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ON THE NORMAL FORMS OF CERTAIN PARAMETRICALLY EXCITED SYSTEMS

W. Y. ZHANG

Department of Systems Design Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1; and Department of Mechanics, Tianjin University, Tianjin 300072, People's Republic of China

K. HUSEYIN

Department of Systems Design Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

AND

M. YE

Department of Mechanics, Tianjin University, Tianjin 300072, People's Republic of China

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In this paper, a modified normal form approach for obtaining normal forms of parametrically excited systems is presented. This approach provides a number of significant advantages over the existing normal form approaches, and improves the associated calculations. The approach lends itself more readily to symbolic calculations, like MAPLE, and the calculations of normal forms, together with the associated coefficients, are carried out much more conveniently. Four examples are presented to illustrate the approach. All examples include a comparison of the results obtained by the methods of normal forms and averaging. Example 4 contains a comparison of the results obtained by the normal form approach and Liapunov–Schmidt method as well.

1. INTRODUCTION

The normal form theory provides a powerful tool for simplifying non-linear differential equations. Using this theory, one can obtain the simplest possible form of a differential equation by sequentially applying certain co-ordinate transformations [1–3]. The formal normal form of a differential equation may be obtained in a relatively simple way [2]; however, calculating the coefficients associated with each term of a normal form can be quite cumbersome. With this motivation, a modified normal form approach has recently been developed in references [4–8] which facilitates the calculation of normal forms as well as the associated coefficients.

These new developments as well as the original normal form theory have been designed mostly for the analysis of autonomous systems. Indeed, published results concerning the normal forms of non-autonomous systems are rather limited [1, 2, 9–11]. If the coefficients of the linear part of a system are time dependent, a direct application of the normal form theory is not possible. However, such a system can be transformed (by using Floquet theory for example) to a system in which the coefficients of the linear part are time invariant. Therefore, in the application of the normal form theory, one can always assume that the

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non-autonomous system under consideration is associated with a time-invariant linear part. In references [1, 2], basic theories for obtaining formal normal forms of non-autonomous systems have been discussed; however, the associated coefficients have not been considered. The normal form theory does not lend itself conveniently to the calculation of the associated coefficients. This is particularly true for higher order normal forms. In reference [9], the authors consider the coefficients associated with normal forms of order 2 or 3, by using Lie Algebra, but mostly focus attention on the analysis of transforming a non-autonomous system, with a time-dependent linear part, to a system with a time-invariant linear part by Floquet theory.

One may consider transforming a non-autonomous system into an autonomous one (by a transformation of the form $z = e^{i\omega t}$), and then applying the normal form theory to the resulting autonomous system. It should, however, be noted that the normal form theory may not be applicable to such systems. The reason for this is that the normal form theory is basically for an asymptotic analysis in the neighbourhood of a point (e.g., the origin), and it yields a sequence of simpler transformed equations by a series of *near-identity transformations* with specific orders. However, applying a transformation of the form $z = e^{i\omega t}$ alters the character of the associated co-ordinate transformations which may no longer be defined as "near identity" (for example the term z^m), thus casting doubts on the validity of the procedure. A detailed discussion concerning this point is given in Example 1 of this paper.

It is convenient to investigate a non-linear non-autonomous system by certain asymptotic methods, like IHB technique [12, 17] or averaging and L–S methods [14–16]. If it could be established *a priori* that the normal form theory and any one of these asymptotic methods (e.g., averaging) yield identical results (at least to a certain specific order), then, the latter method can be used to obtain the full normal form if it is more convenient. In the case of autonomous systems, it has recently been demonstrated analytically [7] that the normal form theory and the averaging techniques produce *equivalent* results. A similar comparison also applies to the IHB technique, which is shown [13] to yield fastest results through the application of a symbolic computation procedure (e.g., MAPLE), compared to the normal form theory and averaging methods.

In this paper, the modified normal form approach developed earlier [4–8] is extended to non-autonomous non-linear systems. This approach provides a number of significant advantages over the existing normal form theory, and improves the associated calculations. This approach lends itself readily to the symbolic computations. Non-resonance as well as a variety of resonances associated with parametrically excited systems are considered. Four examples are presented to illustrate the convenience of the modified normal form approach.

2. BASIC THEORY [1, 2] AND THE NEW APPROACH

In order to develop a convenient approach applicable to non-autonomous systems, a very brief outline of the basic theory concerning non-autonomous systems [1, 2] is first presented.

Consider the following T-periodic differential equation:

$$\dot{u} = g(t, u) = B(t)u + g^2(t, u) + \cdots + g^r(t, u) + O(|u|^{r+1})$$

where g is continuous, $g(t, \cdot) \in \hat{C}^{r+1}(C^n, C^n)$, $r \geqslant 2$, $g(t, 0) = 0$ for all $t \in R$, and there is $T > 0$ such that $g(t + T, u) = g(t, u)$ for all $t \in R$, $u \in C^n$; $B(t)$ is an $n \times n$ matrix with continuous T-periodic entries, $g^k \in H^k_{n,T}$, $k = 2, \ldots, r$,

$$H^{k,s}_{n,T} = \{g \in \hat{C}^s(R \times C^n, C^n) | g(t, \cdot) \in H^k_n \text{ for each } t \in R, \quad g(t+T, u) = g(t, u) \text{ for all } t \in R \text{ and } u \in C^n\}$$

and

$$H^k_{n,T} = \{g \in \hat{C}^0(R \times C^n, C^n) | g(t, \cdot) \in H^k_n \text{ for each } t \in R, \quad g(t+T, u) = g(t, u) \text{ for all } t \in R \quad \text{and } u \in C^n\}$$

where s is a non-negative integer. When $s = 0$, we use $H^k_{n,T}$ instead of $H^{k,0}_{n,T}$. Each $H^{k,s}_{n,T}$ is a linear space and H^k_n is a vector space of homogeneous polynomials of order k in n variables with values in C^n.

It is well known from the Floquet theory that the above T-periodic differential equation can be transformed to the following form [1, 2]:

$$\dot{x} = f(t, x) = Ax + f^2(t, x) + \cdots + f^r(t, x) + O(|x|^{r+1}) \tag{1}$$

by using $u = P(t)x$, where A is a constant $n \times n$ matrix; $P(t) = X(t)\mathrm{e}^{-At}$, $X(t)$ is the fundamental matrix of $\dot{u} = B(t)u$ and $X(0) = \mathrm{I}$.

Suppose A has been transformed into a diagonal form. Consider a near-identity T-periodic transformation

$$x = y + h^k(t, y), \quad y \in \Omega, \quad t \in R, \tag{2}$$

where $h^k \in H^{k,1}_{n,T}$, $2 \leqslant k \leqslant r$, and Ω is a neighbourhood of the origin in C^n on which $I + h^k(t, \cdot)$ is invertible for each $t \in R$. The functions h^k will be determined such that the terms of order k in the transformed form will be simplified as resonant monomials of order k. Substituting equation (2) into equation (1) results in

$$\dot{y} = Ay + f^2(t, y) + \cdots + f^{k-1}(t, y) + (f^k(t, y) - ad^k_A h^k(t, y)) + O(|y|^{k+1}), \; y \in \Omega, \tag{3}$$

where ad^k_A is a linear operator ad^k_A: $H^{k,1}_{n,T} \to H^k_{n,T}$ defined by

$$ad^k_A h(t, y) = \frac{\partial h(t, y)}{\partial t} + h_y(t, y)Ay - Ah(t, y), \; h \in H^{k,1}_{n,T} \tag{4}$$

and $h_y(t, y)$ is the Jacobian matrix of $h(t, y)$.

Equation (3) indicates that the terms of order less than k do not change in form; only those terms of order k or higher change their forms. This is the simplest form for a polynomial of order k if

$$f^k(t, y) - ad^k_A h^k(t, y) = 0, \quad k \geqslant 2. \tag{5}$$

Let M^k_T be the range of the operator ad^k_A in $H^k_{n,T}$ and N^k_T be a complementary subspace to M^k_T in $H^k_{n,T}$, then

$$H^k_{n,T} = M^k_T \oplus N^k_T. \tag{6}$$

If $f^k(t, y) \in M_T^k$ there exists $h^k(t, y)$ such that

$$ad_A^k\, h^k(t, y) = f^k(t, y), \quad k \geqslant 2. \tag{7}$$

This means the polynomial of order k in equation (3) can be transformed to zero. Otherwise, we can only find h^k, which leads to $f^k(t, y) - ad_A^k\, h^k(t, y) \in N_T^k$. Suppose $f^k(t, y) = \xi^k(t, y) + g_{NF}^k(t, y)$, where $\xi^k(t, y) \in M_T^k$, $g_{NF}^k(t, y) \in N_T^k$. If we choose h^k, which leads to $ad_A^k(h^k(t, y)) = \xi^k(t, y)$, then equation (3) can be transformed into the following form:

$$\dot{y} = Ay + f^2(t, y) + \cdots + f^{k-1}(t, y) + g_{NF}^k(t, y) + O(|y|^{k+1}). \tag{8}$$

One can now state the following theorem [1, 2]:

Theorem 1. *Let the decomposition* (6) *be given for* $k = 2, \ldots, r$. *There exists a neighbourhood* Ω *of the origin and a sequence of near-identity* T*-periodic transformations* $x = y + h^k(t, y)$, $y \in \Omega$, $k = 2, \ldots, r$, *such that equation* (1) *takes the form*

$$\dot{y} = Ay + g_{NF}^2(t, y) + \cdots + g_{NF}^k(t, y) + O(|y|^{k+1}), \quad y \in \Omega, \tag{9}$$

where $g_{NF}^k(t, y) \in N_T^k$, $k = 2, 3, \ldots, r$.

