cons}(x) = \sum_{r=1}^{n} w_r(x)\bar{\eta}_r. \tag{31}$$

It is easy to show that the above expression can be reformulated as

$$\bar{w}_{cons}(x) = (\mathbf{w}^{\mathrm{T}}(x)\,\mathbf{H}_{cons}(\Omega^*)\,\mathbf{w}(\gamma L))\,F_0, \tag{32}$$

which in turn, after some rearrangements, leads to

$$\frac{\bar{w}_{cons}(x)}{F_0/(EI/L^3)} = \mathbf{a}^{\mathrm{T}}(x)\,\mathbf{diag}\left(\frac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{G}^*\mathbf{R}^*\mathbf{S}^*\mathbf{a}(\gamma L) \tag{33}$$

with

$$\mathbf{S}^* = \mathbf{I} - \frac{\mathbf{a}(s^*)\,\mathbf{a}^{\mathrm{T}}(s^*)\,\mathbf{diag}(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{G}^*\mathbf{R}^*}{\mathbf{a}^{\mathrm{T}}(s^*)\,\mathbf{diag}(1/(\bar{\beta}_i^4 - \Omega^{*2}))\,\mathbf{G}^*\mathbf{R}^*\mathbf{a}(s^*)}. \tag{34}$$

In the special case $\beta_M = 0$, i.e., no tip mass, it is easy to show that expression (33) reduces to that of reference [3].

Noting that according to equation (30), the real part of $\bar{w}_{cons}(x)\mathrm{e}^{\mathrm{i}\Omega t}$ represents the physical displacements, the amplitude distribution $A(x)$ along the supported beam subject to the harmonic force is obtained as

$$A(x) = \sqrt{\bar{w}_{cons}^2(x)_{\mathrm{Re}} + \bar{w}_{cons}^2(x)_{\mathrm{Im}}}. \tag{35}$$

In the case $F_0 = 1$, the right side of equation (35) represents nothing else but the frequency response function of the beam in Figure 1.

2.2. SOLUTION THROUGH THE BOUNDARY VALUE PROBLEM FORMULATION

In order to prove the validity of expression (35) along with equations (34) and (33), the only way is to compare this with the results of a boundary value problem formulation.

The bending vibrations of the four beam portions shown in Figure 1 are governed by the partial differential equations

$$EIw_i^{\mathrm{IV}}(x,t) + m\ddot{w}_i(x,t) = 0 \quad (i = 1, 2, 3, 4) \tag{36}$$

with the following boundary and matching conditions:

$$w_1(0,t) = w_1'(0,t) = 0, \quad w_1(s^*,t) = w_2(s^*,t) = 0, \quad w_1'(s^*,t) = w_2'(s^*,t),$$

$$w_1''(s^*,t) = w_2''(s^*,t), \quad w_2(\alpha L,t) = w_3(\alpha L,t), \quad w_2'(\alpha L,t) = w_3'(\alpha L,t),$$

$$w_2''(\alpha L,t) = w_3''(\alpha L,t), \quad EIw_2'''(\alpha L,t) - EIw_3'''(\alpha L,t) - c\dot{w}_2(\alpha L,t) = 0,$$

$$w_3(\gamma L,t) = w_4(\gamma L,t), \quad w_3'(\gamma L,t) = w_4'(\gamma L,t), \quad w_3''(\gamma L,t) = w_4''(\gamma L,t),$$

$$w_4''(L,t) = 0, \quad EIw_4'''(L,t) - M\ddot{w}_4(L,t) = 0,$$

$$EIw_3'''(\gamma L,t) - EIw_4'''(\gamma L,t) + F_0\mathrm{e}^{\mathrm{i}\Omega t} = 0. \tag{37}$$

If harmonic solutions of the form

$$w_i(x,t) = W_i(x)\mathrm{e}^{\mathrm{i}\Omega t} \tag{38}$$

are substituted into equation (36), the following ordinary differential equations are obtained for the amplitude functions $W_i(x)$:

$$W_i^{\mathrm{IV}}(x) - \bar{\Lambda}^4 W_i(x) = 0 \quad (i = 1, 2, 3, 4), \tag{39}$$

where

$$\bar{\Lambda}^4 = \frac{m\Omega^2}{EI}. \tag{40}$$

In the expressions above, both $w_i(x, t)$ and $W_i(x)$ represent complex-valued functions. The essential point here is to imagine the actual bending displacements $w_i(x, t)$ as the real parts of some complex-valued functions, for which the same notation is used for the sake of briefness.

