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ANALYSIS OF NON-LINEAR OSCILLATORS HAVING NON-POLYNOMIAL ELASTIC TERMS

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The purpose of this letter is the study of the properties of the periodic solutions to non-linear oscillator equations for which the elastic restoring forces are non-polynomial functions of the displacement. In particular, this term is selected to be

$$f(x) = -x^{1/3}. \tag{1}$$

The work reported here extends that presented in reference [1] which was concerned with, first, showing that the equation

$$\ddot{x} + x^{1/3} = 0 \tag{2}$$

has all periodic solutions and, second, constructing analytical approximations to these solutions using the method of harmonic balance [2] in the lowest approximation.

To proceed, a second order harmonic balance solution will first be calculated for equation (4). This approximation to the solution takes the form [2]

$$x(t) \simeq A\cos(\omega t) + B\cos(3\omega t), \tag{3}$$

where the following initial conditions are used:

$$x(0) = x_0 = \text{given}, \qquad \dot{x}(0) = 0, \tag{4}$$

and the first and second derivatives are taken to be [2]

$$\dot{x}(t) \simeq -\omega A\sin(\omega t) - 3\omega B\sin(3\omega t), \tag{5a}$$

$$\ddot{x}(t) \simeq -\omega^2 A\sin(\omega t) - 9\omega^2 B\sin(3\omega t). \tag{5b}$$

The parameters (A, B, ω) are to be given in terms of x_0. The calculation begins by rewriting equation (2) as

$$(\ddot{x})^3 = -x. \tag{6}$$

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Substituting equation (3) into equation (6) gives

$$\left\{\omega^6\left[\left(\frac{3}{4}\right)A^3+\left(\frac{27}{4}\right)A^2B+\left(\frac{243}{2}\right)AB^2\right]-A\right\}\cos(\omega t)$$
$$+\left\{\omega^6\left[\frac{A^3}{4}+\left(\frac{2187}{4}\right)B^3+\left(\frac{27}{2}\right)A^2B\right]-B\right\}\cos(3\omega t) \quad (7)$$
$$+\text{(higher order harmonics)}=0.$$

Define $H_1(A, B, \omega)$ and $H_2(A, B, \omega)$ to be, respectively, the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$. The harmonic balance method requires

$$H_1(A, B, \omega)=0, \qquad H_2(A, B, \omega)=0. \quad (8)$$

Eliminating ω^6 in equations (8) gives, after some algebraic manipulation, the result

$$A^3+(51)A^2B-(27)AB^2+(1701)B^3=0. \quad (9)$$

Let $z \equiv B/A$, then this equation becomes

$$(1701)z^3-(27)z^2+(51)z+1=0. \quad (10)$$

The required solution in z is the one having the smallest absolute magnitude. An excellent approximation to this root is [2]

$$\bar{z}\simeq-\frac{1}{51}. \quad (11)$$

Replacing, in $H_1(A, B, \omega)$, B by $\bar{z}A$ and solving for ω gives

$$\omega=\frac{1}{[(\frac{3}{4})+(\frac{27}{4})\bar{z}+(\frac{243}{2})\bar{z}^2]^{1/6}A^{1/3}}. \quad (12)$$

It should be noted that the expression of equation (4) automatically satisfies $\dot{x}(0)=0$. The other initial condition, $x(0)=x_0$, yields the result

$$A(1+\bar{z})=x_0 \quad \text{or} \quad A=\frac{x_0}{1+\bar{z}}. \quad (13)$$

Consequently, the second order harmonic balance approximation to the periodic solution of equation (2) is

$$x(t)\simeq\left(\frac{x_0}{1+\bar{z}}\right)[\cos(\omega t)+\bar{z}\cos(3\omega t)], \quad (14)$$

where

$$\omega(x_0)=\frac{1}{[(\frac{3}{4})+(\frac{27}{4})\bar{z}+(\frac{243}{2})\bar{z}^2]^{1/6}}\left(\frac{1+\bar{z}}{x_0}\right)^{1/3}. \quad (15)$$

Comparing equations (14) and (15) with equations (17) and (18) of reference [1], it is clearly seen that the second harmonic balance approximation only provides small corrections to the periodic solution obtained in the first approximation. This is the expected result.

