Journal of Sound and Vibration (2002) **255**(3), 608–609
doi:10.1006/jsvi.5197, available online at http://www.idealibrary.com on IDEAL®

REPLY TO: DISCUSSION ON "NATURE OF STATIONARITY OF THE RAYLEIGH QUOTIENT AT THE NATURAL MODES IN THE RAYLEIGH–RITZ METHOD"

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(*Received* 1 *May* 2001, *and in final form* 14 *November* 2001)

The author thanks Dr Ilanko for his comments on the discussion of the paper [1]. There were two objectives in reference [1]: (i) to show that the Rayleigh Quotient is not necessarily a minimum with respect to the arbitrary coefficients Q_j, that appear in the assumed deflection shape, $w(x) = \sum_{j=1}^{N} Q_j f_j(x)$, and (ii) to show that the natural frequencies are minimum with respect to the corresponding natural modes.

The Rayleigh–Ritz method assumes deflection function in the form of a series, say, for a one-dimensional problem, as

$$w(x) = \sum_{j=1}^{N} Q_j f_j(x),$$

where $f_j(x)$ satisfy at least the geometrical boundary conditions. When we apply the condition $\partial\omega^2/\partial Q_j = 0$, it means that we are looking for the variation of ω^2 with Q_j, while all other Q_k, for which $k \neq j$, are held constant. The condition $\partial\omega^2/\partial Q_j = 0$, does not refer to ω^2 being minimum with respect to Q_j, but only stationary with respect to Q_j, when $Q_k, k \neq j$ are held constant.

Since the best shape function will give the minimum natural frequency, and we do not know them *a priori*, the only way is to search for shape functions which will give still lower values for the natural frequencies. One way to accomplish the change of the deflection shape is to express it as a function of a parameter and then search the parameter space for the minimum natural frequency. The second part of the paper [1] expresses the natural modes as a continuous function of a parameter p, which coincides with the jth natural mode at $p = p_j$, to show that the natural frequencies are minimum at the natural modes.

Another approach is to use an optimizing exponent parameter in the shape function, called the optimum exponent method, that was popularized by Schmidt [2] and Laura *et al.* [3].

The Rayleigh–Ritz method provides the upper bounds for the natural frequencies. There is no effective way to arrive at the minimum natural frequency except by finding a better deflection shape function that would result in a lower natural frequency. However, a method to bracket the natural frequencies with a lower and an upper bound was proposed by Kuttler and Sigillito [4], and is called the "*a posteriori/a priori* method".

There was no claim in reference [1] that by knowing the nature of $\partial^2\omega^2/\partial Q_j^2$, one can estimate the error in the natural frequency. The statement in reference [1], "Hence ω_j^2 is a minimum if the matrix $[K - \omega_j^2 M]$ is a positive definite while it corresponds to a maximum if $[K - \omega_j^2 M]$ is not positive definite'' only refers to the stationary nature of ω_j^2 with respect to the arbitrary coefficients Q_j and not to the minimum nature of ω_j^2 with respect to the natural modes. The exact ''minimum'' natural frequency can be reached only at the ''exact'' natural mode.

Further, the statement in reference [1], ''The above findings lead to a possibility of checking the accuracy of the computed natural frequencies obtained by energy techniques such as the Rayleigh–Ritz method, the Galerkin method, and the finite element methods, by examining whether the computed natural frequencies are a minimum at the computed natural modes'' underlines the fact that the natural frequencies must be minimum at the natural modes. In the first example in reference [5] the special shape functions correspond to the Rayleigh–Ritz normal modes.

Also when the orthogonality condition is included in testing whether ω^2 is a minimum in the neighborhood of Q_j, as done by Ilanko [5], it must show a minimum because in this case Q_j acts as a single parameter that changes the normal mode that was obtained using the Rayleigh–Ritz method. When Q_j is changed with a simultaneous change in other Q's in the normal mode shape, the situation is similar to changing the normal mode as a function of a parameter, Q_j. Hence, Dr Ilanko's findings are fully in agreement with the results of reference [1].

References such as [6, 7] given in reference [5] specify the minimum nature of the natural frequencies with respect to the natural modes, while there are references such as reference [8] which clearly state that the natural frequencies are stationary with respect to the arbitrary coefficients in the series form of the assumed deflection shape functions in the Rayleigh–Ritz method.

By the way, orthogonalization in equations (9a,b) as done in reference [5] to construct the third mode, is only with respect to the mass matrix. Orthogonalization is accomplished automatically for all the four modes when we solve $[K - \omega_j^2 M]\{Q\} = \{0\}$.

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