The corresponding boundary and matching conditions now read as

$$W_1(0) = W_1'(0), \quad W_1(s^*) = W_2(s^*) = 0, \quad W_1'(s^*) = W_2'(s^*),$$

$$W_1''(s^*) = W_2''(s^*), \quad W_2(\alpha L) = W_3(\alpha L), \quad W_2'(\alpha L) = W_3'(\alpha L),$$

$$W_2''(\alpha L) = W_3''(\alpha L), \quad W_2'''(\alpha L) - W_3'''(\alpha L) - \frac{\mathrm{i}c\Omega}{EI} W_2(\alpha L) = 0,$$

$$W_3(\gamma L) = W_4(\gamma L), \quad W_3'(\gamma L) = W_4'(\gamma L), \quad W_3''(\gamma L) = W_4''(\gamma L),$$

$$W_4''(L) = 0, \quad EI\, W_4'''(L) + M\Omega^2 W_4(L) = 0, \quad W_3'''(\gamma L) - W_4'''(\gamma L) + \frac{F_0}{EI} = 0. \tag{41}$$

The general solutions of the differential equations (39) are

$$W_1(x) = c_1 \sin \bar{\Lambda}x + c_2 \cos \bar{\Lambda}x + c_3 \sinh \bar{\Lambda}x + c_4 \cosh \bar{\Lambda}x,$$

$$W_2(x) = c_5 \sin \bar{\Lambda}x + c_6 \cos \bar{\Lambda}x + c_7 \sinh \bar{\Lambda}x + c_8 \cosh \bar{\Lambda}x,$$

$$W_3(x) = c_9 \sin \bar{\Lambda}x + c_{10} \cos \bar{\Lambda}x + c_{11} \sinh \bar{\Lambda}x + c_{12} \cosh \bar{\Lambda}x,$$

$$W_4(x) = c_{13} \sin \bar{\Lambda}x + c_{14} \cos \bar{\Lambda}x + c_{15} \sinh \bar{\Lambda}x + c_{16} \cosh \bar{\Lambda}x, \tag{42}$$

where c_1 to c_{16} are unknown integration constants to be determined which can be complex in general.

Substitution of expressions (42) into conditions (41) yields, after rearrangement, the following set of 16 inhomogeneous equations for the determination of the coefficients c_i:

$$\mathbf{Ac} = \mathbf{b}. \tag{43}$$

The expression of the (16×16) coefficient matrix $\mathbf{A}$ is given in Appendix A. The vectors $\mathbf{c}$ and $\mathbf{b}$ are defined as

$$\mathbf{c}^{\mathrm{T}} = [c_1 c_2 \ldots c_{16}],$$

$$\mathbf{b}^{\mathrm{T}} = \left[0 \ldots 0 - \frac{F_0}{EI\bar{\Lambda}^3} 0\ 0 \right], \tag{44}$$

where only the 14th element of the (16×1) vector $\mathbf{b}$ is non-zero.

Lengthy expressions of the elements c_i of the vector **c**, which were obtained by MATHEMATICA via symbolic computation, are not given here due to space limitations. They are, however, in the database in JSV + [9]. It is important to note that the vector **c** and, therefore, the amplitude functions $W_i(x)$ $(i = 1, 2, 3, 4)$ in equations (42) contain the common factor $F_0/(EI/L^3)$ which has the dimension of length.

