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DQEM VIBRATION ANALYSES OF NON-PRISMATIC SHEAR DEFORMABLE BEAMS RESTING ON ELASTIC FOUNDATIONS

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1. INTRODUCTION

The analysis of non-prismatic shear deformable beams resting on elastic foundations is frequently necessary for modern engineering design. Certain numerical methods can be used for the analysis of this type of engineering structures. A rather efficient method that can be used to develop solution algorithms for the analysis of non-prismatic shear deformable beams is the DQEM [1].

The DQEM adopts the DQ-related discretization techniques. The method of DQ approximates a partial derivative of a variable function with respect to a co-ordinate at a node as a weighted linear sum of the function values at all nodes along that co-ordinate direction [2]. The original DQ can only be used to solve simple problems [3, 4].

The DQ can be generalized which results in obtaining the generic differential quadrature (GDQ) [5, 6]. The weighting coefficients for a grid model defined by a co-ordinate system having arbitrary dimensions can also be generated. The configuration of a grid model can be arbitrary. In the GDQ, a certain order derivative or partial derivative of the variable function with respect to the co-ordinate variables at a node is expressed as the weighted linear sum of the values of function and/or its possible derivatives or partial derivatives at all nodes.

DQ and GDQ have also been extended which results in the extended differential quadrature (EDQ) [7, 8]. In the EDQ discretization, the number of total degrees of freedom attached to the nodes are the same as the number of total discrete fundamental relations required for solving the problem. A discrete fundamental relation can be defined at a point which is not a node. The points or defining fundamental relations are discrete points. A node can also be a discrete point. For the DQ and GDQ, a node is also a discrete point. Then a certain order derivative or partial derivative, of the variable function existing in a fundamental relation, with respect to the co-ordinate variables at an arbitrary discrete point can be expressed as the weighted linear sum of the values of function and/or its possible derivatives or partial derivatives at all nodes. Thus in solving a problem, a discrete fundamental relation can be defined at a discrete point which is not a node. If a point used for defining discrete fundamental relations is also a node, it is not necessary that the number of discrete fundamental relations at that node equals the number of degrees of freedom attached to it. This concept has been used to construct the discrete inter-element transition conditions and boundary conditions in the differential quadrature element analyses of beam bending problem and warping torsion bar problem [9, 10]. Thus, the EDQ can extend the DQ and GDQ to get these two discretization techniques more flexible in treating the boundary conditions or transition conditions defined on the inter-element boundaries of

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two adjacent elements when they are applied to the DQEM and generalized differential quadrature element method (GDQEM) [5, 11].

The author has proposed a discretization method for solving a generic engineering or scientific problem having an arbitrary domain configuration [1]. Like the finite element method (FEM), in this method the analysis domain of a problem is first separated into a certain number of subdomains or elements. Then the DQ or GDQ discretization is carried out on an element basis. The governing differential or partial differential equations defined on the elements, the transition conditions on inter-element boundaries and the boundary conditions on the analysis domain boundary are in computable algebraic forms after the DQ or GDQ discretization. By assembling all discrete fundamental equations the overall algebraic system can be obtained which is used to solve the problem. The interior elements can be regular. However, in order to solve the problem having an arbitrary analysis domain configuration elements connected to or near the analysis domain boundary might need to be irregular. The mapping technique can be used to develop irregular elements. It results in the DQEM. The GDQ can also be used to develop the irregular elements. It results in the GDQEM. The theoretical basis of DQEM and GDQEM is rigorous since all fundamental relations are locally satisfied. Consequently, the convergence properties of these two discrete element analysis methods is excellent.

