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FUNDAMENTAL FREQUENCY OF CRACKED BEAMS IN BENDING VIBRATIONS: AN ANALYTICAL APPROACH

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This paper presents an analytical approach to the fundamental frequency of cracked Euler–Bernoulli beams in bending vibrations. The flexibility influence function method used to solve the problem leads to an eigenvalue problem formulated in integral form. The influence of the crack was represented by an elastic rotational spring connecting the two segments of the beam at the cracked section. In solving the problem, closed-form expressions for the approximated values of the fundamental frequency of cracked Euler–Bernoulli beams in bending vibrations are reached. The results obtained agree with those numerically obtained by the finite element method.

1. INTRODUCTION

Cracks can appear in structural elements as a consequence of initial defects within the material or caused by fatigue during their operational life. Cracks in a structure reduce its natural frequencies because it becomes more flexible, so, the measurement of these frequencies may be used to detect the presence, size and location of cracks in a structural component (See, for instance references [1–3]).

Also, from the natural frequency values and the corresponding vibration mode shapes of cracked beams, the dynamic stress intensity factor for the three-point [4–6] or one-point [7] bending specimens may be derived.

Natural frequencies of cracked beams can be obtained by numerical analysis of the system using the finite element method (FEM). Alternatively, simplified procedures are available to evaluate the influence of the parameters involved in the problem, such as, crack length, crack location, boundary conditions, etc. saving the computing time of full FEM analysis.

Among these simplified methods are those proposed by Christides and Barr [8] and Shen and Pierre [9, 10], using either a two-term Rayleigh-Ritz [8], or the Galerkin method [9, 10]. In both approaches, a crack function representing the perturbation in the stress field induced by the crack is considered.

In other cases, the presence of a crack, and the corresponding reduction of the flexural beam stiffness, has been represented by means of linear springs [11], whose stiffness may be related to the crack length by the Fracture Mechanics theory [12]. This kind of model has been successfully applied to simply supported [13, 14], cantilever [15] and free–free [16] cracked beams.

Fernández-Sáez *et al.* [17] recently proposed a method to calculate the fundamental frequency of cracked Euler–Bernoulli beams, providing a closed-form expression for the

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case of simply supported beam. However, for other boundary conditions, this method could not provide explicit expressions for the fundamental frequency. In this work, the method of flexibility influence functions [18] is applied to obtain approximate values (lower bounds) of the fundamental frequency for bending vibrations of cracked Euller–Bernoulli beams with different boundary conditions. This method was used by Penny and Reed [18] to get approximate values of the fundamental frequency of uncracked beams subjected to bending vibrations. Here the approach is extended to the problem of cracked beams. The crack is represented by means of a rotational spring model and closed-form expressions for the fundamental frequency are obtained. The results are in agreement with those obtained by FEM analysis of the problem.

2. ANALYTICAL APPROACH

Consider an Euler–Bernoulli beam of length L subjected to an external load $p(x, t)$ (Figure 1). At any point x, the beam has a cross-sectional area $A(x)$, an area moment of inertia about the neutral axis, $I(x)$, and a mass per unit length, $m(x)$ ($m(x) = \rho(x)A(x)$, $\rho(x)$ being the mass density of the beam material). The vertical displacement, $y(x, t)$, at any point, x, of the beam and at any time, t can be expressed as [19]

$$y(x, t) = \int_0^L c(x, \xi)p(\xi, t)\,\mathrm{d}\xi, \tag{1}$$

where $c(x, \xi)$ is the flexibility influence function defined as the vertical displacement of the considered point, x, due to a unit load applied at the point of abcissa ξ. For free beam vibration, the load acting on the beam is only that due to inertia forces:

$$p(\xi, t) = -\,m(\xi)\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

Assuming that the vertical displacement at any point of the beam, $y(x, t)$ can be written as

$$y(x, t) = u(x)\sin(\omega t). \tag{3}$$

$u(x)$ being the transverse deflection of the beam and ω the frequency of the harmonic vibration, equation (1) reduces to

$$u(x) = \omega^2\int_0^L c(x, \xi)m(\xi)u(\xi)\,\mathrm{d}\xi. \tag{4}$$

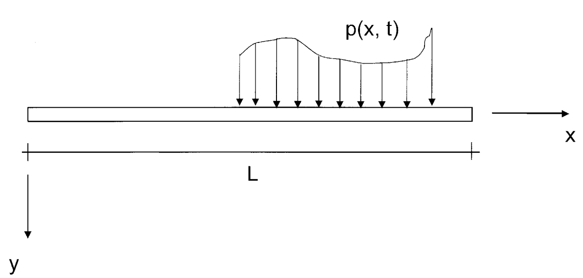

Figure 1. Reference system used for a beam subjected to an external load.

Using new functions, $r(x)$, $K_1(x, \xi)$ and λ, defined through

$$r(x) = u(x)\sqrt{m(x)}, \quad K_1(x, \xi) = c(x, \xi)\sqrt{m(x)}\sqrt{m(\xi)}, \quad \lambda = \omega^2$$

equation (4) becomes

$$r(x) = \lambda \int_0^L K_1(x, \xi) r(\xi)\, d\xi. \tag{5}$$

The last equation is a homogeneous Fredholm integral equation with symmetric kernel, $K_1(x, \xi)$, and represents a classical eigenvalue problem formulated in integral form: solutions $r_n(x)$ (eigenfunctions) exist only for a discrete set of values λ_n (eigenvalues). Although these eigenvalues could be numerically calculated by standard methods (see, for instance reference [20]), Penny and Reed [18] developed a procedure to obtain bounds of the first eigenvalue, λ_1. From the calculation of the nth iterated kernel, $K_n(x, \xi)$, that for $n \geqslant 2$ is defined by