Having obtained $W_i(x)$ $(i = 1, 2, 3, 4)$, it is possible to determine the steady state amplitude at any point x of the beam, due to the harmonic force at a point $x = \gamma L$. Noting that according to equation (38) the real part of $W_i(x)\, e^{i\Omega t}$ represents the physical displacements, the amplitudes distribution $\bar{A}(x)$ along the supported beam subjected to the harmonically varying vertical force at $x = \gamma L$ is obtained as

$$\bar{A}(x) = \sqrt{W_i^2(x)_{\mathrm{Re}} + W_i^2(x)_{\mathrm{Im}}}. \tag{45}$$

In the case $F_0 = 1$, the right side of the above equation represents the frequency response function of the beam in Figure 1.

3. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. In these examples, $\Omega^* = 5$ and $\bar{c} = 0{\cdot}5$ are chosen. These mean that a harmonically varying vertical force of the radian frequency $5\sqrt{EI/mL^4}$ is acting at the location $x = \gamma L$, shown in Figure 1, and the non-dimensionalized damping value is 0·5.

In the first example, the following data $\alpha = 0{\cdot}75$, $\gamma = 1{\cdot}0$ and $\beta_M = 1{\cdot}0$ are chosen which means that the damper and the harmonic force act at the points $x = 0{\cdot}75L$ and at the tip, respectively, and the tip mass is equal to the mass of the beam.

The displacement amplitudes at various sections of the beam, non-dimensionalized by dividing by $F_0/(EI/L^3)$ are given in Table 1. η represents the non-dimensional position of the support, whereas $\bar{x} = x/L$ denotes the non-dimensional position of the point, the vibrational amplitude of which we are interested in. The values in the first columns are

TABLE 1

Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at $\gamma = 1{\cdot}0$. $\Omega = 5\sqrt{EI/mL^4}$, $\alpha = 0{\cdot}75$ and $\beta_M = 1{\cdot}0$ are chosen

	η					
	0·25		0·50		0·75	
$\bar{X}$	Present theory	B–E theory	Present theory	B–E theory	Present theory	B–E theory
0·1	0·000297	0·000297	0·000955	0·000960	0·001053	0·001054
0·2	0·000397	0·000395	0·002865	0·002878	0·003558	0·003558
0·3	0·000853	0·000858	0·004292	0·004312	0·006526	0·006529
0·4	0·003865	0·003877	0·003810	0·003828	0·008992	0·008993
0·5	0·008313	0·008342	0	0	0·009988	0·009993
0·6	0·013847	0·013909	0·008128	0·008174	0·008590	0·008592
0·7	0·020142	0·020256	0·019929	0·020051	0·003882	0·003878
0·8	0·026907	0·027099	0·034368	0·034593	0·004981	0·004982
0·9	0·033911	0·034200	0·050457	0·050817	0·017478	0·017489
1·0	0·040996	0·041394	0·067314	0·067825	0·031803	0·031829

TABLE 2

Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at $\gamma = 1{\cdot}0$. $\Omega = 5\sqrt{EI/mL^4}$, $\eta = 0{\cdot}25$ and $\beta_M = 1{\cdot}0$ are chosen

	α					
	0·25		0·50		0·75	
$\bar{X}$	Present theory	B–E theory	Present theory	B–E theory	Present theory	B–E theory
0·1	0·000297	0·000297	0·000297	0·000297	0·000297	0·000297
0·2	0·000398	0·000396	0·000398	0·000396	0·000397	0·000395
0·3	0·000855	0·000858	0·000855	0·000858	0·000853	0·000858
0·4	0·003875	0·003878	0·003877	0·003878	0·003865	0·003877
0·5	0·008342	0·008343	0·008345	0·008343	0·008313	0·008342
0·6	0·013910	0·013912	0·013912	0·013912	0·013847	0·013909
0·7	0·020260	0·020261	0·020259	0·020261	0·020142	0·020256
0·8	0·027105	0·027105	0·027100	0·027105	0·026907	0·027099
0·9	0·034208	0·034208	0·034199	0·034208	0·033911	0·034200
1·0	0·041404	0·041401	0·041389	0·041402	0·040996	0·041394

values obtained from formula (35), i.e., the present theory, where $n = 15$ is taken in the series expansion (4) and $\bar{\beta}_1$ to $\bar{\beta}_{15}$ in equation (22) taken from reference [7] are correct up to 12 decimal places. These explanations are also valid for Tables 2–4. The values in the second columns are "exact" values (45), obtained by the direct solution of the boundary value problem outlined in Section 2.2, i.e., Bernoulli–Euler theory, indicated in the Tables as B–E theory.