The refinement procedure can be used to the DQEM and GDQEM analyses [1, 5]. There are two refinement methods. One is to increase the elements which is the *h* refinement while the other one is to increase the order of the assumed variable function which is the *p* refinement. The convergence can be assumed by successively carrying out the refinement analysis which adopts a certainly defined refinement indicator and a convergence criterion. The refinement indicator can be the absolute or relative local error of the variable function or a certain physical quantity defined by the derivatives or partial derivatives of the variable function, or an absolute or relative error norm defined by the error of the variable function or a physical quantity defined by the derivatives or partial derivatives of the variable function. The *h* refinement can be achieved by either the enrichment of mesh or the design of a new mesh. The *p* refinement can be achieved by either raising the number of element nodes or possibly adding a certain correction function to the assumed variable function. The adaptive concept can also be introduced into the refinement procedure. There are three methods for solving the overall algebraic system. The first one is to use the direct method to all refinement stages. The second one is to use the iterative method to all refinement stages. The last one is to use the direct method to the initial refinement stage following the use of the iterative method to the other refinement stages. From the point of view of computation cost, the two iterative refinement techniques are effective for solving generic problems. They are especially effective for solving non-linear problems. The adaptive *h* refinement procedure, and the adaptive *p* refinement procedure by raising the number of element nodes was also introduced at the time when the DQEM was proposed [1]. In addition, the repositioning techniques can also be used to improve the solutions.

In generating the mesh, if the external cause is composed of various locally distributed causing functions in order to better approximate the true distribution of the external cause the mesh must be designed in such a way that the external cause in one element will not have two or more locally different distributed functions. The adoption of this adaptive discretization technique will lead to a better solution since the locally different causing functions will lead to locally different response functions. Without adopting this technique it will result in a poor approximation of the element-basis external cause if significantly different causing functions, in quantity or order of distribution, co-existing an element. Moreover, the analysis will be more efficient by adopting this adaptive discretization technique since in subdomains having locally higher order distributed causing functions higher order elements can be used while in subdomains having locally lower order distributed causing functions lower order elements can be used [1]. If the refinement procedure is adopted, it will result in an adaptive refinement analysis.

In treating a concentrated external cause existing in the problem domain, the mesh can be designed in such a way that the concentrated external cause is located on some inter-element boundaries and included in the natural transition conditions or kinematic transition conditions. If the external cause is a force-related quantity, it can also be located in some element domains and approximated by the composition of certain continuous functions based on the rule of force equivalence. However, the solution will not be able to reflect the locally transition response behavior.

It has been proved that DQEM and GDQEM are efficient [1, 5–13]. The convergence rate of DQEM is excellent. The DQEM and GDQEM also have the same advantage as the finite element method in general geometry and systematic boundary treatment. For solving vibration problems, the mass matrix is diagonal which requires a little storage space. Since zeros appear on-diagonal, it is positive semidefinite. The lines with zero mass can be eliminated by some mathematical manipulations. The mass matrix is simpler to form and cheaper to use as compared to the consistent mass matrix used in the FEM analysis. It also gives greater accuracy and fewer spurious oscillations than a consistent mass matrix.

The DQEM vibration analysis model of non-prismatic shear deformable beams resting on elastic foundations has been developed. The numerical algorithm is summarized. Numerical results are also presented. They prove that the developed DQEM analysis model is efficient and reliable.

2. DIFFERENTIAL QUADRATURE DISCRETIZATION

For shear deformable beam problems, assume that N^e is the number of nodes in an element and that a displacement component $\phi^e(\xi)$ associated with the element in the parent space can be approximated by the Lagrange polynomials. Then, the first order weighting coefficients of the differential quadrature approximation can be calculated by using the following equations:

$$D^{\xi}_{\alpha\beta} = \frac{M^{(1)}(\xi_\alpha)}{(\xi_\alpha - \xi_\beta)}, \quad \alpha, \beta = 1, 2, \ldots, N^e, \ \alpha \neq \beta, \tag{1}$$

$$D^{\xi}_{\alpha\alpha} = -\sum_{\gamma=1,\gamma\neq\alpha}^{N^e} D^{\xi}_{\alpha\gamma}, \quad \alpha = 1, 2, \ldots, N^e, \tag{2}$$

where

$$M^{(1)}(\xi_\alpha) = \prod_{\beta=1,\beta\neq\alpha}^{N^e} (\xi_\alpha - \xi_\beta). \tag{3}$$

The second and higher order weighting coefficients can similarly be obtained. They can also be generated through the operation of inner products of the first order weighting coefficient matrix [1]. They can be expressed as

$$\begin{aligned} D^{\xi^m}_{\alpha\beta} &= \sum_{\gamma=1}^{N^e} D^{\xi^{m-1}}_{\alpha\gamma} D^{\xi}_{\gamma\beta}, \\ D^{\xi^{m-1}}_{\alpha\beta} &= \sum_{\gamma=1}^{N^e} D^{\xi^{m-2}}_{\alpha\gamma} D^{\xi}_{\gamma\beta}, \\ &\vdots \\ D^{\xi^2}_{\alpha\beta} &= \sum_{\gamma=1}^{N^e} D^{\xi}_{\alpha\gamma} D^{\xi}_{\gamma\beta}. \end{aligned} \tag{4}$$

The weighting coefficients obtained by the above equation and the weighting coefficients obtained by a recurrence relation derived by Shu and Richards are the same [14].