The following truncated equation of equation (9),

$$\dot{y} = Ay + g_{NF}^2(t, y) + \cdots + g_{NF}^k(t, y), \tag{10}$$

is called a normal form, up to order k, of equation (1).

A monomial $e^{il\gamma t} x^m e_s$, $1 \leqslant s \leqslant n$ is called a resonant monomial of order m if and only if

$$-\lambda_s + \sum_{j=1}^{n} m_j \lambda_j + l\gamma i = 0, \tag{11}$$

where m_j, l are integers, $x^m = x_1^{m_1} \cdots x_n^{m_n}$, $\gamma = 2\pi/T$ and $\Sigma_{j=1}^n |m_j| = m \geqslant 2$. Thus, one obtains the following results:

Theorem 2. $e^{il\gamma t} x^m e_s \in N_T^k$, *if and only if equation* (11) *holds.*

Theorem 3. *If* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$, *then a normal form up to order* $r \geqslant 2$ *can be chosen so that its non-linear part consists of all the resonant monomials up to order* r.

This is the basic theory of normal forms in non-autonomous systems.

For the convenience of discussion, the following two-dimensional equation is now considered:

$$\dot{x} = Ax + \sum_{k=2}^{r} F^k(\gamma t, x) \tag{12}$$

where $x \in C^2$, $F^k \in H^k_{2,T}$, $k = 2, 3, \dots, r$; γ is the frequency associated with the non-autonomous terms; and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} \mathrm{i}\omega & 0 \\ 0 & -\mathrm{i}\omega \end{pmatrix}, \quad F^k(\gamma t, x) = \begin{pmatrix} \sum_{i=0}^{k} \sum_{n=-R}^{R} a_{i(k-i)n} x_1^i x_2^{k-i} \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum_{i=0}^{k} \sum_{n=-R}^{R} b_{i(k-i)n} x_1^i x_2^{k-i} \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}$$

in which $R \in Z$; $a_{i(k-i)n}$ and $b_{i(k-i)n}$ are constants and $\mp \mathrm{i}\omega$ are the eigenvalues.

Suppose

$$x = y + P^2(\gamma t, y), \quad P^2(\gamma t, y) \in H^{2;1}_{2;T}, \tag{13}$$

where $P^2(\gamma t, z)$ is an undefined function, which will be determined such that the terms of order 2 in the transformed form will be simplified as resonant monomials of order 2.

Substituting equation (13) into equation (12), results in

$$\dot{y} = Ay + F^2_1(\gamma t, y) + F^3_1(\gamma t, y) + \text{h.o.t.}, \tag{14}$$

where $F^2_1 = F^2 + AP^2 - DP^2Ay$ and $F^3_1 = F^3 + DF^2P^2 - DP^2F^2_1$.

Suppose $F^2_1(\gamma t, y) = G^2(\gamma t, y)$ in equation (14), where $G^2(\gamma\, t, y)$ is the resonant polynomial of order 2. Solving this equation for $P^2(\gamma t, y)$, the coefficients in $G^2(\gamma t, y)$ can be obtained. Substituting $P^2(\gamma t, y)$ into $F^3_1(\gamma t, y)$ defines F^3_1 as $F^3_1(\gamma t, y)$. Then, suppose

$$y = z + P^3(\gamma t, z), \quad P^3(\gamma t, z) \in H^{3;1}_{2;T}, \tag{15}$$

where $P^3(\gamma t, z)$ is an undefined function, which will be determined such that the terms of order 3 in the transformed form will be simplified as resonant monomials of order 3.

Substituting equation (15) into equation (14) leads to

$$\dot{z} = Az + F^2_1(\gamma t, z) + F^3_2(\gamma t, z) + \text{h.o.t.}, \tag{16}$$

where $F^3_2 = F^3_1 + AP^3 - DP^3Az$.

Suppose $F^3_2(\gamma t, y) = G^3(\gamma t, y)$ in equation (16), where $G^3(\gamma t, y)$ is the resonant polynomial of order 3. Solving this equation for $P^3(\gamma t, y)$ yields the coefficients in $G^3(\gamma t, y)$. Thus, the normal form of equation (12), up to order 3, is given by

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x). \tag{17}$$

It is noted that if there is a resonance in the system, i.e., $p\omega = q\gamma$, where p, q are integers, the calculation of the associated coefficients of normal forms by the above procedure may pose difficulties, particularly in the case of higher order normal forms. In order to determine the normal forms and the associated coefficients more conveniently, a modified normal form approach is presented as follows:

Introducing the transformation

$$x = \mathrm{e}^{At} z \tag{18}$$

into equation (16), one has

$$\dot{z} = \mathrm{e}^{-At} \lfloor F^2_1(\gamma t, \mathrm{e}^{At} z) + F^3_2(\gamma t, \mathrm{e}^{At} z) \rfloor + \text{h.o.t.}, \tag{19}$$

where

$$\mathrm{e}^{At}z = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} z_1 \\ \mathrm{e}^{\lambda_2 t} z_2 \end{pmatrix}.$$

Suppose

$$F^k_{k-1}(\gamma t, y) = \begin{pmatrix} F^k_{k-1(1)}(\gamma t, y) \\ F^k_{k-1(2)}(\gamma t, y) \end{pmatrix} = \begin{pmatrix} \sum\limits_{m+n=k} \sum\limits_{s=-R}^{R} a^{k-1}_{mn(1)s} y_1^m y_2^n \mathrm{e}^{\mathrm{i}s\gamma t} \\ \sum\limits_{m+n=k} \sum\limits_{s=-R}^{R} a^{k-1}_{mn(2)s} y_1^m y_2^n \mathrm{e}^{\mathrm{i}s\gamma t} \end{pmatrix}.$$

then, $\mathrm{e}^{-\lambda_q t} F^k_{k-1(q)}(\gamma t, \mathrm{e}^{At}z)$ can be expressed as

$$\begin{aligned} \mathrm{e}^{-\lambda_q t} F^k_{k-1(q)}(\gamma t, \mathrm{e}^{At}z) &= \sum_{m+n=k} \sum_{s=-R}^{R} \mathrm{e}^{-\lambda_q t} a^{k-1}_{mn(q)s} \mathrm{e}^{m\lambda_1 t} z_1^m \mathrm{e}^{n\lambda_2 t} z_2^n \mathrm{e}^{is\gamma t} \\ &= \sum_{m+n=k} \sum_{s=-R}^{R} a^{k-1}_{mn(q)s} \mathrm{e}^{(m\lambda_1 + n\lambda_2 + \mathrm{i}s\gamma - \lambda_q)t} z_1^m z_2^n, \end{aligned} \tag{20}$$

where $q = 1, 2$.

According to the assumption $F^k_{k-1}(\gamma t, y) = G^k(\gamma t, y)$, functions $F^k_{k-1}(\gamma t, y)$ are composed of resonant monomials, in which

$$m\lambda_1 + n\lambda_2 + \mathrm{i}s\gamma = \lambda_q. \tag{21}$$

According to equations (19–21), one has

$$\mathrm{e}^{-At} F^k_{k-1}(\gamma t, \mathrm{e}^{At}z) = F^k_{k-1}(\gamma t, z) = \underset{t}{M} \{\mathrm{e}^{-At} F^k_{k-1}(\gamma t, \mathrm{e}^{At}z)\}. \tag{22}$$

Similarly, one has

$$\mathrm{e}^{At} F^k_{k-1}(\gamma t, \mathrm{e}^{-At}z) = F^k_{k-1}(\gamma t, z). \tag{23}$$

Thus, equation (19) can be expressed as

$$\dot{z} = G^2(\gamma t, z) + G^3(\gamma t, z) + \text{h.o.t.} \tag{24}$$

Carrying out the inverse transformation $z = \mathrm{e}^{-At}x$ in equation (24), according to equation (23), one has

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x) + \text{h.o.t.} \tag{25}$$

This is the normal form of equation (12).