In Table 1, as expected, the vibration amplitudes of the beam on the left side of the support are increasing while the location of the support is approaching towards the tip while the other parameters are kept constant.

The second example is based on the data $\eta = 0{\cdot}25$, $\gamma = 1{\cdot}0$ and $\beta_M = 1{\cdot}0$ which in turn mean that the beam is supported at $x = 0{\cdot}25L$ and the harmonic force acts again at the tip. The tip mass is again equal to the beam mass. The non-dimensionalized vibration amplitudes at various sections of the beam are given in Table 2 for three different attachment points of the viscous damper to the beam: $x = 0{\cdot}25L$, $0{\cdot}50L$ and $0{\cdot}75L$. The values in the first and second columns are again values obtained from equations (35) and (45).

The effect of the location of the viscous damper on the vibration amplitudes on various sections of the beam is small, as seen from Table 2. In case of $\eta = 0{\cdot}75$, $\alpha = 0{\cdot}75$, $\gamma = 1{\cdot}0$ and $\beta_M = 1{\cdot}0$ as shown in the third column of Table 1, the displacement amplitudes at the tip of the beam increase as expected as compared with the case of $\eta = 0{\cdot}25$, $\alpha = 0{\cdot}25$, $\gamma = 1{\cdot}0$ and $\beta_M = 1{\cdot}0$, which is given in the first column of Table 2.

The third example is concerned with $\eta = 0{\cdot}25$, $\alpha = 0{\cdot}50$ and $\beta_M = 1{\cdot}0$, i.e., the beam is supported at $x = 0{\cdot}25L$ and the damper attachment point is the midpoint of the beam. Tip mass is again equal to the beam mass. The non-dimensionalized amplitudes at various beam sections are given in Table 3 for three acting points of the harmonic force on to the beam: $x = 0{\cdot}50L$, $0{\cdot}75L$ and L. The first columns are values obtained from equation (35), whereas those of the second columns are determined by equation (45).

As γ gets larger, i.e., the location of the harmonic force approaches the tip of the beam, the vibration amplitudes at the tip and in the vicinity of the tip increase as shown in Table 3.

And finally, the fourth example is concerned with $\eta = 0{\cdot}25$, $\alpha = 0{\cdot}50$ and $\gamma = 1{\cdot}0$, i.e., the beam is supported at $x = 0{\cdot}25L$, the damper is attached to the midpoint of the beam, the

TABLE 3

Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at three acting points. $\Omega = 5\sqrt{EI/mL^4}$, $\eta = 0{\cdot}25$, $\alpha = 0{\cdot}50$ and $\beta_M = 1{\cdot}0$ are chosen

	γ					
	0·50		0·75		1·00	
$\bar{X}$	Present theory	B–E theory	Present theory	B–E theory	Present theory	B–E theory
0·1	0·000113	0·000113	0·000029	0·000029	0·000297	0·000297
0·2	0·000152	0·000150	0·000039	0·000039	0·000398	0·000396
0·3	0·000311	0·000313	0·000091	0·000090	0·000855	0·000858
0·4	0·001117	0·001118	0·000554	0·000554	0·003877	0·003878
0·5	0·001521	0·001523	0·001630	0·001628	0·008345	0·008343
0·6	0·000927	0·000929	0·003594	0·003592	0·013912	0·013912
0·7	0·000595	0·000593	0·006732	0·006732	0·020259	0·020261
0·8	0·002807	0·002804	0·011322	0·011320	0·027100	0·027105
0·9	0·005467	0·005464	0·017168	0·017168	0·034199	0·034208
1·0	0·008345	0·008343	0·023634	0·023638	0·041389	0·041402