The second equation to be studied is a modified version of the van der Pol equation, i.e.,

$$\ddot{x} + x^{1/3} = \varepsilon(1 - x^2)\dot{x}, \quad \varepsilon > 0. \tag{16}$$

Written in the form of two first order differential equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x^{1/3} + \varepsilon(1 - x^2)y, \tag{17}$$

it follows that only one fixed-point exists and it is located in the (x, y) phase-plane at (0, 0). The following argument shows that the fixed-point is unstable. Consider the following function of x and y,

$$V(x, y) = ay^2 + bx^2 + cx^{4/3}, \tag{18}$$

where $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$. Taking the derivative with respect to time gives

$$\frac{dV}{dt} = 2ay\frac{dy}{dt} + 2bx\frac{dx}{dt} + \left(\frac{4c}{3}\right)x^{1/3}\frac{dx}{dt}. \tag{19}$$

Using equation (17) in equation (19) yields

$$\frac{dV}{dt} = yx^{1/3}\left(\frac{4c}{3} - 2a\right) + 2a\varepsilon y^2 + 2bxy. \tag{20}$$

By selecting

$$a = \frac{1}{2}, \quad b = 0, \quad c = \frac{3}{4}, \tag{21}$$

it follows that

$$V(x, y) = \left(\frac{1}{2}\right)y^2 + \left(\frac{3}{4}\right)x^{4/3}, \qquad \frac{dV}{dt} = \varepsilon y^2 \geqslant 0. \tag{22a, b}$$

Consequently, the fixed-point, (0, 0), is unstable [3]. Inspection of equation (16) shows that it is of the form for which the Lienard–Levinson–Smith theorem can be applied [4]. Therefore, it can be concluded that equation (16) has a unique, stable limit-cycle toward which all other trajectories asymptotically approach as $t \to \infty$.

The method of harmonic balance can again be used to calculate an analytical approximation to the limit-cycle of equation (16). In order to do so, this equation requires to be rewritten in the form:

$$[-\ddot{x} + \varepsilon(1 - x^2)\dot{x}]^3 = x. \tag{23}$$

For the first approximation,

$$x(t) \simeq A\cos\omega t. \tag{24}$$

Substitution of equation (23) into equation (22) and assuming that $0 < \varepsilon \ll 1$, the following result is obtained:

$$H_1(A, \omega, \varepsilon)\cos\omega t + H_2(A, \omega, \varepsilon)\sin\omega t + (\text{higher order harmonics}) = 0, \tag{25}$$

where

$$H_1(A, \omega, \varepsilon) = \frac{3\omega^6 A^3}{4} - A, \qquad H_2(A, \omega, \varepsilon) = \left(\frac{3\omega^5 \varepsilon A^3}{4}\right)\left(1 - \frac{A^2}{2}\right). \tag{26}$$

Setting $H_1 = 0$ and $H_2 = 0$, first gives $A = 0$ corresponding to the unstable fixed-point at $(\bar{x}, \bar{y}) = (0, 0)$. A second solution is

$$A = \sqrt{2}, \qquad \omega = \left(\frac{4}{6}\right)^{1/6}, \tag{27}$$

which gives

$$x(t) \simeq \sqrt{2}\cos\left[\left(\frac{4}{6}\right)^{1/6} t\right], \tag{28}$$

as the first harmonic balance approximation to the limit-cycle solution to equation (16).

In summary, a second order harmonic balance solution has been calculated to a non-linear conservative oscillator having a fractional power force, i.e., equation (2). A study was also made of a related Van der Pol-type non-linear oscillator, equation (16). After showing that a unique, stable limit cycle exists, harmonic balance was used to determine an approximation to the limit cycle.

Currently, the following third order equation is being investigated:

$$\dddot{x} + \dot{x} + \varepsilon(x^{1/3} - x^3) = 0, \quad \varepsilon > 0. \tag{29}$$

The form of this equation is motivated by a "similar" third order differential equation [5]

$$\dddot{x} + \dot{x} + \varepsilon(x - x^3) = 0, \quad \varepsilon > 0, \tag{30}$$

which models oscillations of a certain class of stars. It is known that equation (30) has three fixed-points and a stable small amplitude limit cycle. The issue to be studied is whether the solutions of equation (29) has these properties.

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