In order to determine the coefficients of the above resonant monomials and the associated transformations more conveniently, consider the following relation (identity):

$$\frac{\partial}{\partial \mathrm{t}} [\mathrm{e}^{-At} P^k(\gamma t, \mathrm{e}^{At}z)] = \mathrm{e}^{-At} \left(\frac{\partial P^k(\gamma t, \xi)}{\partial t} + D_\xi P^k(\gamma t, \xi) A\xi - AP^k(\gamma t, \xi) \right) \Bigg|_{\xi = \mathrm{e}^{At}z}, \quad k \in Z. \tag{26}$$

On the other hand, equations (3) and (4) lead to

$$F^k_{k-1}(\gamma t, x) = F^k_{k-2}(\gamma t, x) + AP^k(\gamma t, x) - D_x P^k(\gamma t, x) Ax - \frac{\partial P^k(\gamma t, x)}{\partial t} \tag{27}$$

and equations (26) and (27) lead to

$$\mathrm{e}^{-At} F^k_{k-1}(\gamma t, \mathrm{e}^{At} z) + \frac{\partial}{\partial \mathrm{t}} [\mathrm{e}^{-\mathrm{A}t} P^k(\gamma t, \mathrm{e}^{At} z)] = \mathrm{e}^{-At} F^k_{k-2}(\gamma t, \mathrm{e}^{At} z) \tag{28}$$

$$\mathrm{e}^{-At}[F^k_{k-2}(\gamma t, \mathrm{e}^{At} z) - F^k_{k-1}(\gamma t, \mathrm{e}^{At} z)] = \frac{\partial}{\partial t} [\mathrm{e}^{-At} P^k(\gamma t, \mathrm{e}^{At} z)] \tag{29}$$

and following the procedures described (p. 615) in reference [4] yields

$$\underset{t}{M} \{\mathrm{e}^{-At}[F^k_{k-2}(\gamma t, \mathrm{e}^{At} z) - F^k_{k-1}(\gamma t, \mathrm{e}^{At} z)]\} = \underset{t}{M} \left\{\frac{\partial}{\partial \mathrm{t}} [\mathrm{e}^{-At} P^k(\gamma t, \mathrm{e}^{At} z)]\right\} = 0. \tag{30}$$

Thus, one has

$$\underset{t}{M} \{\mathrm{e}^{-At} F^k_{k-1}(\gamma t, \mathrm{e}^{At} z)\} = \underset{t}{M} \{\mathrm{e}^{-At} F^k_{k-2}(\gamma t, \mathrm{e}^{At} z)\} \tag{31}$$

Equations (23) and (31) lead to

$$F^k_{k-1}(\gamma t, z) = G^k(\gamma t, z) = \underset{t}{M} \{\mathrm{e}^{-At} F^k_{k-1}(\gamma t, \mathrm{e}^{At} z)\} = \underset{t}{M} \{\mathrm{e}^{-At} F^k_{k-2}(\gamma t, \mathrm{e}^{At} z)\} \tag{32}$$

From equation (32), one has

$$\begin{aligned} P^k(\gamma t, z) = \mathrm{e}^{-At} P^k(\gamma t, \mathrm{e}^{At} z)|_{t=0} &= \int \{\mathrm{e}^{-At}[F^k_{k-2}(\gamma t, \mathrm{e}^{At} z) - F^k_{k-1}(\gamma t, \mathrm{e}^{At} z)]\} \mathrm{d}t|_{t=0} \\ &= \int \{\mathrm{e}^{-At} F^k_{k-2}(\gamma t, \mathrm{e}^{At} z) - G^k(\gamma t, z)\} \mathrm{d}t|_{t=0}. \end{aligned} \tag{33}$$

According to equations (32) and (33), $F^k_{k-1}(\gamma t, z)$ and $P^k(\gamma t, z)$ can be obtained directly from the $(k-2)$th transformed functions $F^k_{k-2}(\gamma t, z)$, where $k \geqslant 2$. Using equation (33), $P^k(\gamma t, z)$ can be expressed as polynomials of order k. This procedure can be conveniently completed with the aid of symbolic calculations. Following the procedures leading to $F^k_m(\gamma t, y)$ and $P^k(\gamma t, y)$ $(m \leqslant k-1)$, one can see that functions $F^k_m(\gamma t, y)$ and $P^k(\gamma t, y)$ obtained from the conventional normal form theory and those from the new approach are identical to each other. Therefore, the normal forms $G^k(\gamma t, z)$ obtained by the conventional normal form theory and those by the new approach are identical to each other. Furthermore, the above procedure can be applied to high-dimensional systems as well. Actually, examples 2 and 4 in this paper are three-dimensional non-autonomous system.

Clearly, the normal forms obtained by the above approach are based on the normal form theory directly and not "on the equivalence of the methods of normal forms and averaging", as assumed in reference [10]. The relation between the results obtained by the methods of normal forms and the averaging are discussed in section 4 in this paper as well. It will be shown that both methods lead to *equivalent* results. However, the modified normal form approach introduced here does not depend on the averaging method.

3. APPLICATIONS TO CERTAIN PARAMETRICALLY EXCITED SYSTEMS

Consider a general system described by equation (12). Many engineering problems can be described by this equation, such as the Euler dynamical buckling problem [16] and certain oscillatory systems [14,15].

In this section, the non-resonant case as well as certain resonances will be discussed.

The system is said to be a non-resonant one if

$$\omega \neq \frac{p}{q}\gamma, \tag{34}$$

where p, q are arbitrary prime integers.

The system is called a resonant system if

$$\omega = \frac{p}{q}\gamma. \tag{35}$$

Furthermore, one has the following popular classifications:

$\gamma = \omega$: primary or main resonance;
$\gamma = q\omega$: subharmonic resonance of order q;
$\gamma = (1/p)\omega$: superharmonic resonance of order p;
$\gamma = (q/p)\omega$: general resonance.

Next, a detailed discussion of the above cases will be presented.

3.1. NON-RESONANT CASE

Following the procedure discussed in section 2, one can obtain a normal form of equation (12) as

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x),$$

where G^k satisfies the following relations:

$$(m_1 - m_2 \mp 1)\omega + n\gamma = 0, \quad q\omega - p\gamma \neq 0. \tag{36}$$

Thus, the solution of equation (36) is $m_1 - m_2 = \pm 1$, $n = 0$. It is not difficult to obtain the formal normal form as follows:

$$\dot{x} = Ax + \begin{pmatrix} \sum_{m=1}^{k} a_m x_1^{m+1} x_2^m \\ \sum_{m=1}^{k} b_m x_1^m x_2^{m+1} \end{pmatrix}. \tag{37}$$

The formal normal form is similar to that of an autonomous system, but the associated coefficients of the normal form account for the non-autonomous terms, as well.

Resonant monomials G^k and transformation functions P^k can be obtained from equations (32) and (33) as

$$G^s(x) = \begin{pmatrix} \displaystyle\sum_{\substack{m+n=s\\ m-n=1}} a^{s-2}_{mn0} x_1^m x_2^n \\ \displaystyle\sum_{\substack{m+n=s\\ m-n=-1}} b^{s-2}_{mn0} x_1^m x_2^n \end{pmatrix}, \quad G^{2r}(x) = 0, \tag{38}$$

where a^{s-2}_{mn0} and b^{s-2}_{mn0} are the coefficients of term $x_1^m x_2^n e^{in_{mn}\gamma t}$ in the transformed functions F^s_{s-2}, $2r \leqslant k$, and

$$P^k(\gamma t, x) = \frac{1}{i} \begin{pmatrix} \displaystyle\sum_{\substack{s=0\\ c_1 \neq 0}}^{k} \sum_{n=-R}^{R} \frac{1}{c_1} e^{i(k-2s-1)\omega t + in\gamma t} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \\ \displaystyle\sum_{\substack{s=0\\ c_2 \neq 0}}^{k} \sum_{n=-R}^{R} \frac{1}{c_2} e^{i(k-2s+1)\omega t + in\gamma t} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \end{pmatrix}, \tag{39}$$

respectively, where $c_1 = (k-2s-1)\omega + n\gamma$, $c_2 = (k-2s+1)\omega + n\gamma$.

3.2. RESONANCES

3.2.1. *Primary resonance*

Following the procedure discussed in section 2, one can obtain a normal form of equation (12) as

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x),$$

where G^k satisfies the following relations:

$$(m_1 - m_2 \mp 1)\omega + n\gamma = 0, \quad \omega = \gamma. \tag{40}$$

Thus, the solution of equation (40) is $m_1 - m_2 + n = \pm 1$. It is not difficult to obtain the formal normal form as follows:

$$\dot{x} = Ax + \begin{pmatrix} \displaystyle\sum_{\substack{m_1+m_2=2\\ m_1-m_2=1-n}}^{k} \sum_{n=-R}^{R} a_{m_1m_2n} x_1^{m_1} x_2^{m_2} e^{in\gamma t} \\ \displaystyle\sum_{\substack{m_1+m_2=2\\ m_1-m_2=-1-n}}^{k} \sum_{n=-R}^{R} b_{m_1m_2n} x_1^{m_1} x_2^{m_2} e^{in\gamma t} \end{pmatrix}, \tag{41}$$

Resonant monomials G^k and the transformation functions P^k can be obtained from equations (32) and (33). Thus,

$$G^k(x) = \begin{pmatrix} \sum\limits_{\substack{s=0 \\ c_1=0}}^{k} \sum\limits_{n=-R}^{R} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0 \\ c_2=0}}^{k} \sum\limits_{n=-R}^{R} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}, \tag{42}$$

where $a^{k-2}_{(k-s)sn}$ and $b^{k-2}_{(k-s)sn}$ are the coefficients of term $x_1^{k-s}x_2^s\mathrm{e}^{\mathrm{i}n\gamma t}$ in the transformed functions F^s_{s-2}, $2n \leqslant k$, and

$$P^k(\gamma t, x) = \frac{1}{\mathrm{i}} \begin{pmatrix} \sum\limits_{\substack{s=0 \\ c_1\neq 0}}^{k} \sum\limits_{n=-R}^{R} \dfrac{1}{c_1} \mathrm{e}^{\mathrm{i}(k-2s-1)t+\mathrm{i}nt} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0 \\ c_2\neq 0}}^{k} \sum\limits_{n=-R}^{R} \dfrac{1}{c_2} \mathrm{e}^{\mathrm{i}(k-2s+1)t+\mathrm{i}nt} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}, \tag{43}$$

respectively, where $c_1 = (k - 2s - 1) + n$, $c_2 = (k - 2s + 1) + n$.

