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EXISTENCE OF NATURAL FREQUENCIES OF SYSTEMS WITH ARTIFICIAL RESTRAINTS AND THEIR CONVERGENCE IN ASYMPTOTIC MODELLING

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A major limitation of the Rayleigh–Ritz method for determining the natural frequencies of a system is the need to choose admissible functions that do not violate the geometric constraints of that system (Courant 1943 *Bulletin of the American Mathematical Society* **49**, 1–23). Several researchers have attempted to overcome this problem by asymptotically modelling the rigid constraints with artificial (imaginary) restraints of very large stiffness (Courant 1943 *Bulletin of the American Mathematical Society* **49**, 1–23; Warburton and Edney 1984 *Journal of Sound and Vibration* **95**, 537–552; Gorman 1989 *Journal of Applied Mechanics* **56**, 893–899; Kim *et al*. 1990 *Journal of Sound and Vibration* **143**, 379–394; Yuan and Dickinson 1992 *Journal of Sound and Vibration* **153**, 203–216; Yuan and Dickinson 1992 *Journal of Sound and Vibration* **159**, 39–55; Cheng and Nicolas 1992 *Journal of Sound and Vibration* **155**, 231–247; Yuan and Dickinson 1994 *Computers and Structures* **53**, 327–334; Lee and Ng 1994 *Applied Acoustics* **42**, 151–163; Amabili and Garziera 1999 *Journal of Sound and Vibration* **224**, 519–539; Amabili and Garziera 2000 *Journal of Fluids and Structures* **14**, 669–690). While the numerical results thus obtained for the systems considered in the literature were in close agreement with exact values for the natural frequencies corresponding to the first few modes, sample calculations show that the error introduced by the asymptotic modelling increases with mode number and therefore to obtain accurate results for higher modes the magnitude of stiffness should also be increased. In any event, the error due to the asymptotic modelling would remain uncertain, except when the correct frequency values are known. However, the use of artificial restraints with negative stiffness, a new concept which was introduced in a recent publication (Ilanko and Dickinson 1999 *Journal of Sound and Vibration* **219**, 370–378) paves the way for estimating the error due to asymptotic modelling. This is possible since in this work, the Rayleigh–Ritz frequencies of the constrained system were found to be bracketed by the frequencies of the asymptotic models with positive and negative restraints. However, the use of artificial restraints with negative stiffness has raised some important questions: would a system with a large negative restraint become unstable, and if so what is the guarantee that the frequencies of the asymptotic model would converge to that of the constrained system? This paper is the result of the author's attempt to answer these questions and gives a proof of existence of natural frequencies for systems with artificial restraints (springs) having positive or negative stiffness coefficients, and their convergence towards constrained systems. Based on Rayleigh's theorem of separation, it has been shown that a vibratory system obtained by the addition of h restraints to an n-degree-of-freedom (d.o.f.) system, where $h < n$, will have at least $(n - h)$ natural frequencies and modes and that as the magnitude of the stiffness of the added restraints becomes very large, these $(n - h)$ natural frequencies will converge to the $(n - h)$ natural frequencies of a constrained system in which the displacements restrained by the springs are effectively constrained.

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1. INTRODUCTION

The Rayleigh–Ritz method is a well-known technique for finding upper-bound estimates of natural frequencies. In his address to the American Mathematical Society [1], Courant describes the difficulty in choosing "the co-ordinate functions" as the "more annoying" of the two "objections" to the Rayleigh–Ritz method, the less annoying one being "the vagueness as to the accuracy of the approximation". This difficulty arises because of the need to satisfy the geometric constraints of a system. In the same reference, Courant suggests the use of artificial stiffness parameters having very large magnitude so that a rigid constraint could be approximately modelled by a restraint. The effectiveness of this approach has since been studied by several researchers for various interesting problems and its applicability has also been extended to analyze rigidly connected systems and systems with cracks [2–11]. The natural frequencies calculated using this asymptotic modelling approach agreed well with benchmark values for the systems considered in references [2–9]. In the literature, the magnitude of non-dimensional stiffness parameters used varies widely. For example, in reference [4], non-dimensional stiffness parameters with magnitudes of the order of 10^6 were found to give results that were within 1% of exact results for the first four natural frequencies of systems comprised of straight and curved beams, but higher non-dimensional stiffness parameters (of the order of 10^9) have been used in other publications [11] in determining up to the first eight natural frequencies of constrained shells, where the magnitude to be used appears to have been determined by a trial and error procedure until numerical convergency was observed. In a numerical experiment carried out by the author, where the first five natural frequencies of a propped cantilever was calculated by modelling it by a spring restrained cantilever, it was observed that the spring stiffness required to obtain convergence increased with the mode number. In general, the stiffness required to suppress the motion of a point may depend on factors such as the proximity of the point to be constrained to a node.

It is worth noting that the problem of lack of knowledge on suitable stiffness values was recognized by Courant who states in reference [1] that it is "worthwhile to study the preferable choices of these artificial parameters" because they must be "large enough to approximate rigidity but small enough to keep the necessary labour within reasonable bounds". In the articles cited here [2–9], calculations were limited to the first few modes and the magnitude of stiffness of restraints required to obtain accurate results was known only because the exact solution to the problems was known. Until recently, there were no techniques available to determine the error due to asymptotic modelling without calculating the frequencies using an alternative method, or by checking the convergence through numerical experimentation. However, the introduction of the use of artificial springs with negative stiffness in a recent article [12] offers a method for calculating the error bounds due to asymptotic modelling. In this article, the stiffness of the restraints was permitted to be of both positive and negative values, the latter representing a new approach. It was shown numerically that the natural frequencies obtained for a given system (by using an exact solution or a Rayleigh–Ritz solution with a fixed number of terms in the displacement series) converged towards those of the corresponding rigidly constrained or connected system from below as the magnitude of the positive artificial spring stiffness was increased, and from above as the absolute value of the negative spring stiffness was increased. These characteristics permit the very useful and simple estimation of the bounds on the error introduced by the asymptotic modelling procedure.