$$K_n(x, \xi) = \int_0^L K_{n-1}(x, \eta) K_1(\eta, \xi)\, d\eta \tag{6}$$

a scalar variable J_n can be calculated as

$$J_n = \int_0^L K_n(x, x)\, dx \tag{7}$$

and upper and lower bounds of the first eigenvalue λ_1 could be deduced as [18]

$$\left(\frac{1}{J_n}\right)^{1/n} < \lambda_1 < \frac{J_{n-1}}{J_n}. \tag{8}$$

For the case $n = 1$, this last equation reduces to

$$\frac{1}{J_1} < \lambda_1 \tag{9}$$

with

$$J_1 = \int_0^L K_1(x, x)\, dx. \tag{10}$$

Since $\lambda_1 = \omega_1^2$, the value $(1/J_1)^{1/2}$ is a first lower bound of the actual value of the fundamental frequency, ω_1. For $n = 2$, a second lower bound of ω_1 can be deduced as $(1/J_2)^{1/4}$. Obviously, the larger the value of n the closer the bounds.

This procedure was used by Penny and Reed [18] for the case of uniform simply supported and fixed–fixed beams, as well as for cantilever beams of varying width and thickness. These authors found, from the analysis of errors involved in the approximation, that the second lower bound is, in practice, a sufficiently good approximation to the true value of λ_1.

3. APPLICATION TO CRACKED BEAMS

This method is applied to the analysis of bending vibrations in the x–y plane of a uniform cracked Euler–Bernoulli beam with different support conditions. The length and width of

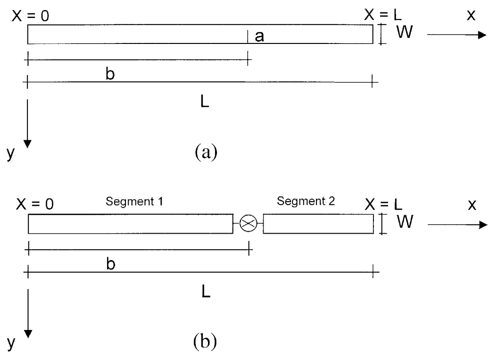

Figure 2. Beam with a transverse edge crack: (a) dimensions of the beam and position of the crack; (b) model for the cracked beam.

the beam are L and W respectively. The beam has a crack of depth a at a distance b from the left support, $x = 0$ as shown in Figure 2(a). The vibrational behaviour of this beam is non-linear in general, due to the opening and closing of the crack. Recently, Chati *et al.* [21] have shown that natural frequencies of a cracked beam could be approximated by combining the natural frequencies obtained in two different assumptions regarding the crack: it always remains open or closed. In this approach, the calculation of the fundamental frequency of the beam for bending vibrations will assume that the crack remains always open.

The cracked beam has been treated as two beams connected by a rotational elastic spring at the crack section (see Figure 2(b)). The stiffness of the spring depends on the crack depth and the geometry of the cracked section. All magnitudes referring to the left segment ($0 \leqslant x \leqslant b$) have the subscript 1 and the right one ($x > b$) is subscripted by 2. If $u(x)$ represents the transverse deflection of the beam and $\theta(x)$ its corresponding slope ($\theta(x) = \mathrm{d}u(x)/\mathrm{d}x$), this model introduces a discontinuity in the slope of the beam at the crack section ($x = b$), which is proportional to the bending moment transmitted through it, $M_f(b)$:

$$u_2'(b) - u_1'(b) = \Delta\theta = C_m M_f(b), \tag{11}$$

where $(\cdot)'$ denotes derivation by x. C_m is the flexibility constant of the spring and can be calculated by

$$C_m = \frac{W}{EI}\,\Theta(\alpha,\ \textit{geometry of the cross-section}) \tag{12}$$

E being Young's modulus of the beam material, I the moment of inertia of the uncracked section and Θ a function depending on the crack ratio $\alpha = a/W$, and the beam cross-section geometry. This function can be evaluated by Fracture Mechanics theory and, in the case of a rectangular section, it takes the form [22]

$$\Theta(\alpha) = 2\left(\frac{\alpha}{1-\alpha}\right)^2 (5{\cdot}93 - 19{\cdot}69\alpha + 37{\cdot}14\alpha^2 - 35{\cdot}84\alpha^3 + 13{\cdot}12\alpha^4). \tag{13}$$

The other kinematic conditions that should be satisfied are the continuity of displacements, bending moments (related to the second derivative of the beam deflection) and shear forces

TABLE 1

Boundary conditions for the different studied beams

Case	$x = 0$	$x = L$
Simply supported beam	$u_1 = 0$ $\dfrac{d^2u_1}{dx^2} = 0$	$u_2 = 0$ $\dfrac{d^2u_2}{dx^2} = 0$
Cantilever beam	$u_1 = 0$ $\dfrac{du_1}{dx} = 0$	$\dfrac{d^2u_2}{dx^2} = 0$ $\dfrac{d^3u_2}{dx^3} = 0$
Fixed–pinned beam	$u_1 = 0$ $\dfrac{du_1}{dx} = 0$	$u_2 = 0$ $\dfrac{d^2u_2}{dx^2} = 0$
Fixed–fixed beam	$u_1 = 0$ $\dfrac{du_1}{dx} = 0$	$u_2 = 0$ $\dfrac{du_2}{dx} = 0$

(related to the third derivative of the beam deflection), i.e.,

$$u_1(b) = u_2(b), \quad u_1''(b) = u_2''(b), \quad u_1'''(b) = u_2'''(b). \tag{14}$$

Additionally, four boundary conditions, two for each beam end, must be satisfied. In this paper four support conditions are considered: two cases are statically determinate (simply supported and cantilever beams) and two cases statically undetermined (fixed–pinned and fixed–fixed beams). Table 1 gives the boundary conditions that apply in each case.