3.2.2. *Subharmonic resonance*

Following the procedure discussed in section 2, one can obtain a normal form of equation (12) as

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x),$$

where G^k satisfies the following relations:

$$(m_1 - m_2 \mp 1)\omega + n\gamma = 0, \quad q\omega = \gamma. \tag{44}$$

Thus, the solution of equation (44) is $m_1 - m_2 + qn = \pm 1$. The formal normal form is given by

$$\dot{x} = Ax + \begin{pmatrix} \sum\limits_{\substack{m_1+m_2=2 \\ m_1-m_2=1-qn}}^{k} \sum\limits_{n=-R}^{R} a_{m_1m_2n} x_1^{m_1} x_2^{m_2} \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{m_1+m_2=2 \\ m_1-m_2=-1-qn}}^{k} \sum\limits_{n=-R}^{R} b_{m_1m_2n} x_1^{m_1} x_2^{m_2} \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix} \tag{45}$$

which is valid generally if $q\omega = \gamma$.

The associated coefficients of normal forms can be obtained by the procedure introduced above. Resonant monomials G^k and the transformation functions P^k can be obtained from equations (32) and (33) as

$$G^k(x) = \begin{pmatrix} \sum\limits_{\substack{s=0\\c_1=0}}^{k} \sum\limits_{n=-R}^{R} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0\\c_2=0}}^{k} \sum\limits_{n=-R}^{R} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}, \tag{46}$$

and

$$P^k(\gamma t, x) = \frac{1}{i} \begin{pmatrix} \sum\limits_{\substack{s=0\\c_1\neq 0}}^{k} \sum\limits_{n=-R}^{R} \frac{1}{c_1} \mathrm{e}^{\mathrm{i}(k-2s-1)t+\mathrm{i}nt} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0\\c_2\neq 0}}^{k} \sum\limits_{n=-R}^{R} \frac{1}{c_2} \mathrm{e}^{i(k-2s+1)t+\mathrm{i}nt} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}, \tag{47}$$

respectively, where $a^{k-2}_{(k-s)sn}$ and $b^{k-2}_{(k-s)sn}$ are the coefficients of term $x_1^{k-s}x_2^s\mathrm{e}^{\mathrm{i}n\gamma t}$ in the transformed functions F^s_{s-2}; $c_1 = (k-2s-1)+nq$, $c_2 = (k-2s+1)+qn$.

3.2.3. *Superharmonic resonance*

Following the procedure discussed in section 2, one can obtain a normal form of equation (12) as

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x),$$

where G^k satisfies the following relations:

$$(m_1 - m_2 \mp 1)\omega + \mathrm{n}\gamma = 0, \quad \omega = p\gamma. \tag{48}$$

Thus, the solution of equation (48) is $(m_1 - m_2)p + n = p$. It is not difficult to obtain the formal normal form as follows:

$$\dot{x} = Ax + \begin{pmatrix} \sum\limits_{\substack{\mathrm{m}_1+m_2=2\\(m_1-m_2)p=p-n}}^{k} \sum\limits_{n=-R}^{R} a_{m_1m_2n} x_1^{m_1} x_2^{m_2} \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{m_1+m_2=2\\(m_1-m_2)p=-p-n}}^{k} \sum\limits_{n=-R}^{R} b_{m_1m_2n} x_1^{m_1} x_2^{m_2} \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix}. \tag{49}$$

Resonant monomials G^k and the transformation functions P^k can be obtained from equations (32) and (33) as

$$G^k(x) = \begin{pmatrix} \sum\limits_{\substack{s=0\\c_1=0}}^{k} \sum\limits_{n=-R}^{R} a^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0\\c_2=0}}^{k} \sum\limits_{n=-R}^{R} b^{k-2}_{(k-s)sn} x_1^{k-s} x_2^s \mathrm{e}^{\mathrm{i}n\gamma t} \end{pmatrix} \tag{50}$$

and

$$P^k(\gamma t, x) = \frac{1}{i}\begin{pmatrix} \sum\limits_{\substack{s=0\\ c_1\neq 0}}^{k}\sum\limits_{n=-R}^{R}\frac{1}{c_1}\mathrm{e}^{\mathrm{i}(k-2s-1)t+\mathrm{i}nt}a^{k-2}_{(k-s)sn}x_1^{k-s}x_2^s\mathrm{e}^{\mathrm{i}n\gamma t} \\ \sum\limits_{\substack{s=0\\ c_2\neq 0}}^{k}\sum\limits_{n=-R}^{R}\frac{1}{c_2}\mathrm{e}^{\mathrm{i}(k-2s+1)t+\mathrm{i}nt}b^{k-2}_{(k-s)sn}x_1^{k-s}x_2^s\mathrm{e}^{\mathrm{i}n\gamma t}\end{pmatrix}, \tag{51}$$

respectively, where $a^{k-2}_{(k-s)sn}$ and $b^{k-2}_{(k-s)sn}$ are the coefficients of term $x_1^{k-s}x_2^s\mathrm{e}^{\mathrm{i}n\gamma t}$ in the transformed functions F^s_{s-2}; $c_1 = (k-2s-1)p + n$, $c_2 = (k-2s+1)p + n$.

3.2.4. *General resonance*

Following the procedure discussed in section 2, one can obtain a normal form of equation (12) as

$$\dot{x} = Ax + G^2(\gamma t, x) + G^3(\gamma t, x),$$

where G^k satisfies

$$(m_1 - m_2 \mp 1)\omega + n\gamma = 0, \quad \omega = \frac{p}{q}\gamma, \tag{52}$$

where p, q are given prime integers and $p, q \neq 0$.

Thus, the solution of above equation is $(m_1 - m_2 \mp 1)p + nq = 0$, i.e., $m_1 - m_2 = qu \pm 1$, $n = -pu$, $u \in Z$. The formal normal form is given by

$$\dot{x} = Ax + \sum_{m=1}^{k} a_{0m}x_1^{m+1}x_2^m$$
$$+\begin{pmatrix} \sum\limits_{m=0}^{M}\sum\limits_{u=1}^{U} a_{m(m+uq-1)(up)}x_1^m x_2^{m+uq-1}\mathrm{e}^{\mathrm{i}up\gamma t} + \sum\limits_{m=0}^{M}\sum\limits_{u=1}^{U} a_{(m+uq+1)m(-up)}x_1^{m+uq+1}x_2^m\mathrm{e}^{-\mathrm{i}up\gamma t} \\ \sum\limits_{m=0}^{M}\sum\limits_{u=1}^{U} \bar{a}_{(m+uq-1)m(-up)}x_1^{m+uq-1}x_2^m\mathrm{e}^{-\mathrm{i}up\gamma t} + \sum\limits_{m=0}^{M}\sum\limits_{u=1}^{U} \bar{a}_{m(m+uq+1)(up)}x_1^m x_2^{m+uq+1}\mathrm{e}^{\mathrm{i}up\gamma t}\end{pmatrix}. \tag{53}$$

Following reference [2], suppose conditions $q - 1 \leqslant 2k + 1 \leqslant q$ apply; in this special case it follows from equation (53) that

$$\dot{x}_1 = \mathrm{i}\omega x_1 + a_{0(q-1)q}x_2^{q-1}\mathrm{e}^{\mathrm{i}q\omega t} + \sum_{m=1}^{k} a_{(m+1)m0}x_1^{m+1}x_2^m$$
$$\dot{x}_2 = -\mathrm{i}\omega x_2 + \bar{a}_{(q-1)0(-q)}x_1^{q-1}\mathrm{e}^{-\mathrm{i}q\omega t} + \sum_{m=1}^{k} \bar{a}_{m(m+1)0}x_2^{m+1}x_1^m. \tag{53*}$$

This result is identical to that in reference [2], and describes higher resonances. It is noted that in reference [2] the associated coefficients are not discussed. In this paper, the associated coefficients can be obtained readily by the procedure introduced above, i.e., from equations (32) and (33). Generally, the lower resonances play more important role than higher resonances in the dynamical analysis of a system. The general normal form (53) embraces both lower and higher order resonances.