It should be mentioned that the analytical functions used to approximate $\phi^e(\xi)$ when calculating the weighting coefficients can also be Hermite polynomials, Chebyshev polynomials, Euler polynomials, Bernoulli polynomials, sinc functions, etc.

Let l^e denote the element length and assume that the range of the parent element is $-0{\cdot}5 \leqslant \xi \leqslant 0{\cdot}5$. Then the first and second order derivatives of a displacement component ϕ^e with respect to the physical co-ordinate x^e can be written as

$$\frac{\mathrm{d}\phi^e}{\mathrm{d}x^e} = \frac{1}{l^e}\frac{\mathrm{d}\phi^e}{\mathrm{d}\xi}, \qquad \frac{\mathrm{d}^2\phi^e}{\mathrm{d}(x^e)^2} = \frac{1}{(l^e)^2}\frac{\mathrm{d}^2\phi^e}{\mathrm{d}\xi^2} \tag{5, 6}$$

using the DQ discretization into the above two equations, the first and second order derivatives of displacement ϕ^e with respect to x^e at discrete points can be expressed as

$$\frac{\mathrm{d}\phi^e_\alpha}{\mathrm{d}x^e} = \frac{1}{l^e}\sum_{\beta=1}^{N^e} D^{\xi}_{\alpha\beta}\phi^e_\beta, \qquad \frac{\mathrm{d}^2\phi^e_\alpha}{\mathrm{d}(x^e)^2} = \frac{1}{(l^e)^2}\sum_{\beta=1}^{N^e} D^{\xi^2}_{\alpha\beta}\phi^e_\beta. \tag{7, 8}$$

3. DIFFERENTIAL QUADRATURE ELEMENT DISCRETIZATION

For reference in the sequel and for establishing notation, the equations of non-prismatic Timoshenko beam theory are first summarized. Consider an x–y–z Cartesian co-ordinate system with x-axis coincident with the beam's neutral axis. The two modal displacements employed are the transverse displacement w and the bending rotation Ψ. The differential eigenvalue equations of shear deformable beams resting on elastic foundations are expressed by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k^2GA\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \Psi\right)\right] - \bar{k}w + \rho A\omega^2 w = 0 \tag{9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right) - k^2GA\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \Psi\right) + \rho I\omega^2\psi = 0, \tag{10}$$

where E is Young's modulus, G the shear modulus, $\bar{k}$ the foundation constant, A the cross-sectional area, I the moment of inertia of the cross-sectional area and k^2 the shear correction coefficient, ρ the mass per unit length and ω the natural frequency. The stress resultants of bending moment and shear force are

$$M = EI\frac{\mathrm{d}\Psi}{\mathrm{d}x}, \qquad Q = k^2GA\left(\Psi + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \tag{11, 12}$$

respectively. The kinematic boundary conditions are $w = \bar{w}$ and $\Psi = \bar{\Psi}$, where $\bar{w}$ and $\bar{\Psi}$ are described transverse displacement and bending rotation respectively. Let $\tilde{w}$ and $\tilde{\Psi}$ denote the two displacement parameters and consider a body having the mass $\bar{M}^n$ and the moment of inertia $\bar{I}^n$ attached to the natural boundary. The natural boundary conditions are

$$k^2GA\left(\tilde{\Psi} + \frac{\partial\tilde{w}}{\partial x}\right) + v^n\bar{M}^n\frac{\partial^2\tilde{w}}{\partial t^2} = 0 \tag{13}$$

and

$$EI\,\frac{\partial \widetilde{\Psi}}{\partial x} + v^n \bar{I}^n\,\frac{\partial^2 \widetilde{\Psi}}{\partial t^2} = 0, \tag{14}$$

where v^n is a sign indicator equal to 1 for the right boundary and -1 for the left boundary.