To calculate the vertical displacement $c(x, \xi)$, as well as the slope of the transverse deflection of the beam, $\theta(x, \xi)$, the area–moment method can be used, taking account of both the boundary and the kinematic conditions, i.e.,

$$\theta(x, \xi) = \theta(0) + \int_0^x \frac{M_f(x, \xi)}{EI}\,dx + \Delta\theta, \tag{15}$$

$$c(x, \xi) = \theta(0)x + \int_0^x \frac{M_f(x, \xi)}{EI}\,x\,dx + \Delta\theta(x - b), \tag{16}$$

where $\theta(0)$ is the slope of the beam at the left support ($x = 0$), $M_f(x, \xi)$ the bending moment at the point x produced by a unit load applied at point ξ, and $\Delta\theta$ the discontinuity of the slope at the cracked section, which is proportional to the bending moment transmitted by this section (equation (11)). The last terms in equations (15) and (16) ($\Delta\theta$ and $\Delta\theta(x - b)$ respectively) need be considered only for the case of $x > b$. In other cases ($x \leqslant b$) these terms must be discarded.

To calculate $\theta(0)$ for the case of a simply supported beam (in the case of fixed–pinned or fixed–fixed beams, $\theta(0) = 0$), as well as the bending moment distributions in the statistically undetermined cases, it is necessary to consider the support conditions of the beam at the other end ($x = L$).

Using the non-dimensional variables:

$$\bar{x} = \frac{x}{L}, \quad \bar{\xi} = \frac{\xi}{L}, \quad \delta = \frac{b}{L}, \quad \beta = \frac{W}{L} \tag{17}$$

the general expression for the function $c(x, \xi)$ can be written as

$$c(x, \xi) = \frac{L^3}{EI} \bar{c}_i(\bar{x}, \bar{\xi}; \delta, \beta, \alpha). \tag{18}$$

For each case considered, the functions $\bar{c}_i(\bar{x}, \bar{\xi}; \delta, \beta, \alpha)$ constitute a set of six different functions ($i = 1, 2, \ldots, 6$) that must be applied depending on the relative position of variables, $\bar{x}$, $\bar{\xi}$ and δ, according to

- $i = 1$. $\bar{\xi} < \delta$ and $0 < \bar{x} < \bar{\xi}$.
- $i = 2$. $\bar{\xi} < \delta$ and $\bar{\xi} < \bar{x} < \delta$.
- $i = 3$. $\bar{\xi} < \delta$ and $\delta < \bar{x} < 1$.
- $i = 4$. $\bar{\xi} > \delta$ and $0 < \bar{x} < \delta$.
- $i = 5$. $\bar{\xi} > \delta$ and $\delta < \bar{x} < \bar{\xi}$.
- $i = 6$. $\bar{\xi} > \delta$ and $\bar{\xi} < \bar{x} < 1$.

The expressions of the functions $\bar{c}_i(\bar{x}, \bar{\xi}; \delta, \beta, \alpha)$ for each case are given in Appendix A.

From the above functions, the kernel of equation (5) can be calculated as

$$K_1(x, \xi) = \frac{mL^3}{EI} \bar{c}_i(\bar{x}, \bar{\xi}). \tag{19}$$

To calculate J_1 (equation (10)), it is necessary to compute, previously, $K_1(x, x)$. For this it is helpful to note that $\bar{c}_1(\bar{x}, \bar{x}) = \bar{c}_2(\bar{x}, \bar{x})$ (necessary to compute $K_1(x, x)$ for case $\bar{x} < \delta$), and $\bar{c}_5(\bar{x}, \bar{x}) = \bar{c}_6(\bar{x}, \bar{x})$ (necessary to calculate $K_1(x, x)$ for case $\bar{x} > \delta$).

Therefore, the function $K_1(x, x)$ has the following expressions:

$$K_1(x, x) = \frac{mL^3}{EI} \bar{c}_1(\bar{x}, \bar{x}), \quad \bar{x} < \delta, \tag{20}$$

$$K_1(x, x) = \frac{mL^3}{EI} \bar{c}_5(\bar{x}, \bar{x}), \quad \bar{x} > \delta. \tag{21}$$

From equations (10), (20) and (21), J_1 becomes

$$J_1 = \frac{mL^4}{EI} \left(\int_0^\delta \bar{c}_1(\bar{x}, \bar{x})\, d\bar{x} + \int_\delta^1 \bar{c}_5(\bar{x}, \bar{x})\, d\bar{x} \right). \tag{22}$$

A second, and closer lower bound of the fundamental frequency can be obtained by computing the second iterated kernel, $K_2(x, \xi)$, and J_2, using equation (7). For the cracked beam the expression for $K_2(x, \xi)$ is

$$K_2(x, \xi) = \left(\frac{mL^3}{EI}\right)^2 L\bar{K}_{2i}(\bar{x}, \bar{\xi}). \tag{23}$$

The functions $\bar{K}_{2i}(\bar{x}, \bar{\xi}; \delta, \beta, \alpha)$ constitute a set of six different functions ($i = 1, 2, \ldots, 6$) that must be applied depending on the relative position of variables, $\bar{x}$, $\bar{\xi}$ and δ, with the same meaning as for $\bar{c}_i(x, \xi)$. Details of the calculation of these functions are given in Appendix B. To calculate J_2 (equation (7)) it is necessary to obtain $K_2(x, x)$. Note that $\bar{K}_{21}(\bar{x}, \bar{x}) = \bar{K}_{22}(\bar{x}, \bar{x})$ (necessary to build $K_2(x, x)$ for case $\bar{x} < \delta$), and $\bar{K}_{25}(\bar{x}, \bar{x}) = \bar{K}_{26}(\bar{x}, \bar{x})$ (necessary to compute $K_2(x, x)$ for case $\bar{x} > \delta$). Therefore, the function $K_2(x, x)$ has the following expressions:

$$K_2(x, x) = \left(\frac{mL^3}{EI}\right)^2 L\bar{K}_{21}(\bar{x}, \bar{x}), \quad \bar{x} < \delta, \tag{24}$$

$$K_2(x, x) = \left(\frac{mL^3}{EI}\right)^2 L\bar{K}_{25}(\bar{x}, \bar{x}), \quad \bar{x} > \delta \tag{25}$$

and, from equations (7), (24) and (25), J_2 can be obtained as

$$J_2 = \left(\frac{mL^4}{EI}\right)^2 \left(\int_0^\delta \bar{K}_{21}(\bar{x}, \bar{x})\,\mathrm{d}\bar{x} + \int_\delta^1 \bar{K}_{25}(\bar{x}, \bar{x})\,\mathrm{d}\bar{x}\right). \tag{26}$$