4. SPECIFIC EXAMPLES AND DISCUSSIONS

Example 1. Determine the normal form and the related coefficients of the following system:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 \mathrm{x} + (ax^2 + bxy + cy^2)\cos \gamma t + dx^2 y. \tag{54}$$

Introducing the transformation

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{\mathrm{i}}{2}\omega(z - \bar{z})$$

into equation (54), one has

$$\begin{aligned}\dot{z} = \mathrm{i}\omega z &- \frac{\mathrm{i}}{8\omega}(a(z+\bar{z})^2 + \mathrm{i}b\omega(z+\bar{z})(z-\bar{z}) - c\omega^2(z-\bar{z})^2)\,(\mathrm{e}^{\mathrm{i}\gamma t} + \mathrm{e}^{-\mathrm{i}\gamma t}) \\ &+ \frac{1}{8\omega}d(z+\bar{z})^2(z-\bar{z}).\end{aligned} \tag{55}$$

Following the procedure introduced in section 3 yields the following results:
If $\omega = 1$, $\gamma = 2$, one has the normal form up to order 3 as follows:

$$\begin{aligned}\dot{z} = \mathrm{i}z &+ \frac{1}{8}dz^2\bar{z} + \frac{\mathrm{i}}{40}a^2z^2\bar{z} + \frac{19\mathrm{i}}{120}acz^2\bar{z} - \frac{1}{48}abz^2\bar{z} - \frac{1}{48}bcz^2\bar{z} - \frac{\mathrm{i}}{16}a^2\bar{z}^3\mathrm{e}^{2\mathrm{i}\gamma t} \\ &+ \frac{\mathrm{i}}{16}ac\bar{z}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3}{32}ab\bar{z}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{\mathrm{i}}{60}c^2z^2\bar{z} + \frac{1}{32}bc\bar{z}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3\mathrm{i}}{80}b^2z^2\bar{z} + \frac{\mathrm{i}}{32}b^2\bar{z}^3\mathrm{e}^{2\mathrm{i}\gamma t}.\end{aligned} \tag{56}$$

Assuming that $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $\theta = t + \vartheta$, leads to the polar form,

$$\begin{aligned}\dot{r} &= r^3(\tfrac{1}{8}d - \tfrac{1}{48}ab - \tfrac{1}{48}bc - \tfrac{1}{16}a^2\sin 4\vartheta + \tfrac{1}{16}ac\sin 4\vartheta + \tfrac{1}{32}b^2\sin 4\vartheta \\ &\quad - \tfrac{3}{32}ab\cos 4\vartheta + \tfrac{1}{32}bc\cos 4\vartheta), \\ \dot{\vartheta} &= r^2(\tfrac{1}{40}a^2 + \tfrac{19}{120}ac - \tfrac{1}{60}c^2 - \tfrac{3}{80}b^2 - \tfrac{1}{16}a^2\cos 4\vartheta + \tfrac{1}{16}ac\cos 4\vartheta \\ &\quad + \tfrac{1}{32}b^2\cos 4\vartheta - \tfrac{1}{32}bc\sin 4\vartheta + \tfrac{3}{32}ab\sin 4\vartheta).\end{aligned} \tag{57}$$

This result is obtained in 1 s with the aid of MAPLE.

It may be interesting to the reader to see how the averaging methods work compared to the normal form procedure presented above. Thus, introducing the scaling $z = \varepsilon x$ into equation (55) results in

$$\dot{x} = Ax + \varepsilon F^2(x, \gamma t) + \varepsilon^2 F^3(x, \gamma t), \tag{58}$$

where ε is a small perturbation parameter.

Substituting $x = e^{At}y$ into equation (58), one has

$$\dot{y} = \varepsilon f^2(y, t) + \varepsilon^2 f^3(y, t) \tag{59}$$

in which $f^2(y, t) = e^{-At}F^2(e^{At}y)$ and $f^3(y, t) = e^{-At}F^3(e^{At}y)$.

Introducing the transformation

$$y = u + \sum_{m=1}^{s} \varepsilon^m \phi_m(u, t)$$

and assuming

$$\dot{u} = \sum_{m=1}^{s} \varepsilon^m G_A^m(u, t),$$

where $\underset{t}{M}\{G_A^m(u, t)\} = 0$, i.e., explicit time averaging of function $G_A^m(u, t)$ is equal to zero, equation (59) leads to the identity

$$\left(I + \sum_{m=1}^{s} \varepsilon^m \frac{\partial \phi_m}{\partial u}\right)\left(\sum_{m=1}^{s} \varepsilon^m G_A^m\right) + \sum_{m=1}^{s} \varepsilon^m \frac{\partial \phi_m}{\partial t} = \varepsilon f^2\left(\left(u + \sum_{i=1}^{s} \varepsilon^i \phi_i\right), t\right)$$
$$+ \varepsilon^2 f^3\left(\left(u + \sum_{i=1}^{s} \varepsilon^i \phi_i\right), t\right). \tag{60}$$

Comparing the coefficients of similarly-ordered terms in equation (60) produces

$$\varepsilon\colon\ G_A^1 + \frac{\partial \phi_1}{\partial t} = f_A^2,$$

$$\varepsilon^2\colon\ G_A^2 + \frac{\partial \phi_2}{\partial \mathrm{t}} = f_A^3,$$

$$\vdots$$

$$\varepsilon^{k-1}\colon\ G_A^{k-1} + \frac{\partial \phi_{k-1}}{\partial t} = f_{\mathrm{A}}^{\mathrm{k}}, \tag{61}$$

where

$$f_A^2 = f^2 = 0, \quad f_A^3 = f^3 + D_u f^2 \phi_1 - D_u \phi_1 \mathrm{G}_A^1,$$

$$\vdots$$

$$f_A^k = \sum_{m=2}^{k} \sum_{p=1}^{m} \frac{D_x^p f^m}{p!} \sum_{\substack{\bar{s}=p \\ \delta=k-m}} a_{s_1 s_2 \cdots s_s} \phi_1^{s_1} \phi_2^{s_2} \cdots \phi_s^{s_s} - \sum_{m=1}^{k-1} D_u \phi_{k-m-1} G_A^m$$

and

$$\phi_q = \int (f_A^{q+1} - G_A^q)\,\mathrm{d}t + \sum_{r=1}^{k} \begin{pmatrix} \tilde{c}_{r(1)} G_{A(1)}^r \\ \vdots \\ \tilde{c}_{r(n)} G_{A(\mathrm{n})}^r \end{pmatrix}, \quad G_A^r = \begin{pmatrix} G_{A(1)}^r(x) \\ \vdots \\ G_{A(n)}^r(x) \end{pmatrix}.$$

For convenience, $\tilde{c}_{r(1)}, \tilde{c}_{r(2)}, \ldots, \tilde{c}_{r(n)}$ are usually chosen as zero, and a "*simplified form*" of equation (58) is obtained.

Thus, if $\omega = 1$, $\gamma = 2$, following the averaging analysis as above, one has the following "*simplified form*" of equation (58) up to order 3:

$$\begin{aligned}\dot{u} &= \mathrm{i}u + \frac{1}{8}\varepsilon^2 du^2\bar{u} + \frac{\mathrm{i}}{40}\varepsilon^2 a^2u^2\bar{u} + \frac{19\mathrm{i}}{120}\varepsilon^2\, acu^2\,\bar{u} - \frac{1}{48}\varepsilon^2 abu^2\bar{u} - \frac{1}{48}\varepsilon^2\, bcu^2\bar{u} \\ &- \frac{\mathrm{i}}{16}\varepsilon^2 a^2\bar{u}^3\mathrm{e}^{2\mathrm{i}\gamma t} + \frac{\mathrm{i}}{16}\varepsilon^2 ac\bar{u}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3}{32}\varepsilon^2\, ab\bar{u}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{\mathrm{i}}{60}\varepsilon^2 c^2u^2\bar{u} \\ &+ \frac{1}{32}\varepsilon^2\, bc\bar{u}^3\mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3\mathrm{i}}{80}\varepsilon^2 b^2u^2\bar{u} + \frac{\mathrm{i}}{32}\varepsilon^2 b^2\bar{u}^3\mathrm{e}^{2\mathrm{i}\gamma t}.\end{aligned} \tag{62}$$

This result is obtained in 0.25 s with the aid of MAPLE.

It should be noted that the results obtained by both methods are identical (in this example) up to order 3. Further investigation shows that the results obtained by both methods are identical (in this example) up to order 6, and appear to be different in higher orders.

For example, if $b = 0$, $c = 0$, $d = 0$, and $\omega = 1$, $\gamma = 2$, one can obtain the associated normal form up to order 7 (by normal form theory) as follows:

$$\begin{aligned}\dot{z} &= \frac{1}{40}a^2z^2\bar{z} - \frac{\mathrm{i}}{16}a^2\bar{z}^3\mathrm{e}^{2\gamma t} + \frac{33\,689\mathrm{i}}{2\,304\,000}a^4z^3\bar{z}^2 + \frac{2057\mathrm{i}}{115\,200}a^4z\bar{z}^4\mathrm{e}^{2\gamma t} \\ &+ \frac{9\mathrm{i}}{2560}a^4z^5\mathrm{e}^{-2\gamma t} + \frac{290\,469\,377\mathrm{i}}{92\,897\,280\,000}a^6z^2\bar{z}^5\mathrm{e}^{2\gamma t} + \frac{174\,337\mathrm{i}}{154\,828\,800}a^6z^6\bar{z}\mathrm{e}^{-2\gamma t} \\ &- \frac{57\,347\mathrm{i}}{22\,118\,400}a^6\bar{z}^7\mathrm{e}^{4\gamma t} - \frac{890\,580\,469}{851\,558\,400\,000}a^6z^4\bar{z}^3.\end{aligned} \tag{63}$$

This result is obtained in 50 s with the aid of MAPLE.