Since the publication of reference [12], the author has participated in oral and e-mail discussions with several interested researchers concerning the existence and convergence of the natural frequencies for a system when springs of negative stiffness are introduced. While the use of restraints with positive stiffness appears to have been generally accepted, with the use of negative stiffness, the possibility of a system becoming unstable has raised the questions, whether or not the introduction of the large negative stiffness could result in instability, and if so which frequencies would vanish. These questions have prompted the author to do further research, the results of which are presented as two theorems, namely the proof of existence of natural frequencies of negatively restrained systems and their convergence towards the frequencies of the corresponding fully constrained system. Before the analytical derivations, an explanation of how the asymptotic modelling works is given.

When using springs with positive stiffness approaching but less than plus infinity to model rigid constraints in a discrete system, the number of degrees of freedom (d.o.f.) of the unconstrained system is effectively decreased by the number of constraints so modelled; and the frequencies associated with the same number of modes approach infinity. Each time a restraint with very high stiffness is added to a system, the highest natural frequency of that system would approach infinity. For a closed-form or infinite series solution of the equations governing the motion of continuous systems, there are no finite highest natural frequencies and hence this is of no consequence. However, for a truncated series solution, all frequencies of the system will be finite and a number of the frequencies, equal to the number of constraints being modelled by the springs, will approach infinity. This is of little consequence practically, since the truncated series solution will normally only be accurate for the lower modes of vibration, and with the exception of systems where coincident roots occur, it is the frequencies corresponding to highest modes that become infinite.

When springs of negative stiffness are used to model constraints, as the absolute values of their stiffness are increased toward infinity, again the number of d.o.f. of the system is reduced by the number of springs introduced. This time, however, some of the frequencies become imaginary as the spring stiffnesses reach certain values and the square of each of these frequencies (proportional to the eigenvalues of the problem) becomes negative infinity as the corresponding stiffness tends towards minus infinity. With the addition of each negative restraint, the lowest natural frequency would become imaginary indicating a state of instability.

In both of the above cases, some of the finite, real frequencies will converge towards a natural frequency of the corresponding rigidly constrained system as the absolute value of the stiffness of each artificial spring tends towards infinity. (Here, the "corresponding rigidly constrained system" would be the true system if an exact solution were being used or a system represented by a truncated series solution if such a solution were being used. The effect of any truncation is not modified.) The number of frequencies that asymptotically approach those of the corresponding constrained system is equal to the number of d.o.f. of the unconstrained system minus the number of restraints introduced.

The validity of the above statements can be inferred from the excellent and famous work of Lord Rayleigh [13], in which he describes several theorems and proofs dealing with the convergence of the periods of free vibration as the potential or kinetic energies of a system are increased or diminished in some manner. However, he does not specifically refer to the use of springs of either positive or negative stiffness for achieving the changes in the potential energies. In the opinion of the author, it is not easy to derive concise proofs of the existence and convergence of the natural frequencies of asymptotically modelled systems directly from Rayleigh's arguments. Hence, the author has derived the alternative proofs presented here. A numerical example illustrating the behaviour of a discrete system with the addition of springs of positive and negative stiffness is also presented.

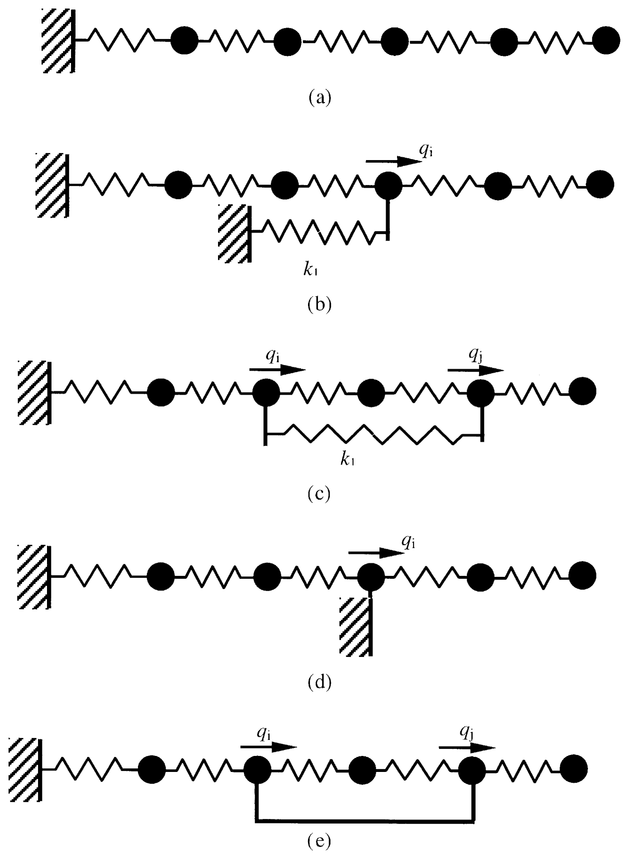

Figure 1. (a) System A; (b) System $A_1(q_i$ restrained); (c) an alternative System A_1 ($q_i - q_j$ restrained); (d) system $\tilde{A}_1$ (q_1 constrained); (e) an alternative system $\tilde{A}_1$ ($q_i - q_i$ constrained).

2. THEORETICAL DERIVATIONS

2.1. THE NATURAL FREQUENCIES OF AN n-d.o.f. SYSTEM WITH ONE ADDITIONAL POSITIVE OR NEGATIVE RESTRAINT

Consider an n-d.o.f. system. Let us refer to this as the "original system" or system A. Figure 1a shows a typical spring–mass system, which may be used as an illustration. If we now add to this system, a spring having a stiffness k_1, the total potential energy of the system will have an additional term

$$\Delta V = \tfrac{1}{2} k_i q_i^2, \tag{1a}$$

where q_i is the displacement that is being restrained by the spring (see Figure 1(b)). In a case, where the spring restrains the relative displacement of two co-ordinates q_i and q_j, as shown in Figure 1(c), the additional potential energy term will take the form

$$\Delta V = \tfrac{1}{2} k_1 (q_i - q_j)^2. \tag{1b}$$

Let us refer to the new system, which has been obtained by modifying system A by adding one spring, as system A_1.