From the above calculations, approximate values (lower bounds) of the fundamental frequency can be obtained as

First lower bound:

$$\omega_1 > \left(\frac{1}{J_1}\right)^{1/2} = f_1(\delta, \beta, \alpha)\sqrt{\frac{EI}{mL^4}}. \tag{27}$$

Second lower bound:

$$\omega_1 > \left(\frac{1}{J_1}\right)^{1/4} = f_2(\delta, \beta, \alpha)\sqrt{\frac{EI}{mL^4}}. \tag{28}$$

The expressions of the functions f_1 and f_2, which depend on the boundary conditions, are given in Appendix C.

In this way, closed-form expressions for two lower bounds of the natural frequency of the cracked beam were obtained.

4. COMPARISON WITH NUMERICAL APPROACH

To validate the proposed method, the results were compared with those obtained by numerical simulation using the ABAQUS finite-element code [23]. Several cracked beams with different boundary conditions and $\alpha = a/W$ values were considered:
The cases analyzed were:

- Case (a): simply supported cracked beam ($\delta = 0{\cdot}50$).
- Case (b): cantilever cracked beam ($\delta = 0{\cdot}75$).
- Case (c): fixed–pinned cracked beam ($\delta = 0{\cdot}50$).
- Case (d): fixed–fixed cracked beam ($\delta = 0{\cdot}75$).

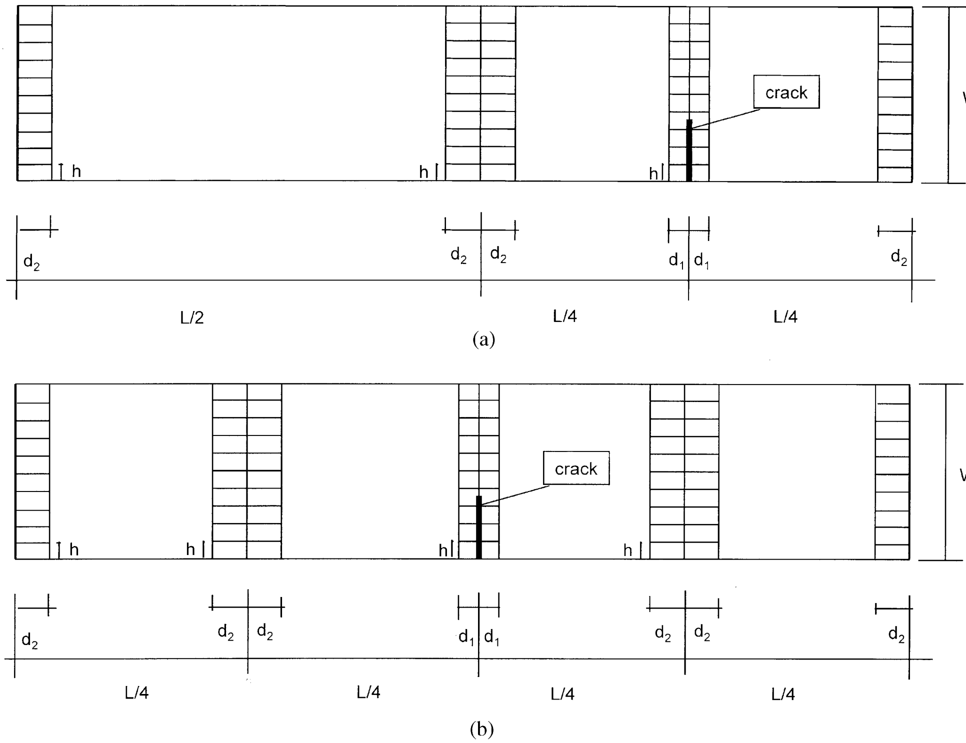

Figure 3. Meshes used in the numerical analysis: (a) cases of $b/L = 0{\cdot}75$; (b) cases of $b/L = 0{\cdot}50$.

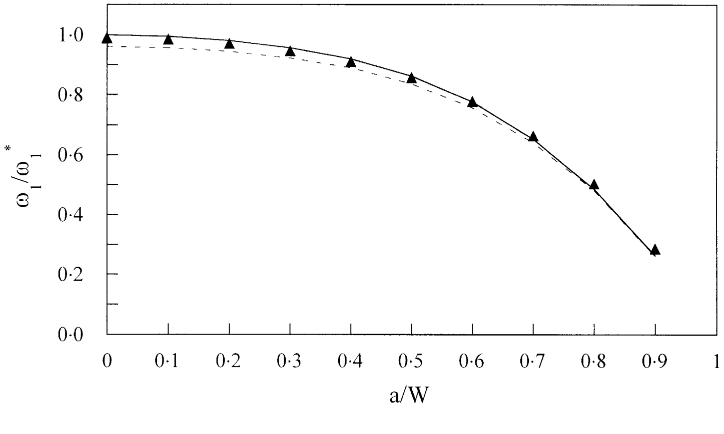

Figure 4. Variation of fundamental frequency with crack ratio of a simply supported beam ($b/L = 0{\cdot}50$, $\omega_1^* = 3595{\cdot}3$ rad/s). ▲ Numerical results; —— second lower bound; ---- first lower bound.