The "*simplified form*" of equation (58) up to order 7 (obtained by averaging method) is obtained as

$$\begin{aligned}\dot{u} &= \frac{1}{40}\varepsilon^2a^2u^2\bar{u} - \frac{\mathrm{i}}{16}\varepsilon^2a^2\bar{u}^3\mathrm{e}^{2\gamma t} + \frac{33\,689\mathrm{i}}{2\,304\,000}\varepsilon^4a^4u^3\bar{u}^2 + \frac{2057\mathrm{i}}{115\,200}\varepsilon^4a^4u\bar{u}^4\mathrm{e}^{2\gamma t} \\ &+ \frac{9\mathrm{i}}{2560}\varepsilon^4a^4u^5\mathrm{e}^{-2\gamma t} + \frac{236\,279\,297\mathrm{i}}{92\,897\,280\,000}\varepsilon^6a^6u^2\bar{u}^5\mathrm{e}^{2\gamma t} - \frac{6887\mathrm{i}}{3\,870\,720\,000}\varepsilon^6a^6u^6\bar{u}\;\mathrm{e}^{-2\gamma t} \\ &- \frac{6671\mathrm{i}}{110\,592\,000}\varepsilon^6a^6\bar{u}^7\mathrm{e}^{4\gamma t} - \frac{646\,504\,753}{170\,311\,680\,000}\varepsilon^6a^6u^4\bar{u}^3.\end{aligned} \tag{64}$$

This result is obtained in 24 s with the aid of MAPLE.

Equations (63) and (64) are apparently not identical to each other. However, it can be shown that the complete results (including terms of order 11) are linked together by a near-identity transformation. One can reach similar conclusions in all the following examples. A detailed discussion concerning the relationship and differences between the methods of normal forms and averaging is presented in reference [7].

As remarked in the introduction, in order to determine the normal form and the associated coefficients of non-autonomous systems, one might consider transforming the non-autonomous terms into autonomous ones by a transformation of the form $z = \mathrm{e}^{\mathrm{i}\omega t}$; then, the normal form theory is employed to study the resulting transformed (autonomous) system. However, the applicability of the normal form theory to such systems requires justification.

Suppose

$$z_2 = \mathrm{e}^{\mathrm{i}\gamma t}, \quad \bar{z}_2 = \mathrm{e}^{-\mathrm{i}\gamma t}. \tag{65}$$

Introducing equation (65) and $x = \frac{1}{2}(z_1 + \bar{z}_1), \quad y = (\mathrm{i}/2)\omega(z_1 - \bar{z}_1)$ into equation (54), one has

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}\omega z_1 - \frac{\mathrm{i}}{8\omega}\left(a(z_1 + \bar{z}_1)^2 + \mathrm{i}b\omega(z_1^2 - \bar{z}_1^2) - c\omega^2(z_1 - \bar{z}_1)^2\right)(z_2 + \bar{z}_2) \\ &\quad + \frac{1}{8\omega} d(z_1 + \bar{z}_1)^2(z_1 - \bar{z}_1), \\ \dot{z}_2 &= \mathrm{i}\gamma z_2. \end{aligned} \tag{66}$$

This is an autonomous system, and the normal form theory can be used to obtain the associated normal form.

As an example, let $\omega = 1$, $\gamma = 2$; then, one has the normal form up to order 3 as follows:

$$\dot{z}_1 = \mathrm{i}z_1 + \tfrac{1}{8} d z_1^2 \bar{z}_1, \quad \dot{z}_2 = \mathrm{i}\gamma\, z_2 \tag{67}$$

and the normal form up to order 5 as

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}z_1 + \frac{\mathrm{i}}{32} a^2 z_1^2 \bar{z}_1 + \frac{19\mathrm{i}}{96} ac z_1^2 \bar{z}_1 - \frac{5}{192} ab z_1^2 \bar{z}_1 - \frac{\mathrm{i}}{48} c^2 z_1^2 \bar{z}_1 - \frac{5}{192} bc z_1^2 \bar{z}_1 \\ &\quad - \frac{3\mathrm{i}}{64} b^2 z_1^2 \bar{z}_1 - \frac{11\mathrm{i}}{256} d^2 z_1^3 \bar{z}^2 + \frac{1}{16} a^2 \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} + \frac{3\mathrm{i}}{64} a^2 \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3\mathrm{i}}{64} ac \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} \\ &\quad + \frac{9}{128} ab \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} - \frac{1}{16} ac \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3\mathrm{i}}{32} ab \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} + \frac{\mathrm{i}}{32} bc \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} \\ &\quad - \frac{3}{128} bc \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} - \frac{1}{32} b^2 \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t} - \frac{3\mathrm{i}}{128} b^2 \bar{z}_1^3 \mathrm{e}^{2\mathrm{i}\gamma t}. \end{aligned} \tag{68}$$

Comparing results (56) and (67) (both of order 3), it appears that they are completely different; equation (67) does not seem to contain enough terms. On the other hand, results (56) and (68) (of order 5) contain similar as well as different terms. It seems impossible to find a near-identity transformation to link results (56) and (68).

It is noted that normal forms are not unique. However, one normal form can be linked to another normal form by a *near-identity transformation*. In other words, the *near-identity* co-ordinate transformations are the basis of the normal form theory. If a "*simplified form*" cannot be linked to a normal form by certain near-identity transformations, this "*simplified form*" is not a normal form.

Example 2. Determine the normal form and the related coefficients of the following non-autonomous system with a damping term:

$$\dot{x} = y, \quad \dot{y} = -x + ax^3 \cos 3t - 2by. \tag{69}$$

In engineering systems, damping may be assumed to be small, i.e., $b = \varepsilon b$, where ε is a small parameter. In this case, suppose $\varepsilon = z$, then equation (69) can be transformed into

$$\dot{x} = y, \quad \dot{y} = -x + ax^3 \cos 3t - 2bzy, \quad \dot{z} = 0. \tag{70}$$

It is noted that equation (70) is an approximate expression for equation (69), and it is valid only if b is small. If $a = 0$, the normal form up to order 20 of equation (70) is given as follows:

$$\dot{r} = -\varepsilon br,$$

$$\dot{\theta} = 1 - \tfrac{1}{2}\varepsilon^2 b^2 - \tfrac{1}{8}\varepsilon^4 b^4 - \tfrac{1}{16}\varepsilon^6 b^6 - \tfrac{5}{128}\varepsilon^8 b^8 - \tfrac{7}{256}\varepsilon^{10} b^{10} - \tfrac{21}{1024}\varepsilon^{12} b^{12} - \tfrac{33}{2048}\varepsilon^{14} b^{14}$$
$$- \tfrac{429}{32768}\varepsilon^{16} b^{16} - \tfrac{715}{65536}\varepsilon^{18} b^{18} - \tfrac{2431}{262144}\varepsilon^{20} b^{20} + o(b^{22}). \tag{71}$$

The variable z is replaced by ε in the above result. This result is obtained in 0.5 s with the aid of MAPLE.

Normal form (71) is actually the Taylor expansion of solution $\ddot{x} + 2\varepsilon b\dot{x} + x = 0$ up to order 20. If $a \neq 0$ and $\varepsilon \in \Omega$, where Ω is the neighbourhood of the origin in C^n, the normal form of equation (70) is identical to normal form of equation (69).

In the case of $a \neq 0$, but still in the vicinity of the origin, introduce

$$x = \frac{1}{2}(z_1 + \bar{z}_1), \quad y = \frac{\mathrm{i}}{2}\omega(z_1 - \bar{z}_1) \quad \text{and} \quad z = z_2$$

into equation (70) to obtain

$$\dot{z}_1 = -\mathrm{i}z_1 - \tfrac{1}{16}\mathrm{i}a(z_1 + \bar{z}_1)^3(\mathrm{e}^{\mathrm{i}\gamma t} + \mathrm{e}^{-\mathrm{i}\gamma t}) - 2(z_1 - \bar{z}_1)z_2, \quad \dot{z}_2 = 0. \tag{72}$$

Following the procedure introduced in section 3, one obtains the normal form up to order 6 as follows:

$$\dot{z}_1 = \mathrm{i}z_1 + \varepsilon bz_1 - \frac{\mathrm{i}}{2}\varepsilon^2 b^2 z_1 - \frac{\mathrm{i}}{8}\varepsilon^4 b^4 z_1 - \frac{3\mathrm{i}}{128}a^2 \bar{z}_1^5 \mathrm{e}^{2\mathrm{i}\gamma t} + \frac{69\mathrm{i}}{1120}a^2 z_1^3 \bar{z}_1^2$$
$$- \frac{559}{9800}\varepsilon a^2 bz_1^3 \bar{z}_1^2 + \frac{9}{128}\varepsilon a^2 b\bar{z}_1^5 \mathrm{e}^{2\mathrm{i}\gamma t} \tag{73}$$

Here, variable z_2 has been replaced by ε. Suppose $\varepsilon b = b$ (back scaling), $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$, $\theta = t + \vartheta$; then, one has the normal form in polar form as follows:

$$\dot{r} = -rb + r^5(-\tfrac{3}{128}a^2 \sin 6\vartheta - \tfrac{559}{9800}a^2 b + \tfrac{9}{128}a^2 b \cos 6\vartheta),$$
$$\dot{\vartheta} = -\tfrac{1}{2}b^2 - \tfrac{1}{8}b^4 + r^4(\tfrac{69}{1120}a^2 - \tfrac{3}{128}a^2 \cos 6\vartheta - \tfrac{9}{128}a^2 b \sin 6\vartheta). \tag{74}$$

This result is obtained in 1 s with the aid of MAPLE.