For positive stiffness values, increasing the magnitude of stiffness cannot decrease the Rayleigh Quotient, and hence the natural frequencies of the modified system (A_1) will be upper bounds to the natural frequencies of the original system (A) [13]. Denoting the mth natural frequencies of systems A and A_1 by ω_m and $\omega_{m,1}$, respectively, the above statement may be expressed as

$$\omega_m \leqslant \omega_{m,1} \quad \text{for } k_1 > 0. \tag{2}$$

The natural frequencies may be expressed in terms of kinetic and potential energies as follows:

$$(\omega_m)^2 = \min\left(\frac{V}{\psi}\right) \quad \text{subject to } \bar{m}.u \cdot u_i = 0 \ \text{ for } i = 1, 2, \dots, (m-1), \tag{3a}$$

$$(\omega_{m,1})^2 = \min\left(\frac{V + \Delta V}{\psi}\right) \quad \text{subject to } \bar{m}.u \cdot u_i = 0 \ \text{ for } i = 1, 2, \dots, (m-1), \tag{3b}$$

where V is the potential energy of system A, and ψ is a kinetic energy function for systems A or A_1. The kinetic energy function consists of terms such as $\frac{1}{2}\bar{m}_i q_i^2$ and is of the same form for both systems, although its actual value would depend on the displacement form u that is used in the analysis. The statement $\bar{m}.u \cdot u_i = 0$, for $i = 1, 2, \dots, (m-1)$ refers to the conditions of orthogonality of the displacement form with respect to the first $(m-1)$ modes.

The highest natural frequency of System A_1 ($\omega_{n,1}$) will increase monotonically with stiffness k_1 and, as

$$k_1 \to \infty, \qquad \omega_{n,1} \to \infty. \tag{4}$$

However, this does not happen with other natural frequencies for the following reasons.

As the stiffness approaches infinity, equation (3b) can be satisfied in either of the two ways. One possibility is where q_i or $(q_i - q_j)$ remains finite while the change in potential energy ΔV, and hence the right-hand side of equation (3b), approach infinity thus making the frequency approach infinity. The alternative is that ΔV takes a definite value as a result of the displacement q_i or the relative displacement $(q_i - q_j)$ becoming zero. In this case, finite values for natural frequencies are obtainable. This case is of more interest, because it represents a system where a displacement of the point that is attached to the spring (or the relative displacement between two points that are attached to the spring) is effectively constrained, and this fact has been used by several researchers who have used asymptotic models with springs of very large stiffness to obtain estimates of natural frequencies of constrained systems using the Rayleigh–Ritz procedure [1–12]. It will be shown that this convergence to the constrained system occurs for $(n-1)$ modes of a system with one added restraint.

Considering the limiting case where k_1 approaches infinity, the constrained system that is being approached will have $n-1$ d.o.f., and hence $n-1$ natural modes and frequencies. Let us refer to this system, which is obtained from the original system by adding a constraint corresponding to the spring restraint, as $\tilde{A}_1$. A schematic representation of the constrained systems corresponding to Figure 1(b) and 1(c) are shown in Figure 1(d) and 1(e) respectively. We may consider the asymptotic model A_1 as slightly less stiff compared with the constrained system $\tilde{A}_1$. Therefore, all $n-1$ natural frequencies of the asymptotic model will be equal to, or slightly lower than, the corresponding frequencies of the constrained system. Denoting the mth natural frequency of system $\tilde{A}_1$ by $\tilde{\omega}_{m,1}$, we may state this as

$$\omega_{m,1} \leqslant \tilde{\omega}_{m,1} \quad \text{for } m < n \quad \text{and} \quad k_1 > 0. \tag{5}$$

Rayleigh has described the above inequality and its potential application in modelling constrained systems in Articles 88 and 92(a) of reference [13] and the explanation given above is similar to that in reference [13]. He has also stated in Article 88 that not only restraints but also masses of very large magnitude could be used to model a constraint, and has illustrated it through an example of a 2-d.o.f. system in Article 115.

Equation (5) may also be obtained analytically by applying Rayleigh's theorem of separation [13, 14]. According to this theorem, the natural frequencies of a constrained system are separated by the natural frequencies of the original system. Lord Rayleigh's proof of the theorem of separation is based on Lagrange's equations of motion and uses added restraints (alternatively the use of added masses or dashpots are also mentioned) which when given infinite values result in constrained systems. An alternative proof of this theorem, which the author finds simpler (it does not use restraints), is found in reference [14] (pp. 35–39). Applying Rayleigh's theorem of separation to system A, we get,

$$\omega_m \leqslant \tilde{\omega}_{m,1} \leqslant \omega_{m+1} \quad \text{for } 1 \leqslant m < n. \tag{6}$$

This is illustrated in Figure 2(a), where the solid lines correspond to the natural frequencies of the original system (A) and the dotted lines show the natural frequencies of system $\tilde{A}_1$. It should be noted here that while the inequality in equation (6) is applicable for any constraint, we are only interested in the particular case where the constraint corresponds to the restraint in system A_1.