The beam was assumed to be $L = 200$ mm long, and rectangular cross-section (width, $W = 10$ mm and thickness, $B = 10$ mm). The beam material had a mass density $\rho = 7850\ \text{kg/m}^3$ and Young's modulus $E = 200$ GPa.

Details of the mesh used in the numerical analysis are given in Figure 3(a) (Cases (b) and (d), $\delta = 0{\cdot}75$) and Figure 3(b) (Cases (a) and (c), $\delta = 0{\cdot}50$). Finite element meshes of 1000 eight-node plane stress element were used. Four zones can be distinguished in every mesh:

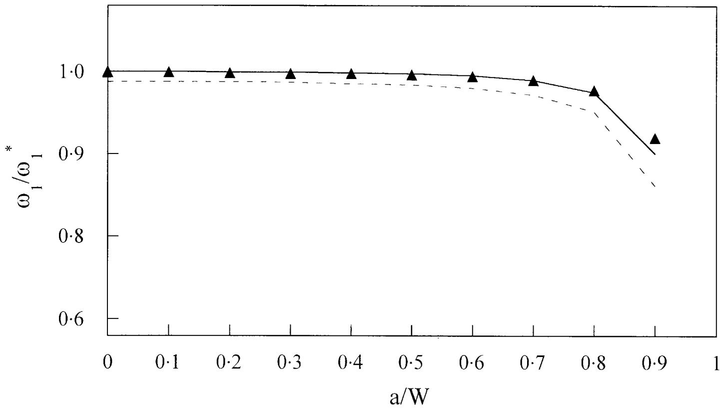

Figure 5. Variation of fundamental frequency with crack ratio of a Cantilever beam ($b/L = 0{\cdot}75$, $\omega_1^* = 1280{\cdot}8$ rad/s). ▲ Numerical results; —— second lower bound; ---- first lower bound.

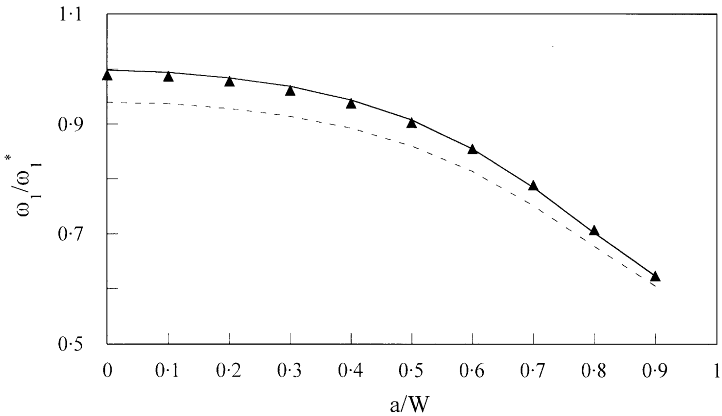

Figure 6. Variation of fundamental frequency with crack ratio of a fixed–pinned beam. ($b/L = 0{\cdot}50$, $\omega_1^* = 5616{\cdot}5$ rad/s). ▲ Numerical results; —— second lower bound; ---- first lower bound.

the two zones at each side of the cracked section (300 elements each) and two other zones (200 elements each). The height of all elements was $h/L = 1/200$ and their width varied from $d_1/L = 1/200$ near the crack to $d_2/L = 1/80$ far away. For each case, 10 analyses were made for values of the crack ratio, α, varying from 0 (uncracked beam) to 0·9.

Comparison of the numerical results with the two lower bounds of the fundamental frequency obtained by the proposed method are shown in Figure 4 (simply supported beam), Figure 5 (cantilever beam), Figure 6 (fixed–pinned beam) and Figure 7 (fixed–fixed beam). In these figures the ratio between the fundamental frequency of the cracked (ω_1) and uncracked beam (ω_1^*) as a function of the parameter $\alpha = a/W$ is shown. The numerical value of ω_1^* for each case appears in the corresponding figure.

In all the cases considered, the numerical values of the frequency ratio are less than 1 for values of $\alpha = 0$, (no crack). This is because the beam is numerically modelled as a two-dimensional solid and thus shear effect in the beam was implicitly taken into account. Therefore, although the length/width ratio of the beam is large, there is a small influence of the shear force in the numerical value of the fundamental frequency in comparison with

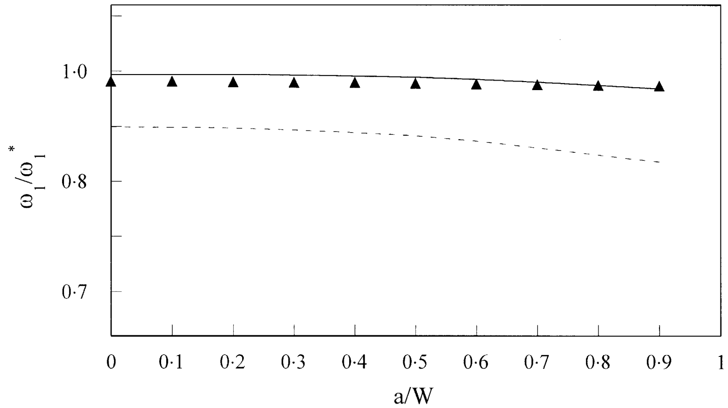

Figure 7. Variation of fundamental frequency with crack ratio of a fixed–fixed beam ($b/L = 0{\cdot}75$, $\omega_1^* = 8150{\cdot}0$ rad/s). ▲ Numerical results; —— second lower bound; - - - - first lower bound.

those derived from the one-dimensional Euler–Bernoulli beam theory, in which no shear effect is considered. This effect, that tends to reduce the fundamental frequency, becomes more important in cases of fixed–pinned (Figure 6) and fixed–fixed (Figure 7) beams than in the other two analyzed cases (simply supported and cantilever beams).