Following the procedure discussed in Example 1, the "*simplified equation*" of (70) can be obtained by averaging method. The results obtained by both methods are identical (in this example) up to order 4, and appear to be different in higher orders. However, it can be shown that the complete results are linked by a near-identity transformation.

Example 3. Determine the normal form and the related coefficients of the following non-autonomous system:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x + ax^2 y \cos 3\omega t + bx^3. \tag{75}$$

Introducing the transformation

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{\mathrm{i}}{2}\,\omega(z - \bar{z})$$

into equation (75), one has

$$\dot{z} = \mathrm{i}\omega z + \frac{1}{16}a(z + \bar{z})^2(z - \bar{z})\,(\mathrm{e}^{2\mathrm{i}\omega} + \mathrm{e}^{-2\mathrm{i}\omega}) - \frac{1}{8\omega}\,\mathrm{i}b(z + \bar{z})^3. \tag{76}$$

Following the procedure introduced in section 3, and assuming that $\omega = 1$, the normal form (up to order 5) of equation (76) can be obtained as follows:

$$\dot{z} = \mathrm{i}z - \frac{3\mathrm{i}}{8}bz^2\bar{z} + \frac{\mathrm{i}}{64}a^2\bar{z}^5\mathrm{e}^{2\gamma t} - \frac{\mathrm{i}}{1120}a^2z^3\bar{z}^2 - \frac{51\mathrm{i}}{256}b^2z^3\bar{z}^2. \tag{77}$$

The normal form in polar form is given by

$$\begin{aligned} \dot{r} &= r^5\,\tfrac{1}{64}\,a^2\sin 6\vartheta, \\ \dot{\vartheta} &= -\tfrac{3}{8}\,br^2 - \tfrac{1}{1120}\,a^2r^4 - \tfrac{51}{256}\,b^2r^4 + \tfrac{1}{64}\,a^2r^4\cos 6\vartheta\,. \end{aligned} \tag{78}$$

With the aid of MAPLE, the above result is obtained in 0·32 s.

Following the procedure discussed in Example 1, the "*simplified equation*" of equation (76) can be obtained by averaging method. It is interesting to note that the results obtained by the methods of normal forms and averaging are identical up to order 8, and appear to be different after order 9. However, a further investigation shows that the complete results are linked together by a near-identity transformation of order 8.

Example 4. Determine the normal form and related coefficients of the following Mathieu–Duffing equation:

$$\dot{x} = y, \quad \dot{y} = -\,\omega^2x - ax\cos(\gamma t) - 2by - cx^3. \tag{79}$$

It is noted that the coefficients of the linear system of above equation are time dependent; thus, the direct application of normal form theory is not possible. Often, the coefficients *b*, *c* may be assumed to be small so that one can set

$$a = \varepsilon a, \quad b = \varepsilon b, \tag{80}$$

where ε is a small parameter.

Then, equation (79) takes the form

$$\dot{x} = y, \quad \dot{y} = -\,\omega^2x - \varepsilon ax\cos(\gamma t) - \varepsilon 2by - cx^3. \tag{81}$$

Let $\varepsilon \in \Omega$, where Ω is the neighbourhood of the origin in C^n; then, one can assume that

$$\varepsilon = z \tag{82}$$

and equation (81) takes the form

$$\dot{x} = y, \quad \dot{y} = -\,\omega^2x - ax\cos(\gamma t)z - 2byz - cx^3, \quad \dot{z} = 0. \tag{83}$$

Introducing the transformation

$$x = \frac{1}{2}(z_1 + \bar{z}_1), \quad y = \frac{\mathrm{i}}{2}\omega(z_1 - \bar{z}_1) \quad \text{and} \quad z = z_2$$

into equation (83), one has

$$\dot{z}_1 = i\omega z_1 + \frac{\mathrm{i}a}{4}(z_1 + \bar{z}_1)(\mathrm{e}^{\mathrm{i}\gamma t} + \mathrm{e}^{-\mathrm{i}\gamma t})z_2 - b(z_1 - \bar{z}_1)z_2 + \frac{\mathrm{i}c}{8}(z_1 + \bar{z}_1)^3, \quad \dot{z}_2 = 0. \qquad (84)$$

Let $\omega = 1$ and $\gamma = 2$; then, the normal form of equation (84) is given (up to order 4) by

$$\begin{aligned}\dot{z}_1 = \mathrm{i}z_1 - b\varepsilon z_1 + \frac{\mathrm{i}a\varepsilon}{4}\bar{z}\mathrm{e}^{\mathrm{i}\gamma t} + \frac{3\mathrm{i}c}{8}z_1^2\bar{z}_1 - \frac{\mathrm{i}}{64}a^2\varepsilon^2 z_1 - \frac{1}{4}ab\varepsilon^2\bar{z}_1\mathrm{e}^{\mathrm{i}\gamma t} - \frac{\mathrm{i}}{2}b^2\varepsilon^2 z_1 \\ - \frac{5\mathrm{i}}{128}ac\varepsilon z_1^3\mathrm{e}^{-\mathrm{i}\gamma t} - \frac{15\mathrm{i}}{128}ac\varepsilon z_1\bar{z}_1^2\mathrm{e}^{\mathrm{i}\gamma t} - \frac{7\mathrm{i}}{1024}a^3\varepsilon^3\bar{z}_1\mathrm{e}^{\mathrm{i}\gamma t},\end{aligned} \qquad (85)$$

where z_2 has been replaced by ε.

With the aid of MAPLE, this result can be obtained in 0.21 s (PC 266).

It is noted that $z_2(=\varepsilon)$ appears with coefficients a, b only. For convenience, suppose that $a\varepsilon = a$, $b\varepsilon = b$; then, equation (85) can be expressed as

$$\begin{aligned}\dot{z}_1 = iz_1 - bz_1 + \frac{\mathrm{i}a}{4}\bar{z}\mathrm{e}^{\mathrm{i}\gamma t} + \frac{3\mathrm{i}c}{8}z_1^2\bar{z}_1 - \frac{\mathrm{i}}{64}a^2 z_1 - \frac{1}{4}ab\bar{z}_1\mathrm{e}^{\mathrm{i}\gamma t} - \frac{\mathrm{i}}{2}b^2 z_1 \\ - \frac{5\mathrm{i}}{128}acz_1^3\mathrm{e}^{-\mathrm{i}\gamma t} - \frac{15\mathrm{i}}{128}acz_1\bar{z}_1^2\mathrm{e}^{\mathrm{i}\gamma t} - \frac{7\mathrm{i}}{1024}a^3\bar{z}_1\mathrm{e}^{\mathrm{i}\gamma t}\end{aligned} \qquad (86)$$

This is the normal form (up to order 4) of equation (83), and it is valid only when a and b are small.

Suppose $z_1 = r\mathrm{e}^{\mathrm{i}\theta}, \bar{z}_1 = r\mathrm{e}^{-\mathrm{i}\theta}$, $\theta = \frac{1}{2}\gamma t + \vartheta$; then, the polar normal form is given as follows:

$$\begin{aligned}\dot{r} &= -br - \tfrac{1}{4}abr\cos 2\vartheta + (-\tfrac{5}{64}acr^3 - \tfrac{7}{1024}a^3 r + \tfrac{1}{4}ar)\sin 2\vartheta, \\ \dot{\vartheta} &= \tfrac{3}{8}cr^2 - \tfrac{1}{64}a^2 - \tfrac{1}{2}b^2 + (-\tfrac{5}{32}acr^2 - \tfrac{7}{1024}a^3 + \tfrac{1}{4}a)\cos 2\vartheta + \tfrac{1}{4}ab\sin 2\vartheta.\end{aligned} \qquad (87)$$

Higher order normal forms can similarly be obtained. Clearly, the resonant monomials (when $\omega = 1$, $\gamma = 2$), defined by equation (11), can be obtained as follows:

$$G(z_1, \bar{z}_1) = z_1(\sum\alpha_j|z_1|^{2j}) + \bar{z}_1(\sum\beta_j|z_1|^{2j}) + \sum[z_1^{2s+1}(\sum\zeta_{sj}|z_1|^{2j}) + \bar{z}_1^{2s+1}(\sum\eta_{sj}|z_1|^{2j})], \quad (88)$$

where α_j, β_j, ζ_{sj}, η_{sj} are in terms of ε and e^{imt}.

Equation (88) is the formal normal form of equation (84), in which $\varepsilon = z_2$.

This system has been investigated by several authors (e.g., Bogoliubov and Mitropolsky [14], Nayfeh and Mook [15], and Chen and Langford [16]). The asymptotic analyses employed in references [14, 15] produce only lower order results. On the other hand, the analyses in reference [16] is based on a simplified equation, obtained by the L-S method, which is given by

$$G(z_1, \bar{z}_1) = z_1(\sum \alpha_j |z_1|^{2j}) + \bar{z}_1(\sum \beta_j |z_1|^{2j}) \tag{89}$$

where $\alpha, \beta \in C$, $j = 0, 1, \ldots$.