For analyzing the non-prismatic shear deformable beam problems, a two-node element can be used for elements having no distributed external cause. But the number of element nodes must be at least three for elements having distributed external cause. Using equations (7) and (8), equations (9) and (10) can be discretized which show to have the following forms:

$$\left[\frac{k^2 G^e}{(l^e)^2}\left(\sum_{\bar{\beta}=1}^{N^e} D^{\xi}_{(\alpha)\bar{\beta}} A^e_{\bar{\beta}} \sum_{\beta=1}^{N^e} D^{\xi}_{\alpha\beta} + A^e_{(\alpha)} \sum_{\beta=1}^{N^e} D^{\xi 2}_{\alpha\beta}\right) - \bar{k}_{(\alpha)}\delta_{\alpha\beta}\right] w^e_\beta + \frac{k^2 G^e A^e_{(\alpha)}}{l^e} \sum_{\beta=1}^{N^e} D^{\xi}_{\alpha\beta}\Psi^e_\beta$$

$$+ \frac{k^2 G^e}{l^e} \sum_{\bar{\beta}=1}^{N^e} D^{\xi}_{(\alpha)\bar{\beta}} A^e_{\bar{\beta}} \Psi^e_\alpha + \omega^2(\rho A w)_\alpha = 0, \quad \alpha = 2, 3, \ldots, N^e - 1, \tag{15}$$

$$-\frac{k^2 G^e A^e_{(\alpha)}}{l^e} \sum_{\beta=1}^{N^e} D^{\xi}_{\alpha\beta} w^e_\beta - k^2 G^e A^e_{(\alpha)} \Psi^e_\alpha + \frac{E^e}{(l^e)^2}$$

$$\times\left(\sum_{\bar{\beta}=1}^{N^e} D^{\xi}_{(\alpha)\bar{\beta}} I^e_{\bar{\beta}} \sum_{\beta=1}^{N^e} D^{\xi}_{\alpha\beta} + I^e_{(\alpha)} \sum_{\beta=1}^{N^e} D^{\xi 2}_{\alpha\beta}\right)\Psi^e_\beta + \omega^2(\rho I \Psi)_\alpha = 0, \quad \alpha = 2, 3, \ldots, N^e - 1. \tag{16}$$

The transition conditions of two adjacent elements are the kinematic and the natural transition conditions. The kinematic transition conditions are the continuities of transverse displacement and bending rotation. Let $x = x^{i,i+1}$ denote the inter-element boundary of two adjacent elements i and $i+1$. The discrete kinematic transition conditions are $w^i_{N^i} = w^{i+1}_1$ and $\Psi^i_{N^i} = \Psi^{i+1}_1$.

The natural transition conditions are the equilibrium conditions of internal forces and inertia forces on the inter-element boundary between two adjacent elements i and $i+1$. Figure 1 shows the forces on the inter-element boundary in which $\bar{M}^{i,i+1}$ and $\bar{I}^{i,i+1}$ are the

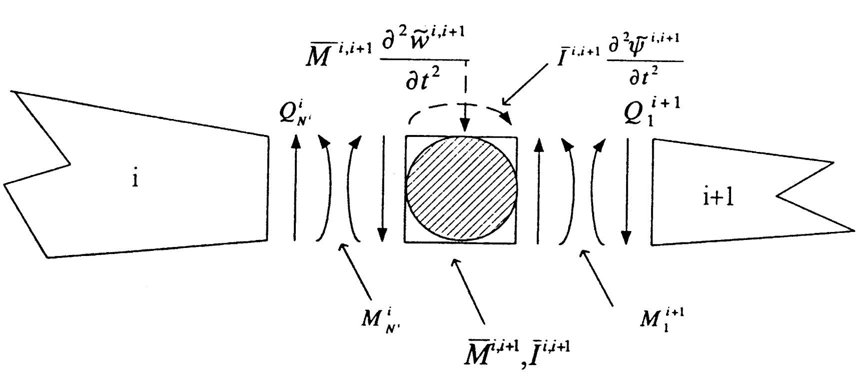

Figure 1. Forces at the inter-element boundary of the two adjacent elements i and $i+1$.