In all the cases considered, the first lower bound underestimates the numerical value of the fundamental frequency and the second lower bound agrees very well with the numerical results. So this second lower bound can be considered an excellent approximation to the fundamental frequency of bending vibrations of cracked Euler–Bernoulli beams.

5. SUMMARY AND CONCLUSIONS

The method of flexibility influence functions is used to approximate the fundamental frequency for bending vibrations of cracked Euler–Bernoulli beams with different boundary conditions. The presence of the crack is considered by means of a rotational elastic spring and, so the flexibility influence functions are constructed taking into account the boundary conditions and the discontinuity in the slope of the beam at the cracked section. This procedure leads to an eigenvalue problem formulated in integral form and its solution provides closed-form expressions for the successive lower bounds of the fundamental frequency. The results are compared with those obtained numerically by a finite element computer code and in all cases the results predicted from the closed-form expressions for the second lower bound are very close to those obtained from the FEM.

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APPENDIX A: FLEXIBILITY FUNCTIONS

A.1. SIMPLY SUPPORTED BEAM

$$\bar{c}_1 = \tfrac{1}{6}\bar{x}(2\bar{\xi} + \bar{\xi}(\bar{x}^2 + \bar{\xi}^2) - \bar{x}^2 - 3\bar{\xi}^2 + 6\beta(1-\delta)^2\bar{\xi}\Theta), \tag{A1}$$

$$\bar{c}_2 = \tfrac{1}{6}\bar{\xi}(2\bar{x} + \bar{x}(\bar{x}^2 + \bar{\xi}^2) - 3\bar{x}^2 - \bar{\xi}^2 + 6\beta(1-\delta)^2\bar{x}\Theta), \tag{A2}$$

$$\bar{c}_3 = \tfrac{1}{6}\bar{\xi}(1-\bar{x})(2\bar{x} - \bar{x}^2 - \bar{\xi}^2 + 6\beta(1-\delta)\delta\Theta), \tag{A3}$$

$$\bar{c}_4 = -\tfrac{1}{6}\bar{x}(1-\bar{\xi})(\bar{x}^2 + \bar{\xi}(\bar{\xi}-2) - 6\beta(1-\delta)\delta\Theta), \tag{A4}$$

$$\bar{c}_5 = \tfrac{1}{6}(1-\bar{\xi})(\bar{x}(-\bar{x}^2 + 2\bar{\xi} - \bar{\xi}^2) + 6\beta(1-\bar{x})\delta^2\Theta), \tag{A5}$$

$$\bar{c}_6 = \tfrac{1}{6}(1-\bar{x})(\bar{\xi}(-\bar{x}^2 + 2\bar{x} - \bar{\xi}^2) + 6\beta(1-\bar{\xi})\delta^2\Theta). \tag{A6}$$

A.2. CANTILEVER BEAM

$$\bar{c}_1 = -\tfrac{1}{6}\bar{x}^2(\bar{x} - 3\bar{\xi}) \tag{A7}$$

$$\bar{c}_2 = \tfrac{1}{6}\bar{\xi}^2(3\bar{x} - \bar{\xi}) \tag{A8}$$

$$\bar{c}_3 = \tfrac{1}{6}\bar{\xi}^2(3\bar{x} - \bar{\xi}) \tag{A9}$$

$$\bar{c}_4 = -\tfrac{1}{6}\bar{x}^2(\bar{x} - 3\bar{\xi}) \tag{A10}$$

$$\bar{c}_5 = -\tfrac{1}{6}(\bar{x}^2(\bar{x} - 3\bar{\xi}) + 6\beta(\bar{x}-\delta)(\delta-\bar{\xi})\Theta) \tag{A11}$$

$$\bar{c}_6 = \tfrac{1}{6}(\bar{\xi}^2(3\bar{x} - \bar{\xi}) - 6\beta(\bar{x}-\delta)(\delta-\bar{\xi})\Theta) \tag{A12}$$

A.3. FIXED–PINNED BEAM

$$\bar{c}_1 = -\frac{\bar{x}^2((1-\bar{\xi})(2\bar{x} - 6\bar{\xi} - \bar{x}\bar{\xi}^2 + \bar{\xi}(2\bar{x} + 3\bar{\xi})) + 6\beta(1-\delta)^2(\bar{x} - 3\bar{\xi})\Theta)}{12(1 + 3\beta(1-\delta)^2\Theta)}, \tag{A13}$$

$$\bar{c}_2 = \frac{\bar{\xi}^2((1-\bar{x})(6\bar{x} - 2\bar{\xi} + \bar{x}^2\bar{\xi} - \bar{x}(3\bar{x} + 2\bar{\xi})) + 6\beta(1-\delta)^2(3\bar{x} - \bar{\xi})\Theta)}{12(1 + 3\beta(1-\delta)^2\Theta)}, \tag{A14}$$

$$\bar{c}_3 = \frac{\bar{\xi}^2(1-\bar{x})(6\bar{x} - 2\bar{\xi} + \bar{x}^2\bar{\xi} - \bar{x}(3\bar{x} + 2\bar{\xi}) + 6\beta(1-\delta)^2(3\delta - \bar{\xi})\Theta)}{12(1 + 3\beta(1-\delta)^2\Theta)}, \tag{A15}$$

$$\bar{c}_4 = \frac{\bar{x}^2(1-\bar{\xi})(6\bar{\xi} - 2\bar{x} + \bar{x}\bar{\xi}^2 - \bar{\xi}(2\bar{x} + 3\bar{\xi}) + 6\beta(\bar{x} - 3\delta)(\delta - 1)\Theta)}{12(1 + 3\beta(1-\delta)^2\Theta)}, \tag{A16}$$