We can also apply Rayleigh's theorem of separation to system A_1. This would give

$$\omega_{m,1} \leqslant \tilde{\omega}^*_{m,1} \leqslant \omega_{m+1,1} \quad \text{for } 1 \leqslant m < n, \tag{7a}$$

where $\tilde{\omega}^*_{m,1}$ is the mth natural frequency of system A_1 subject to *any* additional constraint. If, however, the additional constraint corresponds to the restraint (spring) in A_1, then for this particular constrained system (we will refer to this as "the appropriately constrained system" in this part of the derivations but subsequently it will be assumed that the constraints would correspond to the restraints), the restraint (spring) is redundant, as the associated displacement is zero. Thus, if the system is appropriately constrained, the resulting value of the mth natural frequency $\tilde{\omega}_{m,1}$ is a special case of $\tilde{\omega}^*_{m,1}$, i.e.,

$$\tilde{\omega}_{m,1} \in \tilde{\omega}^*_{m,1}. \tag{7b}$$

Therefore, from equations (7a) and (7b) it is clear that

$$\omega_{m,1} \leqslant \tilde{\omega}_{m,1} \leqslant \omega_{m+1,1} \quad \text{for } 1 \leqslant m < n. \tag{7c}$$

It should be noted here that equations (7a)–(7c) are valid for both positive and negative values of stiffness as Rayleigh's theorem of separation is applicable to any system. The left-hand side inequality of equation (7c) confirms equation (5).

Combining equations (2) and (5) gives

$$\omega_m \leqslant \omega_{m,1} \leqslant \tilde{\omega}_{m,1} \quad \text{for } 1 \leqslant m < n \quad \text{and} \quad k_1 > 0. \tag{8}$$

Since there are $(n-1)$ natural frequencies for system $\tilde{A}_1$, $(n-1)$ natural frequencies of the spring-restrained system A_1 will be bounded by the natural frequencies of the original system and the constrained system as given by equation (8). This is to say that no matter how large the magnitude of stiffness k_1 is, for system A_1 there exist at least $(n-1)$ natural frequencies.

This also means, from equation (3b), that for the first $(n-1)$ modes, as the magnitude of the stiffness parameter k_1 approaches infinity, $(\Delta V/\psi)$ has a limit. From this and equations

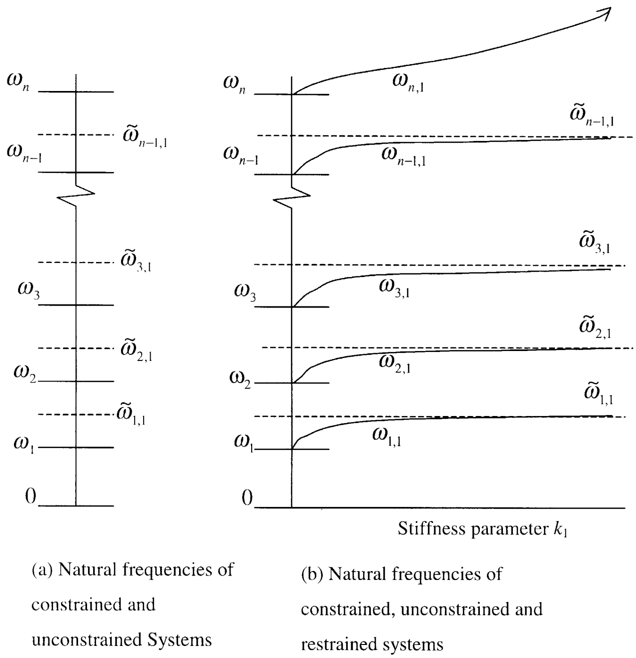

Figure 2. (a, b) Sketch of the variation of natural frequencies with stiffness.

(1a) or (1b), we may deduce that for the first $(n-1)$ modes, as $k_1 \to \infty$, either

$$q_i \to 0, \tag{9a}$$

or

$$(q_i - q_j) \to 0 \tag{9b}$$

depending on whether the spring restrains a single displacement or a relative displacement between two points, as otherwise $(\Delta V/\psi) \to \infty$ which is only true for the nth mode. This means for the first $(n-1)$ modes, as

$$k_1 \to \infty, \qquad A_1 \to \tilde{A}_1. \tag{9c}$$

Using the right-hand side inequality of equation (8) and the fact that $\omega_{m,1}$ cannot decrease with increasing k_1, we can state that, as

$$k_1 \to \infty, \qquad \omega_{m,1} \to \tilde{\omega}_{m,1} \quad \text{for } 1 \leqslant m < n. \tag{10}$$

For very high values of stiffness, these natural frequencies approach that of the constrained system as given by equation (10). However, the highest natural frequency of system A_1 has no such bounds, and increases monotonically with stiffness as given by equation (4). A sketch of the variation of the natural frequencies of system A_1 with the stiffness parameter k_1 is shown in Figure 2(b).

Let us now consider the case of adding a restraint (for example, a spring) having a negative stiffness. If, for system A_1, the stiffness parameter k_1 were assigned a negative value, then the resulting change in the potential energy would be negative. This means that the natural frequencies of system A_1 would be less than or equal to the corresponding frequencies of the original system.

If $\omega_{m,1}$, exists,

$$\omega_{m,1} \leqslant \omega_m \quad \text{for } k_1 < 0 \quad \text{and} \quad m \leqslant n. \tag{11}$$

Since a decrease in stiffness cannot cause an increase in natural frequencies, if $\omega_{m+1,1}$ exists

$$\omega_{m+1,1} \leqslant \omega_{m+1} \quad \text{for } 1 \leqslant m < n \quad \text{and} \quad \text{for } k_1 < 0. \tag{12a}$$

Combining equation (12a) and the right-hand side inequality in equation (7c) gives

$$\tilde{\omega}_{m,1} \leqslant \omega_{m+1,1} \leqslant \omega_{m+1} \quad \text{for } 1 \leqslant m < n \quad \text{and} \quad k_1 < 0. \tag{12b}$$

From the above equation, it is clear that since there are $(n-1)$ natural frequencies $(\tilde{\omega}_{m,1})$ for the constrained system, there will also exist at least $(n-1)$ natural frequencies for the system with one negative spring. Similar to the arguments used for the positive stiffness case, for at least $(n-1)$ modes, as $k_1 \to -\infty$,

$$\text{either } q_i \to 0, \tag{13a}$$

or

$$(q_i - q_j) \to 0. \tag{13b}$$

This means for $(n-1)$ modes, as

$$k_1 \to -\infty, \qquad A_1 \to \tilde{A}_1. \tag{13c}$$

Since the natural frequencies cannot increase with any decrease in stiffness, using equations (12b) and (13c), as

$$k_1 \to -\infty, \qquad \omega_{m+1,1} \to \tilde{\omega}_{m,1} \quad \text{for } 1 \leqslant m < n. \tag{14}$$

Note that the lowest natural frequency of the constrained system occurs when $m = 1$, and this is approached by the second mode of the system with a spring having negative stiffness. The first natural frequency of the spring-restrained system would vanish if the magnitude of the negative stiffness were sufficiently large. These results may be summarized by the following theorems.