mass and moment of inertia, respectively, of a body attached to the inter-element boundary. The equilibrium of lateral forces is expressed by

$$k^2 G^i A^i_{N^i}\left(\tilde{\Psi}^i_{N^i} + \frac{\partial \tilde{w}^i_{N^i}}{\partial x}\right) - k^2 G^{i+1} A^{i+1}_1\left(\tilde{\Psi}^{i+1}_1 + \frac{\partial \tilde{w}^{i+1}_1}{\partial x}\right) + \bar{M}^{i,i+1}\frac{\partial^2 \tilde{w}^{i,i+1}}{\partial t^2} = 0. \quad (17)$$

Considering the harmonic motion and using the DQ discretization, the above equation leads to

$$k^2 G^i A^i_{N^i}\left(\Psi^i_{N^i} + \frac{1}{l^i}\sum_{\beta=1}^{N^i} D^{\xi}_{N^i\beta} w^i_\beta\right) - k^2 G^{i+1} A^{i+1}_1\left(\Psi^{i+1}_1 + \frac{1}{l^{i+1}}\sum_{\beta=1}^{N^{i+1}} D^{\xi}_{1\beta} w^{i+1}_\beta\right)$$
$$- \bar{M}^{i,i+1}\omega^2 w^{i,i+1} = 0. \quad (18)$$

The equilibrium of moments is expressed by

$$E^i I^i_{N^i}\frac{\partial \tilde{\Psi}^i_{N^i}}{\partial x^i} - E^{i+1} I^{i+1}_1 \frac{\partial \tilde{\Psi}^{i+1}_1}{\partial x^{i+1}} + \bar{I}^{i,i+1}\frac{\partial^2 \tilde{\Psi}}{\partial t^2} = 0. \quad (19)$$

Considering the harmonic motion and using the DQ discretization, the above equation leads to the following equation:

$$\frac{E^i I^i_{N^i}}{l^i}\sum_{\beta=1}^{N^i} D^{\xi}_{N^i\beta}\Psi^i_\beta - \frac{E^{i+1} I^{i+1}_1}{l^{i+1}}\sum_{\beta=1}^{N^{i+1}} D^{\xi}_{1\beta}\Psi^{i+1}_\beta - \bar{I}^{i,i+1}\omega^2 w^{i,i+1} = 0. \quad (20)$$

For the pin-connected inter-element boundary, the continuity of Ψ and equation (19) are replaced by the following two equations:

$$\frac{E^i I^i_{N^i}}{l^i}\sum_{\beta=1}^{N^i} D^{\xi}_{N^i\beta}\Psi^i_\beta = 0, \qquad \frac{E^{i+1} I^{i+1}_1}{l^{i+1}}\sum_{\beta=1}^{N^i+1} D^{\xi}_{1\beta}\Psi^{i+1}_\beta = 0, \quad (21, 22)$$

forced displacements are imposed on an inter-element boundary, the discrete natural transition conditions (17) and (19) are replaced by the condition equations of forced displacement $w^i_{N^i} = \bar{w}^{i,i+1}$ and displacement gradient $(1/l^i)\sum_{\beta=1}^{N^i} D^{\xi}_{N^i\beta} w^i_\beta = \mathrm{d}\bar{w}^{i,i+1}/\mathrm{d}x$ where $\bar{w}^{i,i+1}$ and $\mathrm{d}\bar{w}^{i,i+1}/\mathrm{d}x$ are the prescribed values.

Letting element m be an element consisting of the kinematic boundary, the kinematic boundary conditions can be rewritten as $w^m_I = \bar{w}^m_I$, $I = 1$ or N^m and $\Psi^m_I = \bar{\Psi}^m_I$, $I = 1$ or N^m. Also, letting element n be an element consisting of the natural boundary and v^n equal 1 for the right boundary and -1 for the left boundary, the discrete natural boundary conditions can be obtained from equations (12) and (13):

$$k^2 G^n A^n_{(I)}\left(\Psi^n_I + \frac{1}{l^n}\sum_{\beta=1}^{N^n} D^{\xi}_{I\beta} w^n_\beta\right) - v^n \bar{M}^n \omega^2 w^n_I = 0, \quad I = 1 \text{ or } N^n, \quad (23)$$

$$\frac{E^n I^n_{(I)}}{l^n}\sum_{\beta=1}^{N^n} D^{\xi}_{I\beta}\Psi^n_\beta - v^n \bar{I}^n \omega^2 \Psi^n_I = 0, \quad I = 1 \text{ or } N^n. \quad (24)$$