Terms in the simplified equation by the L–S method are defined by [16]

$$G(-z, -\bar{z}) = -G(z, \bar{z}).$$

A more complete simplified equation (by the L–S method) of equation (81) is given by

$$G(z, \bar{z}) = \sum_{m+n=2s-1} \zeta_{mn} z^m \bar{z}^n \tag{90}$$

which can be expressed as

$$G(z, \bar{z}) = z(\sum \alpha_j |z|^{2j}) + \bar{z}(\sum \beta_j |z|^{2j}) + \sum [z^{2s+1}(\sum \zeta_{sj} |z|^{2j}) + \bar{z}^{2s+1}(\sum \eta_{sj} |z|^{2j})]. \tag{91}$$

This result is identical to equation (88) obtained by the normal form theory.

It should be noted that the *forms* of the simplified equations obtained by different methods, such as, the methods of L–S, averaging and normal forms and the IHB technique, are the same. They consist of resonant monomials, defined by equation (11). However, the associated coefficients of a specific order, obtained by different approaches, may not be equivalent to each other [7, 13], because of the different procedures.

Here, it is evident that results (88) and (89) are not topologically equivalent in the neighbourhood of the origin in C^n, because they cannot be linked by near-identity transformations. The third order terms, for example, appearing in equation (91) are associated with the parametric excitation in equation (81) and they are fundamentally important in the analysis of the behaviour of the parametrically excited system.

Furthermore, equations (88) and (89) lead to quite different bifurcation equations obtained by $G(z_1, \bar{z}_1) = 0$. Let $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$, then, equation (89) leads to

$$G(z_1, \bar{z}_1) = r\mathrm{e}^{\mathrm{i}\theta}(\sum \alpha_j r^{2j}) + r\mathrm{e}^{-\mathrm{i}\theta}(\sum \beta_j r^{2j}) = 0. \tag{92}$$

The associated bifurcation equation can be obtained as

$$\sum (\alpha_j \bar{\alpha}_j - \beta_j \bar{\beta}_j)\, r^{2j} = 0 \tag{93}$$

which is relatively simple in form. The *simplified equation* of order 3 leads to

$$(\alpha_0 \bar{\alpha}_0 - \beta_0 \bar{\beta}_0) + (\alpha_1 \bar{\alpha}_1 - \beta_1 \bar{\beta}_1) r^2 + (\alpha_2 \bar{\alpha}_2 - \beta_2 \bar{\beta}_2) r^4 = 0. \tag{93*}$$

However, complete equation (88) leads to

$$G(z_1, \bar{z}_1) = r\mathrm{e}^{\mathrm{i}\theta}(\sum \alpha_j r^{2j}) + r\mathrm{e}^{-\mathrm{i}\theta}(\sum \beta_j r^{2j}) + \sum r^{2s+1}[\mathrm{e}^{(2s+1)\mathrm{i}\theta}(\sum \zeta_{sj} r^{2j}) + \mathrm{e}^{-(2s+1)\mathrm{i}\theta}(\sum \eta_{sj} r^{2j})]. \tag{94}$$

The associated bifurcation equation cannot be obtained in a general simple form like equation (93).

However, it can be determined for special cases. Consider, for example, the fourth order normal form given by equation (86) which can also be expressed as

$$\dot{r} + \mathrm{i}r\dot{\vartheta} = \left(\omega - \frac{1}{2}\gamma\right)\mathrm{i}r - br + \frac{\mathrm{i}a}{4}r\mathrm{e}^{-2\mathrm{i}\vartheta} + \frac{3\mathrm{i}c}{8}r^3 - \frac{\mathrm{i}}{64}a^2r - \frac{1}{4}abr\mathrm{e}^{-2\mathrm{i}\vartheta} - \frac{\mathrm{i}}{2}b^2r - \frac{5\mathrm{i}}{128}acr^3\mathrm{e}^{2\mathrm{i}\vartheta} - \frac{15\mathrm{i}}{128}acr^3\mathrm{e}^{-2\mathrm{i}\vartheta} - \frac{7\mathrm{i}}{1024}a^3r\mathrm{e}^{-2\mathrm{i}\vartheta}. \tag{95}$$

The steady states are given by

$$r(\alpha_0 + \alpha_1\mathrm{e}^{-2\mathrm{i}\vartheta} + \alpha_2\mathrm{e}^{2\mathrm{i}\vartheta}) = 0, \tag{96}$$

where

$$\alpha_0 = \left(\omega - \frac{1}{2}\gamma\right)\mathrm{i} - b + \mathrm{i}\left(\frac{3c}{8}r^2 - \frac{1}{2}b^2 - \frac{1}{64}a^2\right), \quad \alpha_1 = -\frac{1}{4}ab + \mathrm{i}\left(\frac{a}{4} - \frac{15}{128}acr^2 - \frac{7}{1024}a^3\right),$$

$$\alpha_2 = -\frac{5\mathrm{i}}{128}acr^2.$$

Solving the equations

$$\alpha_0 + \alpha_1\mathrm{e}^{-2\mathrm{i}\vartheta} + \alpha_2\mathrm{e}^{2\mathrm{i}\vartheta} = 0, \quad \bar{\alpha}_0 + \bar{\alpha}_1\mathrm{e}^{2\mathrm{i}\vartheta} + \bar{\alpha}_2\mathrm{e}^{-2\mathrm{i}\vartheta} = 0$$

for $\mathrm{e}^{2\mathrm{i}\vartheta}$ and $\mathrm{e}^{-2\mathrm{i}\vartheta}$, and eliminating ϑ in the solution by $\mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{-2\mathrm{i}\theta} = 1$, one has the bifurcation equation as follows:

$$(\alpha_0\bar{\alpha}_1 - \bar{\alpha}_0\alpha_2)(\bar{\alpha}_0\alpha_1 - \alpha_0\bar{\alpha}_2) = (\alpha_1\bar{\alpha}_1 - \alpha_2\bar{\alpha}_2)^2, \quad a \neq 0. \tag{97}$$

It is evident that the above equation is real although α_0, α_1, α_2 are complex. Thus, the response equation can be obtained from equation (97), which is in the form of

$$m_8r^8 + m_6r^6 + m_4r^4 + m_2r^2 + m_0 = 0, \tag{98}$$

where

$$m_0 = -\tfrac{125}{32}a^4 + \tfrac{57\,125}{131\,072}a^6b^2 - \tfrac{3875}{1024}a^4b^4 - \tfrac{42\,875}{8\,388\,608}a^8b^2 + \tfrac{6125}{524\,288}a^6b^4 + \tfrac{125}{2}a^2b^2 - \tfrac{2625}{256}a^4b^2 + \tfrac{625}{8}a^2b^4 + \tfrac{125}{8}a^2b^6 - \tfrac{9625}{524\,288}a^8 + \tfrac{177\,625}{536\,870\,912}a^{10} + \tfrac{3625}{8192}a^6 - \tfrac{300\,125}{137\,438\,953\,472}a^{12},$$

$$m_2 = -\tfrac{1625}{16}a^2b^2c + \tfrac{1125}{128}a^4b^2c - \tfrac{3375}{64}a^2b^4c + \tfrac{541\,625}{33\,554\,432}a^8c + \tfrac{3375}{512}a^4c - \tfrac{210\,875}{1\,048\,576}a^6b^2c + \tfrac{4375}{16\,384}a^4b^4c - \tfrac{643\,125}{4\,294\,967\,296}a^{10}c - \tfrac{37\,375}{65\,536}a^6c,$$

$$m_4 = \frac{1875}{128}a^2c^2\gamma - \frac{1875}{64}a^2c^2\omega - \frac{3125}{512}a^2b^2c^2\omega + \frac{3125}{1024}a^2b^2c^2\gamma + \frac{3125}{512}a^2c^2\omega^2$$

$$- \frac{625}{2048}a^4c^2\gamma - \frac{3125}{512}a^2c^2\omega\gamma + \frac{3125}{2048}a^2c^2\gamma^2 \frac{3125}{2048}a^2b^4c^2 + \frac{1125}{128}a^2c^2$$

$$+ \frac{2\,237\,625}{8\,388\,608}a^6c^2 + \frac{125}{2}a^2b^2c^2 - \frac{81\,625}{16\,384}a^4c^2 - \frac{1\,990\,625}{536\,870\,912}a^8c^2 - \frac{1875}{1024}a^4b^2c^2,$$

$$m_6 = \frac{9375}{2048}a^2c^3\omega - \frac{9375}{4096}a^2c^3\gamma + \frac{395\,625}{262\,144}a^4\mathrm{c}^3 - \frac{328\,125}{8\,388\,608}a^6\mathrm{c}^3 - \frac{5625}{1024}a^2c^3 - \frac{9375}{4096}a^2b^2c^3,$$

$$m_8 = \frac{28\,125}{32\,768}a^2c^4 - \frac{78\,125}{524\,288}a^4c^4.$$

It is clear that equation (93*) is a special case of the bifurcation (98). Higher order normal forms may lead to more complicated bifurcation equations.

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