$$\begin{aligned}\bar{c}_5 = {} & \frac{(1-\bar{\xi})}{12(1 + 3\beta(1-\delta)^2\Theta)}(\bar{x}^2(6\bar{\xi} - 2\bar{x} + \bar{x}\bar{\xi}^2 - \bar{\xi}(2\bar{x} + 3\bar{\xi})) - 6\beta(2(\bar{x}-\delta)(\delta - \bar{\xi}) \\ & + \bar{x}^3 - 3\bar{x}^2\delta + 2\bar{x}\delta\bar{\xi} - 2\delta^2\bar{\xi} + \bar{x}\bar{\xi}^2 - \delta\bar{\xi}^2 + \delta(-\bar{x}^3 + 3\bar{x}^2\delta - \bar{x}\bar{\xi}^2 + \delta\bar{\xi}^2))\Theta),\end{aligned} \tag{A17}$$

$$\begin{aligned}\bar{c}_6 = {} & \frac{(1-\bar{x})}{12(1 + 3\beta(1-\delta)^2\Theta)}(\bar{\xi}^2(6\bar{x} - 2\bar{\xi} + \bar{x}^2\bar{\xi} - \bar{x}(3\bar{x} + 2\bar{\xi})) - 6\beta(2(\bar{x}-\delta)(\delta - \bar{\xi}) \\ & - \delta\bar{x}^2\bar{\xi}^3 - 2\bar{x}\delta^2 + \bar{x}^2\bar{\xi} + 2\bar{x}\bar{\xi}\delta - 3\delta\bar{\xi}^2 + \bar{\xi}^3\delta(\bar{x}^2\delta - \bar{x}^2\bar{\xi} + 3\delta\bar{\xi}^2 - \bar{\xi}^3))\Theta).\end{aligned} \tag{A18}$$

A.4. FIXED–FIXED BEAM

$$\bar{c}_1 = -\frac{\bar{x}^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}((1-\bar{\xi})^2(\bar{x}-3\bar{\xi}+2\bar{x}\bar{\xi}) + 2\beta(2\bar{x}-6\bar{\xi}-6\delta(\bar{x}-3\bar{\xi})$$

$$+ 6\delta^2(\bar{x}-3\bar{\xi}) - (\bar{x}-3\delta)(3\delta-\bar{\xi})\bar{\xi}^2)\Theta), \tag{A19}$$

$$\bar{c}_2 = \frac{\bar{\xi}^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}((1-\bar{x})^2(3\bar{x}-\bar{\xi}-2\bar{x}\bar{\xi}) + 2\beta(6\bar{x}-2\bar{\xi}+6\delta^2(3\bar{x}-\bar{\xi})$$

$$+ \bar{x}^2(\bar{x}-3\delta)(3\delta-\bar{\xi}) + 6\delta(\bar{\xi}-3\bar{x}))\Theta), \tag{A20}$$

$$\bar{c}_3 = \frac{(1-\bar{x})^2\bar{\xi}^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}(3\bar{x}-\bar{\xi}-2\bar{x}\bar{\xi}+2\beta(2+\bar{x}-3\delta)(3\delta-\bar{\xi})\Theta), \tag{A21}$$

$$\bar{c}_4 = -\frac{(1-\bar{\xi})^2\bar{x}^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}(\bar{x}-3\bar{\xi}+2\bar{x}\bar{\xi}+2\beta(\bar{x}-3\delta)(2-3\delta+\bar{\xi})\Theta), \tag{A22}$$

$$\bar{c}_5 = -\frac{(1-\bar{\xi})^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}(\bar{x}^3(2\bar{\xi}+1) - 3\bar{x}^2\bar{\xi} + 2\beta(3(\bar{x}-\delta)(\delta-\bar{\xi})$$

$$- \bar{x}^2(\bar{x}-3\delta)(3\delta-\bar{\xi}) + 2(\bar{x}^3-3\bar{x}^2\delta+3\bar{x}\delta\bar{\xi}-3\delta^2\bar{\xi}))\Theta), \tag{A23}$$

$$\bar{c}_6 = -\frac{(1-\bar{x})^2}{6(1+4\beta(1-3\delta+3\delta^2)\Theta)}(\bar{\xi}^3(2\bar{x}+1) - 3\bar{x}\bar{\xi}^2 + 2\beta(3(\bar{x}-\delta)(\delta-\bar{\xi})$$

$$- \bar{\xi}^2(\bar{x}-3\delta)(3\delta-\bar{\xi}) + 2(\bar{\xi}^3-3\bar{\xi}^2\delta+3\bar{x}\delta\bar{\xi}-3\delta^2\bar{x}))\Theta). \tag{A24}$$

APPENDIX B: CALCULATION OF $\bar{K}_{2i}$

$$\bar{K}_{21}(\bar{x},\bar{\xi}) = \int_0^{\bar{x}} \bar{c}_2(\bar{x},\eta)c_1(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{x}}^{\bar{\xi}} \bar{c}_1(\bar{x},\eta)\bar{c}_1(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\bar{\xi}}^{\delta} \bar{c}_1(\bar{x},\eta)\bar{c}_2(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\delta}^{1} \bar{c}_4(\bar{x},\eta)\bar{c}_3(\eta,\bar{\xi})\,\mathrm{d}\eta, \tag{B1}$$

$$\bar{K}_{22}(\bar{x},\bar{\xi}) = \int_0^{\bar{\xi}} \bar{c}_2(\bar{x},\eta)c_1(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{\xi}}^{\bar{x}} \bar{c}_2(\bar{x},\eta)\bar{c}_2(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\bar{x}}^{\delta} \bar{c}_1(\bar{x},\eta)\bar{c}_2(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\delta}^{1} \bar{c}_4(\bar{x},\eta)\bar{c}_3(\eta,\bar{\xi})\,\mathrm{d}\eta, \tag{B2}$$