With the kinematic transition conditions in mind, then assemble the discrete element governing equations (14) and (15) for elements having more than two nodes, the discrete natural transition conditions (17) and (19), and the discrete natural boundary conditions (22) and (23), the discrete eigenvalue equation system of non-prismatic shear deformable beam problems can be obtained. Consider the kinematic boundary conditions, the discrete eigenvalue equation system can be expressed as

$$([K] - \omega^2[M])\{D\} = \{0\}, \tag{25}$$

where $[K]$ is the overall stiffness matrix, $[M]$ the overall mass matrix and $\{D\}$ the overall modal displacement vector. $[K]$ is a sparse matrix. $[M]$ is a diagonal matrix with zeros appearing on-diagonal. This is positive semidefinite. Equation (24) is a generalized eigenvalue problem with infinite frequencies existing. Premultiplication of equation (24) by $[K]^{-1}$ leads to

$$([A] - \lambda[I])\{D\} = \{0\}, \tag{26}$$

where $[A] = [K]^{-1}[M]$ and $\lambda = 1/\omega^2$. Equation (25) can be solved by using either an exact solution technique or an approximate solution technique. If the order of the eigenvalue system is large, the approximation algorithms which calculate the eigenpairs in descending order can reduce the expense.

Some d.o.f. can be eliminated before solving equation (24). If the rotary inertia is neglected, the displacement parameters Ψs at all element interior nodes can be eliminated. If no mass is attached to an inter-element boundary or natural boundary, the displacement parameters associated with it can also be eliminated. Considering the non-existence of inertia forces for some component equations existing in equation (24), the equation can be rewritten as

$$\left(\begin{bmatrix} [K_{aa}] & [K_{ab}] \\ [K_{ba}] & [K_{bb}] \end{bmatrix} - \omega^2 \begin{bmatrix} [M_{aa}] & [0] \\ [0] & [0] \end{bmatrix}\right) \begin{Bmatrix} \{D_a\} \\ \{D_b\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{27}$$

with $[M_{aa}]$ being a diagonal matrix without zeros appearing on-diagonal. From the lower part of equation (26), the following relation can be obtained:

$$\{D_b\} = [K_{bb}]^{-1}[K_{ba}]\{D_a\}. \tag{28}$$

The substitution of equation (27) into the upper part of equation (26) yields

$$([\bar{K}_{aa}] - \omega^2[M_{aa}])\{D_a\} = \{0\}, \tag{29}$$

where

$$[\bar{K}_{aa}] = [K_{aa}] + [K_{ab}][K_{bb}]^{-1}[K_{ba}]. \tag{30}$$

Equation (28) can be treated and solved by the same procedure that transfers equation (24) into equation (25). It can also be solved by adopting the advantage of the diagonality of $[M_{aa}]$. Defining $[M_{aa}] = [L]^2$ and $\{D_a\} = [L]^{-1}\{Y\}$, substituting them into equation (28), and then premultiplying by $[L]^{-1}$, the following eigenvalue problem can be obtained:

$$([H]-\omega^2[I])\{Y\}=\{0\}, \tag{31}$$

where $[H]=[L]^{-1}\ [\bar{K}_{aa}][L]^{-1}$. For economically solving a large eigenvalue problem, the approximation algorithms which calculate the eigenpairs in ascending order can be used.