TABLE 4

Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at $\gamma = 1{\cdot}0$. $\Omega = 5\sqrt{EI/mL^4}$, $\eta = 0{\cdot}25$ and $\alpha = 0{\cdot}50$ are chosen

	β_M					
	0·50		1·50		2·50	
$\bar{X}$	Present theory	B–E theory	Present theory	B–E theory	Present theory	B–E theory
0·1	0·000616	0·000615	0·000196	0·000196	0·000116	0·000116
0·2	0·000824	0·000820	0·000262	0·000261	0·000156	0·000155
0·3	0·001772	0·001778	0·000564	0·000565	0·000335	0·000336
0·4	0·008033	0·008037	0·002555	0·002555	0·001519	0·001519
0·5	0·017289	0·017292	0·005500	0·005498	0·003270	0·003269
0·6	0·028825	0·028834	0·009169	0·009167	0·005451	0·005450
0·7	0·041976	0·041994	0·013351	0·013352	0·007938	0·007938
0·8	0·056151	0·056179	0·017860	0·017862	0·010619	0·010619
0·9	0·070861	0·070901	0·022538	0·022542	0·013400	0·013402
1·0	0·085759	0·085811	0·027277	0·027283	0·016218	0·016220

harmonic force acts at the tip of the beam. The non-dimensionalized amplitudes at various beam sections are given in Table 4 for three different tip mass ratios: $\beta_M = 0{\cdot}50$, 1·50 and 2·50. The first columns are values obtained from equation (35), whereas those of the second columns are determined by equation (45).

As seen in Table 4, for larger β_M ratios, (i.e., for heavier end masses), smaller vibration amplitudes at the various sections of the beam are observed while the other parameters are kept constant.

The agreement of the values in both columns in Table 1–4 justifies expression (35) along with equations (33) and (34), obtained on the basis of a formula established for the receptance matrix of viscously damped discrete systems subject to several constraint equations. It is worth noting that the agreement of the numbers in both columns becomes excellent if many more decimal places are considered in $\bar{\beta}_i$ values.

4. CONCLUSIONS

This study is concerned with the determination of the frequency response function of a viscously damped, cantilevered Bernoulli–Euler beam, which is simply supported in-span and carries a tip mass. The frequency response function is obtained through a formula, which was established for the receptance matrix of discrete systems subjected to linear constraint equations. The comparison of the numerical results obtained with those via a boundary value problem formulation justifies the approach used here.

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APPENDIX

The matrix **A** in equation (43):