$$\bar{K}_{23}(\bar{x},\bar{\xi}) = \int_0^{\bar{\xi}} \bar{c}_3(\bar{x},\eta)\bar{c}_1(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{\xi}}^{\delta} \bar{c}_3(\bar{x},\eta)\bar{c}_2(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\delta}^{\bar{x}} \bar{c}_3(\bar{x},\eta)\bar{c}_3(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{x}}^{1} \bar{c}_5(\bar{x},\eta)\bar{c}_3(\eta,\bar{\xi})\,\mathrm{d}\eta, \tag{B3}$$

$$\bar{K}_{24}(\bar{x},\bar{\xi}) = \int_0^{\bar{x}} \bar{c}_2(\bar{x},\eta)\bar{c}_4(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{x}}^{\delta} \bar{c}_1(\bar{x},\eta)\bar{c}_4(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\delta}^{\bar{\xi}} \bar{c}_4(\bar{x},\eta)\bar{c}_5(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{\xi}}^{1} \bar{c}_4(\bar{x},\eta)\bar{c}_6(\eta,\bar{\xi})\,\mathrm{d}\eta, \tag{B4}$$

$$\bar{K}_{25}(\bar{x},\bar{\xi}) = \int_0^{\delta} \bar{c}_3(\bar{x},\eta)c_4(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\delta}^{\bar{x}} \bar{c}_6(\bar{x},\eta)\bar{c}_5(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\bar{x}}^{\bar{\xi}} \bar{c}_5(\bar{x},\eta)\bar{c}_5(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{\xi}}^{1} \bar{c}_5(\bar{x},\eta)\bar{c}_6(\eta,\bar{\xi})\,\mathrm{d}\eta, \tag{B5}$$

$$\bar{K}_{26}(\bar{x},\bar{\xi}) = \int_0^{\delta} \bar{c}_3(\bar{x},\eta)\bar{c}_4(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\delta}^{\bar{\xi}} \bar{c}_6(\bar{x},\eta)\bar{c}_5(\eta,\bar{\xi})\,\mathrm{d}\eta$$

$$+ \int_{\bar{\xi}}^{\bar{x}} \bar{c}_6(\bar{x},\eta)\bar{c}_6(\eta,\bar{\xi})\,\mathrm{d}\eta + \int_{\bar{x}}^{1} \bar{c}_5(\bar{x},\eta)\bar{c}_6(\eta,\bar{\xi})\,\mathrm{d}\eta. \tag{B6}$$

APPENDIX C. EXPRESSIONS FOR f_1 AND f_2

C.1. SIMPLY SUPPORTED BEAM

$$f_1 = 9{\cdot}48683\left(\frac{1}{1 + 30\beta(1-\delta)^2\delta^2\Theta}\right)^{1/2}, \tag{C1}$$

$$f_2 = 9{\cdot}85957\left(\frac{1}{1 + 20\beta\delta^2(2 - 7\delta^2 + 14\delta^4 - 12\delta^5 + 3\delta^6)\Theta + 1050\beta^2(1-\delta)^4\delta^4\Theta^2}\right)^{1/4}. \tag{C2}$$

C.2. CANTILEVER BEAM

$$f_1 = 3{\cdot}4641\left(\frac{1}{1 + 4\beta(1-\delta)^3\Theta}\right)^{1/2}, \tag{C3}$$

$$f_2 = 3{\cdot}5154\left(\frac{1}{1 + \frac{8}{33}\beta(1-\delta)^4(33 + 41\delta + 29\delta^2 + 2\delta^3 + 70\beta(1-\delta)^2\Theta)\Theta}\right)^{1/4}. \tag{C4}$$

C.3. FIXED–PINNED BEAM

$$f_1 = 14{\cdot}4914\left(\frac{1 + 3\beta(1-\delta)^2\Theta}{1 + \frac{7}{2}\beta(1-\delta)^2(2 - 8\delta + 12\delta^2 + 12\delta^3 - 3\delta^4)\Theta}\right)^{1/2}, \tag{C5}$$

$$f_2 = 15{\cdot}3814\left(\frac{(1 + 3\beta(1-\delta)^2\Theta)^2}{1 + \frac{7}{104}\beta(1-\delta)^2(h_1 + 33\beta(1-\delta)^2 h_2\Theta)\Theta}\right)^{1/4}, \tag{C6}$$

$$h_1 = 2(104 - 352\delta + 176\delta^2 + 704\delta^3 + 2552\delta^4 - 2728\delta^5 - 1276\delta^6 + 1496\delta^7 - 187\delta^8), \tag{C7}$$

$$h_2 = 24 - 192\delta + 672\delta^2 - 1024\delta^3 + 80\delta^4 + 1856\delta^5 + 272\delta^6 - 232\delta^7 + 29\delta^8. \tag{C8}$$

C.4. FIXED–FIXED BEAM

$$f_1 = 20{\cdot}4939\left(\frac{1 + 4\beta(1 - 3\delta + 3\delta^2)\Theta}{1 + 4\beta(2 - 14\delta + 42\delta^2 - 35\delta^3 - 35\delta^4 + 63\delta^5 - 21\delta^6)\Theta}\right)^{1/2}, \tag{C9}$$

$$f_2 = 22{\cdot}2699\left(\frac{(1 + 4\delta(1 - 3\delta + 3\delta^2)\Theta)^2}{1 + \frac{8}{71}\beta(h_1 + 6h_2\beta\Theta)\Theta}\right)^{1/4}, \tag{C10}$$

$$h_1 = 142 - 882\delta + 1806\delta^2 + 3465\delta^4 - 22869\delta^5 + 27027\delta^6 + 6930\delta^7 - 31185\delta^8 + 19635\delta^9 - 3927\delta^{10}, \tag{C11}$$

$$h_2 = 104 - 1456\delta + 9464\delta^2 - 35574\delta^3 + 79002\delta^4 - 82698\delta^5 - 26103\delta^6 + 163086\delta^7 - 139755\delta^8 - 7700\delta^9 + 75229\delta^{10} - 40194\delta^{11} + 6699\delta^{12}. \tag{C12}$$