4. NUMERICAL EXAMPLES

The first problem solved involves the free vibration of a shear deformable uniform cantilever beam. The non-dimensional rotary inertia parameter of the beam defined as $r=\gamma/L$ with γ being the radius of gyration of the cross-section is selected to be 0·05, while the non-dimensional shear flexibility parameter defined as $s=(EI/k^2AGL^2)^{1/2}$ is selected to be 0·1. A representative beam has the following material and geometrical properties: rectangular cross-section with the depth of the cross-section $h=0{\cdot}17320508$ m, with width $b=0{\cdot}57735027$ m, the beam length $L=1$ m, the shear correction coefficient $k^2=0{\cdot}66666667$, mass per unit length $\rho=0{\cdot}66666667\times10^{-4}$ kg/m, Young's modulus $E=2{\cdot}66666667$ Pa and shear modulus $G=1$ Pa. The effect of rotary inertia is considered. Let b_i denote the exact solutions of the natural frequencies of the Bernoulli–Euler beam. Numerical results and exact solutions of the ratios of the first five natural frequencies of the shear deformable beam are summarized and listed in Table 1 [15]. It shows that the results can converge fast to the exact solutions by either increasing the d.o.f. per element or the number of elements. It also shows that the computation of increasing the d.o.f. per element performs better than the computation of increasing the number of elements. In addition, the DQEM results of higher modes are affected by the consideration of shear deformation and rotary inertia even more.

The second problem solved involves the free vibration of a cantilever beam composed of two prismatic elements with the length of each element being 0·5 m. The depth of the element with the fixed boundary is 0·15 m, while the other element has the depth 0·2 m. All

TABLE 1

The first five natural frequencies of a shear deformable prismatic cantilever beam considering the effect of rotary inertia

d.o.f. per element	Number of elements	ω_1/b_1	ω_2/b_2	ω_3/b_3	ω_4/b_4	ω_5/b_5
3	4	1·3035592	1·1827782	1·0604391	0·0872562	0·8784579
	6	1·1309024	1·0028342	0·8846225	0·7771790	0·6749201
5	2	0·9544300	0·7495082	0·6532353	0·5108864	0·4714955
	4	0·9711311	0·8370633	0·7052438	0·5810131	0·5063390
	6	0·9720688	0·8431605	0·7195133	0·6084758	0·5189333
7	2	0·9723885	0·8475282	0·7305820	0·6464362	0·5927673
	4	0·9716780	0·8447689	0·7237495	0·6175196	0·5336837
	6	0·9723245	0·8447462	0·7234423	0·6162287	0·5321596
9	2	0·9720762	0·8446972	0·7231421	0·6144475	0·5319935
	4	0·9731479	0·8447698	0·7234108	0·6160771	0·5317388
	6	0·9710939	0·8446963	0·7234039	0·6161000	0·5317894
Exact sol.		0·9723	0·8447	0·7234	0·6161	0·5318

TABLE 2

The first five natural frequencies of a shear deformable non-prismatic cantilever beam composed of two prismatic elements considering the effect of rotary inertia (rad/s)

d.o.f. per element	Number of elements	ω_1	ω_2	ω_3	ω_4	ω_5
3 node	2	0·5863078D + 01	0·3798482D + 02	0·1599888D + 03	0·2037513D + 03	
5 node	2	0·2768917D + 01	0·1584239D + 02	0·4127034D + 02	0·6109054D + 02	0·9206028D + 02
7 node	2	0·2813875D + 01	0·1749714D + 02	0·4578586D + 02	0·7700861D + 02	0·1280842D + 03
9 node	2	0·2813415D + 01	0·1745975D + 02	0·4547572D + 02	0·7405805D + 02	0·1067211D + 03
11 node	2	0·2813242D + 01	0·1746005D + 02	0·4548344D + 02	0·7418299D + 02	0·1065002D + 03

TABLE 3

The first five natural frequencies of a shear deformable non-prismatic cantilever beam composed of two prismatic elements resting on an elastic foundation and considering the effect of rotary inertia (rad/s)

d.o.f. per element	Number of elements	ω_1	ω_2	ω_3	ω_4	ω_5
3 node	2	0·3627203D + 02	0·5688747D + 02	0·1310366D + 03	0·2529273D + 03	
5 node	2	0·3591770D + 02	0·4057903D + 02	0·5616760D + 02	0·7204069D + 02	0·1006299D + 03
7 node	2	0·3590355D + 02	0·4129053D + 02	0·5948750D + 02	0·8575786D + 02	0·1128474D + 03
9 node	2	0·3590296D + 02	0·4127607D + 02	0·5925225D + 02	0·8323640D + 02	0·1124629D + 03
9 node	6	0·3590264D + 02	0·4127608D + 02	0·5925805D + 02	0·8333370D + 02	0·1124339D + 03