Theorem 1(a). *The addition of one restraint with positive or negative stiffness to an n-d.o.f. system where* $n > 1$, *results in a system, for which there exist at least* $(n-1)$ *natural modes and frequencies.*

Theorem 1(b). *As the magnitude of the stiffness parameter approaches infinity, the natural frequencies and modes of the modified restrained system would asymptotically approach those of an* $(n-1)$*-d.o.f. system, which is obtained from the original system by the addition of an appropriate constraint.*

2.2. THE NATURAL FREQUENCIES OF AN n-d.o.f. SYSTEM WITH h ADDITIONAL POSITIVE OR NEGATIVE RESTRAINTS

We can now seek to generalize the above theorems for a system with h additional restraints whose stiffnesses may be negative or positive. Since the proof is by induction, it is necessary to state the general theorems first.

Theorem 2(a). *If h restraints of positive or negative stiffness are added to an n-d.o.f. system* (A) *where* $h < n$, *then for the resulting system* (A_h), *there exist at least* $(n-h)$ *natural frequencies and modes.*

Theorem 2(b). *Furthermore, as the h stiffness parameters approach infinity, the* $(n-h)$ *natural frequencies and modes of system* A_h *would asymptotically approach those of the n-d.o.f. system subject to h constraints* $(\tilde{A}_h)$.

2.2.1. *Proof by mathematical induction*

Let us denote the n-d.o.f. system subject to r restraints (where $n > r > 1$) having positive or negative stiffness values by A_r.

If Theorems 2(a) and (b) are true for A_r then:

There will exist $(n-r)$ natural frequencies and modes for A_r; Statement 1(a)

and

As the r stiffness parameters approach infinity, the natural frequencies and modes of system A_r would asymptotically approach that of the n-d.o.f. system subject to r constraints $(\tilde{A}_r)$. Statement 1(b)

Applying Theorem 1(a) to system A_r, and using Statement 1(a), we can state that adding one more spring restraint to A_r will result in a new system A_{r+1} for which there exist $(n-r-1)$ natural frequencies and modes. i.e.,

Theorem 2(a) is true for $h = r+1$ if it is true for $h = r$. Statement 2(a)

Applying Theorem 1(b) to $\tilde{A}_r$ with an extra restraint, we can state that as the magnitude of the stiffness of the $(r+1)$th restraint (newly added) approaches infinity, the resulting system frequencies and modes would approach that of $\tilde{A}_{r+1}$. From Statement 1(b), as the stiffness parameters for the r restraints approach infinity, the frequencies and modes of A_r would approach those of $\tilde{A}_r$. Therefore, if the magnitude of stiffness of all $r+1$ restraints were to approach infinity, the natural frequencies and modes of A_{r+1} would asymptotically approach that of the n-d.o.f. system subject to $r+1$ constraints $(\tilde{A}_{r+1})$. i.e.,

Theorem 2(b) is true for $h = r+1$, if it is true for $h = r$. Statement 2(b)

From Theorems 1(a) and (b), Theorems 2(a) and (b) are true for $h = 1$. From Statements 2(a) and (b) they are true for $h = r+1$ if they are true for $h = r$.

Hence by induction, Theorems 2(a) and (b) are true for any $h > 0$.

2.2.2. *Applicability of Theorems* 1 *and* 2 *for continuous systems*

The above arguments hold for continuous systems, with the exception of any reference to the highest mode and the highest mode number n. For continuous systems, since a highest mode does not exist, the condition that $m < n$, does not apply, and when using positive stiffness values for additional springs, all natural frequencies and modes will be bounded on both sides by the natural frequencies of the corresponding constrained systems. This means equation (4) does not hold as n (highest mode number) does not exist, and equations (5)–(8) are true for all $m > 0$.

However, when finding the natural frequencies and modes using the Rayleigh–Ritz procedure, continuous systems are effectively discretized and the highest mode number n corresponds to the number of terms used in the Rayleigh–Ritz formulation for the displacement.

3. ILLUSTRATIVE EXAMPLE

Let us now consider a 5 d.o.f. linear spring–mass system to illustrate the above derivations. The system is subject to two additional spring elements having stiffness values of k_1 and k_2 as shown in Figure 3(a). Let us refer to this system as system B_2. The magnitudes of the stiffness of other springs are denoted by $\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4$, and $\bar{k}_5$. Let the displacements of the five masses $\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{m}_4$, and $\bar{m}_5$ be q_1, q_2, q, q_4, and q_5 respectively. Of the additional restraints, the spring having stiffness k_1 restrains the displacement of the second mass (q_2) while the spring with the stiffness k_2 restrains the relative displacement between masses 3 and 5 (i.e. (q_3–q_5)).