$$
\mathbf{A}=\left[\begin{array}{cccccccccccccccc}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\sin\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L & \sinh\bar{\Lambda}\eta L & \cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sin\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L & \sinh\bar{\Lambda}\eta L & \cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\cos\bar{\Lambda}\eta L & -\sin\bar{\Lambda}\eta L & \cosh\bar{\Lambda}\eta L & \sinh\bar{\Lambda}\eta L & -\cos\bar{\Lambda}\eta L & \sin\bar{\Lambda}\eta L & -\cosh\bar{\Lambda}\eta L & -\sinh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\sin\bar{\Lambda}\eta L & -\cos\bar{\Lambda}\eta L & \sinh\bar{\Lambda}\eta L & \cosh\bar{\Lambda}\eta L & \sin\bar{\Lambda}\eta L & \cos\bar{\Lambda}\eta L & -\sinh\bar{\Lambda}\eta L & -\cosh\bar{\Lambda}\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sin\bar{\Lambda}\alpha L & \cos\bar{\Lambda}\alpha L & \sinh\bar{\Lambda}\alpha L & \cosh\bar{\Lambda}\alpha L & -\sin\bar{\Lambda}\alpha L & -\cos\bar{\Lambda}\alpha L & -\sinh\bar{\Lambda}\alpha L & -\cosh\bar{\Lambda}\alpha L & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cos\bar{\Lambda}\alpha L & -\sin\bar{\Lambda}\alpha L & \cosh\bar{\Lambda}\alpha L & \sinh\bar{\Lambda}\alpha L & -\cos\bar{\Lambda}\alpha L & \sin\bar{\Lambda}\alpha L & -\cosh\bar{\Lambda}\alpha L & -\sinh\bar{\Lambda}\alpha L & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\sin\bar{\Lambda}\alpha L & -\cos\bar{\Lambda}\alpha L & \sinh\bar{\Lambda}\alpha L & \cosh\bar{\Lambda}\alpha L & \sin\bar{\Lambda}\alpha L & \cos\bar{\Lambda}\alpha L & -\sinh\bar{\Lambda}\alpha L & -\cosh\bar{\Lambda}\alpha L & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & A_1 & A_2 & A_3 & A_4 & \cos\bar{\Lambda}\alpha L & -\sin\bar{\Lambda}\alpha L & -\cosh\bar{\Lambda}\alpha L & -\sinh\bar{\Lambda}\alpha L & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\bar{\Lambda}\gamma L & \cos\bar{\Lambda}\gamma L & \sinh\bar{\Lambda}\gamma L & \cosh\bar{\Lambda}\gamma L & -\sin\bar{\Lambda}\gamma L & -\cos\bar{\Lambda}\gamma L & -\sinh\bar{\Lambda}\gamma L & -\cosh\bar{\Lambda}\gamma L\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\bar{\Lambda}\gamma L & -\sin\bar{\Lambda}\gamma L & \cosh\bar{\Lambda}\gamma L & \sinh\bar{\Lambda}\gamma L & -\cos\bar{\Lambda}\gamma L & \sin\bar{\Lambda}\gamma L & -\cosh\bar{\Lambda}\gamma L & -\sinh\bar{\Lambda}\gamma L\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\bar{\Lambda}\gamma L & -\cos\bar{\Lambda}\gamma L & \sinh\bar{\Lambda}\gamma L & \cosh\bar{\Lambda}\gamma L & \sin\bar{\Lambda}\gamma L & \cos\bar{\Lambda}\gamma L & -\sinh\bar{\Lambda}\gamma L & -\cosh\bar{\Lambda}\gamma L\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\cos\bar{\Lambda}\gamma L & \sin\bar{\Lambda}\gamma L & \cosh\bar{\Lambda}\gamma L & \sinh\bar{\Lambda}\gamma L & \cos\bar{\Lambda}\gamma L & -\sin\bar{\Lambda}\gamma L & -\cosh\bar{\Lambda}\gamma L & -\sinh\bar{\Lambda}\gamma L\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\bar{\Lambda}L & -\cos\bar{\Lambda}L & \sinh\bar{\Lambda}L & \cosh\bar{\Lambda}L\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & B_2 & B_3 & B_4
\end{array}\right],
$$

where

$$
A_1=-\mathrm{i}\,\frac{c\Omega}{EI\bar{\Lambda}^3}\sin\bar{\Lambda}\alpha L-\cos\bar{\Lambda}\alpha L,\quad A_2=-\mathrm{i}\,\frac{c\Omega}{EI\bar{\Lambda}^3}\cos\bar{\Lambda}\alpha L+\sin\bar{\Lambda}\alpha L,\quad A_3=-\mathrm{i}\,\frac{c\Omega}{EI\bar{\Lambda}^3}\sinh\bar{\Lambda}\alpha L+\cosh\bar{\Lambda}\alpha L,
$$

$$
A_4=-\mathrm{i}\,\frac{c\Omega}{EI\bar{\Lambda}^3}\cosh\bar{\Lambda}\alpha L+\sinh\bar{\Lambda}\alpha L,
$$

$$
B_1=-\cos\bar{\Lambda}L+\beta_M\bar{\Lambda}L\sin\bar{\Lambda}L,\quad B_2=\sin\bar{\Lambda}L+\beta_M\bar{\Lambda}L\cos\bar{\Lambda}L,\quad B_3=\cosh\bar{\Lambda}L+\beta_M\bar{\Lambda}L\sinh\bar{\Lambda}L,\quad B_4=\sinh\bar{\Lambda}L+\beta_M\bar{\Lambda}L\cosh\bar{\Lambda}L.
$$