other material and geometrical constants are the same as those of the first problem. The effect of rotary inertia is considered. In this analysis, Chebyshev polynomials are used to define the DQ discretization. Chebyshev polynomials can be generated from the following recursion formula: $T_{n+1}(\xi) = 2\xi T_n(\xi) - T_{n-1}(\xi)$ with the two initial members $T_0(\xi) = 1$ and $T_1(\xi) = \xi$. Certain procedures for adopting an analytical function to define the DQ discretization can be used to calculate the weighting coefficients [6]. In an element, the DQ nodes are defined by the roots of Chebyshev polynomials. With the range being $-1 \leqslant \xi \leqslant 1$, the roots of Chebyshev polynomials are: $\xi_1 = -1$, $\xi_\alpha = -\cos\alpha\pi/(N^e + 1)$ for $\alpha = 2, 3, \ldots, N^e - 1$ and $\xi_{N^e} = 1$. Numerical results of the first five natural frequencies for the structure are summarized and listed in Table 2. The results of the beam resting on the elastic foundation with the foundation modulus = 1 N/m are also listed in Table 3. It also shows that the convergence properties are excellent.

The last problem solved involves the free vibration of a non-prismatic cantilever beam resting on an elastic foundation. With the fixed end A being the origin of the co-ordinate system, the variation of depth is $d(x) = d_0(1 - x/L + x^2/2L^2)$ with $d_0 = 0{\cdot}2$ m and $L = 1$ m. All other material and geometrical constants are the same as those of the first problem. The effect of rotary inertia is considered. In carrying out the DQEM analysis, Lagrange polynomials are used to calculate the weighting coefficients. Elements and nodes in an element are equally spaced. The DQEM results are summarized and listed in Table 4. It also shows that the developed DQEM vibration analysis model has excellent convergence properties.

TABLE 4

The first five natural frequencies of a shear deformable non-prismatic cantilever beam resting on an elastic foundation and considering the effect of rotary inertia (rad/s)

d.o.f. per element	Number of elements	ω_1	ω_2	ω_3	ω_4	ω_5
3 node	2	0·4443144D + 02	0·5689754D + 02	0·2001291D + 03	0·2132939D + 03	
	4	0·4448764D + 02	0·4971019D + 02	0·7629094D + 02	0·1121468D + 03	0·2208612D + 03
	6	0·4393554D + 02	0·4877331D + 02	0·6706347D + 02	0·1009917D + 03	0·1404324D + 03
5 node	2	0·4273247D + 02	0·4749201D + 02	0·5705289D + 02	0·7311480D + 02	0·1011024D + 03
	4	0·4321339D + 02	0·4817797D + 02	0·5842086D + 02	0·7776579D + 02	0·1049493D + 03
	6	0·4324357D + 02	0·4823142D + 02	0·5910570D + 02	0·8066513D + 02	0·1066619D + 03
7 node	2	0·4327764D + 02	0·4835176D + 02	0·5925708D + 02	0·8222986D + 02	0·1041471D + 03
	4	0·4325222D + 02	0·4824901D + 02	0·5931240D + 02	0·8168726D + 02	0·1095850D + 03
	6	0·4325158D + 02	0·4824598D + 02	0·5930851D + 02	0·8161701D + 02	0·1094984D + 03
9 node	2	0·4325257D + 02	0·4825859D + 02	0·5929692D + 02	0·8156748D + 02	0·1101219D + 03
	4	0·4325147D + 02	0·4824596D + 02	0·5930803D + 02	0·8160818D + 02	0·1094385D + 03
	6	0·4325126D + 02	0·4824601D + 02	0·5930836D + 02	0·8161135D + 02	0·1094422D + 03

5. CONCLUSIONS

The DQEM free vibration analysis model of non-prismatic shear deformable beams resting on the elastic foundation was developed. The numerical model was summarized and presented. Numerical results proved that the DQEM has excellent convergence properties. Since the theoretical basis of this DQEM is rigorous, the performance of the developed DQEM analysis model is excellent.

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