The eigenvalue equation for the system is

$$[K]\{q\} - \omega^2[M]\{q\} = \{0\}, \tag{15}$$

where the stiffness and mass matrices are

$$[K] = \begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 & 0 & 0 & 0 \\ -\bar{k}_2 & \bar{k}_2 + \bar{k}_3 + k_1 & -\bar{k}_3 & 0 & 0 \\ 0 & -\bar{k}_3 & \bar{k}_3 + \bar{k}_4 + k_2 & -\bar{k}_4 & -k_2 \\ 0 & 0 & -\bar{k}_4 & \bar{k}_4 + \bar{k}_5 & -\bar{k}_5 \\ 0 & 0 & -k_2 & -\bar{k}_5 & \bar{k}_5 + k_2 \end{bmatrix}, \tag{16a}$$

$$[M] = \begin{bmatrix} \bar{m}_1 & 0 & 0 & 0 & 0 \\ 0 & \bar{m}_2 & 0 & 0 & 0 \\ 0 & 0 & \bar{m}_3 & 0 & 0 \\ 0 & 0 & 0 & \bar{m}_4 & 0 \\ 0 & 0 & 0 & 0 & \bar{m}_5 \end{bmatrix}. \tag{16b}$$

These equations were solved for various combinations of stiffness and mass parameters, and in all cases the results confirmed the existence of natural frequencies and modes and their convergence as predicted by Theorems 1 and 2. Only some sample results are presented here, for the case of, $\bar{k}_i = 100$ N/m and $\bar{m}_i = 0{\cdot}1$ kg for $i = 1, 2, \ldots, 5$, and for various values for k_1 and k_2 in the range -10^9 to $+10^9$ N/m. The results are shown graphically in Figures 4 and 5, and numerically in Tables 1 and 2.

3.1. RESULTS FOR ONE ADDITIONAL CONSTRAINT

Figure 4 and Table 1 show the variation of natural frequencies with k_1, for $k_2 = 0$. It may be seen that for $k_1 < -50$ N/m, there exist only four natural frequencies and modes, and as k_1 takes very large negative values, all of them converge to the natural frequencies of the 5-d.o.f system subject to the constraint $q_2 = 0$ (say system $\tilde{B}_1$, which is actually a 4-d.o.f.

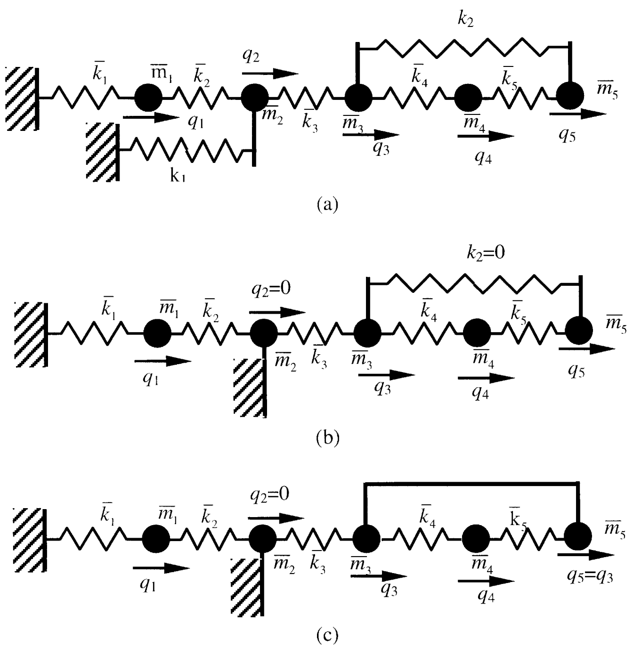

Figure 3.

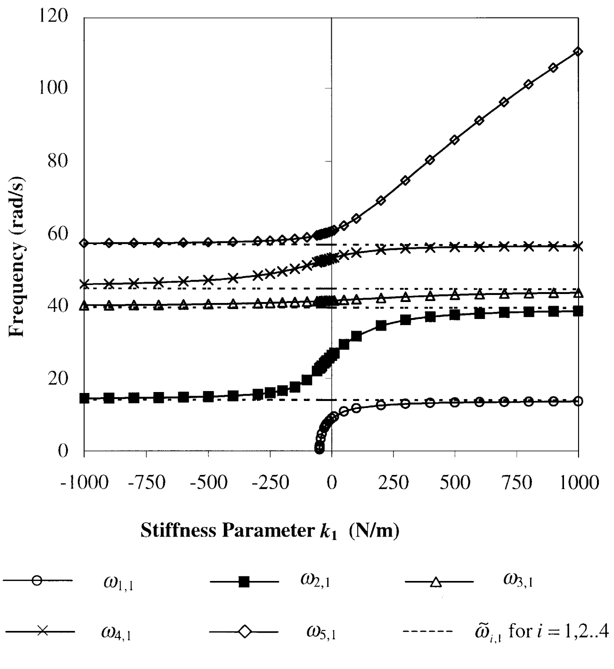

Figure 4. Variation of natural frequencies with stiffness parameter k_1 for $k_2 = 0$.

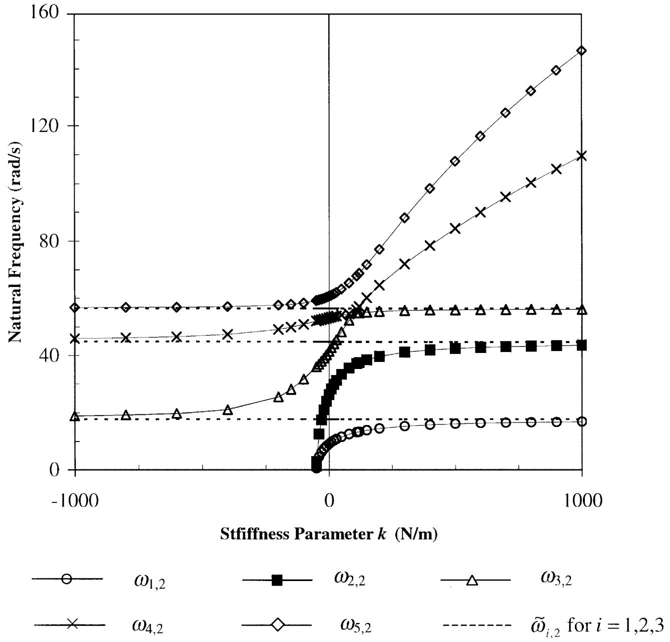

Figure 5. Variation of natural frequencies with stiffness parameter for $k_1 = k_2 = k$.

TABLE 1

Natural frequencies of system B_2 for $k_2 = 0$

k_1 (N/m)	$\omega_{1,1}$ (rad/s)	$\omega_{2,1}$ (rad/s)	$\omega_{3,1}$ (rad/s)	$\omega_{4,1}$ (rad/s)	$\omega_{5,1}$ (rad/s)
– 1E + 09		14·07346	39·43296	44·72136	56·98227
– 1 000 000		14·07384	39·43365	44·72248	56·98258
– 1000		14·47915	39·99207	45·89878	57·2834
– 100		19·54395	41·09613	51·16673	59·00165
– 50		22·70006	41·24934	52·18469	59·66538
– 49·95	0·34293	22·70364	41·2495	52·18573	59·6662
– 49·7	0·839183	22·72158	41·2503	52·19094	59·6703
– 49	1·527944	22·77184	41·25256	52·20552	59·68181
– 45	3·363623	23·06036	41·26548	52·28885	59·74899
– 40	4·665215	23·42328	41·28178	52·39293	59·83629
– 30	6·348633	24·15137	41·31478	52·60032	60·02251
– 25	6·964984	24·51402	41·33149	52·70332	60·12171
– 20	7·488754	24·87423	41·34834	52·80567	60·22514
– 10	8·338074	25·58359	41·38243	53·0078	60·44521
0	9·000781	26·27315	41·41703	53·20555	60·68366
10	9·533173	26·93839	41·45211	53·39781	60·94127
100	11·79806	31·62278	41·78395	54·77226	64·14755
1000	13·71328	38·60747	43·73037	56·67423	110·4405
1 000 000	14·07308	39·43227	44·72024	56·98197	3162·59
1E + 09	14·07346	39·43296	44·72136	56·98227	100 000·0

TABLE 2

Natural modes of system $\tilde{B}_1$

Mode No. (m)	$\tilde{\omega}_{m,1}$(rad/s)	q_1	q_2	q_3	q_4	q_5
1	14·07346	0	0	0·445042	0·801938	1
2	39·43296	0	0	1	0·445042	− 0·80194
3	44·72136	1	0	0	0	0
4	56·98227	0	0	− 0·80194	1	− 0·44504

system, as shown in Figure 3(b)). It is important to note that the convergence to the frequencies of the constrained system is from above, for negative k_1. It may also be noted from Table 1 and Figure 4 that as k_1 takes very large positive values the first four natural frequencies converge to the natural frequencies of the constrained system $\tilde{B}_1$ from below, while the highest natural frequency continues to increase monotonically. By comparing the results for positive and negative values of stiffness, the maximum deviation of the frequencies of the restrained system from those of the constrained system may be found. For example, the first natural frequency of the constrained system $\tilde{B}_1$ is 14·07346 rad/s, and it is bracketed by the natural frequencies of B_1 for $k_1 = -1000$ and 1000 N/m, which give the lowest natural frequencies as 14·47915 and 13·71328 rad/s respectively. By increasing the magnitude of stiffness to 10^6 N/m, the lowest frequencies of B_1 is estimated as 14·07384 and 14·07308 rad/s for negative and positive values respectively. The results have converged to the frequency of the constrained system to the fourth significant figure.

The natural modes of the constrained system $\tilde{B}_1$ ($q_2 = 0$) are shown in Table 2. The modes are normalized by setting the highest displacement to unity. These figures agree with those obtained for the restrained system B_1 with $k_1 = 10^9$ N/m, to four significant figures. It may be noted that constraining q_2 effectively divides the physical system into two subsystems. For the third mode, only the first mass vibrates while for all other modes the vibration is confined to the third, fourth and fifth masses.

3.2. RESULTS FOR TWO ADDITIONAL CONSTRAINTS

Results for various combinations of k_1 and k_2 were obtained, and in all cases when very large values were used for the magnitudes of k_1 and k_2, three natural frequencies were found to converge to those of system $\tilde{B}_2$ shown in Figure 3(c). This constrained system has three modes and frequencies, as two modes of the original system (B_1) have been suppressed by the constraints. The results of increasing both k_1 and k_2 at the same rate (i.e., a common stiffness factor k was increased where $k_1 = k_2 = k$), are shown in Figure 5 and Table 3. The natural frequencies and modes corresponding to the system with two constraints ($\tilde{B}_2$) are shown in Table 4.

It is interesting to note that if one were to plot the variation of frequency with stiffness by following particular modes, two of the natural frequency curves actually appear to cross each other at a stiffness value of 100 N/m. This may not be apparent because the symbols on the figure are used to represent the various frequencies according to their rank only. This was done in order to be consistent with the notation $\omega_{m,h}$ which denotes the mth natural frequency of a system subject to h constraints. By following the modes, one can see that the frequency curves do cross each other and swap ranks. However, once the curves corresponding to the frequencies that monotonically increase cross the line representing the

TABLE 3

Natural frequencies of system B_2 for $k_1 = k_2 = k$

K (N/m)	$\omega_{1,2}$ (rad/s)	$\omega_{2,2}$ (rad/s)	$\omega_{3,2}$ (rad/s)	$\omega_{4,2}$ (rad/s)	$\omega_{5,2}$ (rad/s)
− 1000			18·86284	45·8148	56·67515
− 800			19·18655	46·07655	56·73231
− 600			19·76077	46·49845	56·82708
− 400			21·03744	47·27902	57·01354
− 200			25·44589	49·06429	57·53048
− 150			28·0757	49·85748	57·82907
− 100			31·62278	50·85715	58·31401
− 50			35·95682	52·02986	59·1608
− 49·9	0·452968	1·261079	35·96636	52·03229	59·16304
− 49·5	1·011851	2·818742	36·00456	52·04202	59·17204
− 40	4·419599	12·47887	36·93324	52·27282	59·39893
− 30	6·100281	17·43285	37·96044	52·51412	59·66703
− 20	7·29641	21·05246	39·04649	52·75148	59·9684
− 10	8·233108	23·92362	40·19788	52·98266	60·30625
0	9·000781	26·27315	41·41703	53·20555	60·68366
10	9·647369	28·2219	42·70031	53·41828	61·10339
80	12·399	35·57945	54·55515	52·26533	65·36389
110	13·0826	37·10449	54·86658	55·95276	67·90471
120	13·27381	37·51206	54·95259	57·08374	68·83539
150	13·76487	38·52397	55·16861	60·19689	71·82751
200	14·38295	39·72924	55·42623	64·64725	77·22295
300	15·18232	41·17203	55·72868	72·06286	88·17948
400	15·67511	41·99	55·89608	78·57359	98·45278
500	16·00852	42·50971	56·0011	84·57331	107·9254
1000	16·77823	43·59996	56·2196	109·9501	146·86

highest frequency of the constrained system, no cross overs occur. This means that if positive stiffness values are used in an asymptotic modelling, care should be taken to ensure that the stiffness values used are sufficiently large, so that only the modes that converge to the constrained modes will be in the range of interest. However, the convergence may be verified by using results for large negative stiffness values, and delimit the frequency between the two estimates (one obtained for positive stiffness and another for negative stiffness).

The values of negative stiffness at which the determinant of the stiffness matrix becomes zero can be found by solving det $[K] = 0$. In general, for two negative restraints, this may occur at two different values of stiffness parameters, but for the particular choice of stiffness parameters, this happens only at one point when $k_1 = k_2 = -50$ N/m. Once the critical values of k (say $k_{c,i}$ for $i = 1, \ldots, h$) at which the determinant of the stiffness matrix vanishes are found, then by using stiffness parameters which are lower than all critical stiffness values ($k_{c,i}$) it is possible to ensure that all frequencies calculated are the ones that asymptotically approach the frequencies of the constrained system.

4. CONCLUSIONS

A mathematical proof has been presented to show that if h restraints of positive or negative stiffness are added to an n-d.o.f. system (A), where $h < n$, then for the resulting system (A_h), there exist at least ($n - h$) natural frequencies and modes and that as the

TABLE 4

Natural modes of system $\tilde{B}_2$

Mode No. (m)	$\tilde{\omega}_{m,2}$(rad/s)	q_1	q_2	q_3	q_4	q_5
1	17·71607	0	0	0·84307	1	0·84307
2	44·72136	1	0	0	0	0
3	56·44591	0	0	− 0·59307	1	− 0·59307

h stiffness parameters approach infinity, the $(n - h)$ natural frequencies and modes of system A_h would asymptotically approach those of the n-d.o.f. system subject to h constraints $(\tilde{A}_h)$. The errors due to asymptotic modelling of rigid constraints and connections may be determined by using both positive and negative values for the stiffness of the restraints in the asymptotic models. Alternatively, the magnitude of stiffness required to keep the error due to asymptotic modelling within desired tolerance may be determined. Therefore, asymptotic modelling may be safely used in natural frequency calculations using the Rayleigh–Ritz procedure, without the need to select admissible functions which satisfy geometric constraint conditions.

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REFERENCES

1. R. COURANT 1943 *Bulletin of the American Mathematical Society* **49**, 1–23. Variational methods for the solution of problems of equilibrium and vibration.
2. G. B. WARBURTON and S. L. EDNEY 1984 *Journal of Sound and Vibration* **95**, 537–552. Vibration of rectangular plates with elastically restrained edges.
3. D. J. GORMAN 1989 *Journal of Applied Mechanics* **56**, 893–899. A comprehensive study of the free vibration of rectangular plates resting on symmetrically-distributed uniform elastic edge supports.
4. C. S. KIM, P. G. YOUNG and S. M. DICKINSON 1990 *Journal of Sound and Vibration* **143**, 379–394. On the flexural vibration of rectangular plates approached by using simple polynomials in the Rayleigh–Ritz Method.
5. J. YUAN and S. M. DICKINSON 1992 *Journal of Sound and Vibration* **153**, 203–216. On the use of artificial springs in the study of the free vibrations of systems comprised of straight and curved beams.
6. J. YUAN and S. M. DICKINSON 1992 *Journal of Sound and Vibration* **159**, 39–55. The flexural vibration of rectangular plate systems approached by using artificial springs in the Rayleigh–Ritz method.
7. L. CHENG and J. NICOLAS 1992 *Journal of Sound and Vibration* **155**, 231–247. Free vibration analysis of a cylindrical shell-circular plate system with general coupling and various boundary conditions.

8. J. YUAN and S. M. DICKINSON 1994 *Computers and Structures* **53**, 327–334. Natural frequencies of circular and annular plates with radial or circumferential cracks.
9. H. P. LEE and T. Y. NG 1994 *Applied Acoustics* **42**, 151–163. Natural frequencies and modes for the flexural vibration of a cracked beam.
10. M. AMABILI and R. GARZIERA 1999 *Journal of Sound and Vibration* **224**, 519–539. A technique for the systematic choice of admissible functions in the Rayleigh–Ritz method, doi: 10.1006/jsvi.1999.2198.
11. M. AMABILI and R. GARZIERA 2000 *Journal of Fluids and Structures* **14**, 669–690. Vibrations of circular cylindrical shells with uniform constraints, elastic bed and added mass. Part I: empty and fluid filled shells, doi: 10.1006/jfls.2000.0288.
12. S. ILANKO and S. M. DICKINSON 1999 *Journal of Sound and Vibration* **219**, 370–378. Asymptotic modelling of rigid boundaries and connections in the Rayleigh–Ritz Method, doi:10.1006/jsvi.1998.1847.
13. LORD J. W. S. RAYLEIGH 1877 *Theory of Sound*. New York: Dover; first American edition; 1945.
14. S. H. GOULD 1966 *Variational Methods for Eigenvalue Problems*. London: Oxford University Press.