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LIMIT CYCLE FLUTTER OF AIRFOILS IN STEADY AND UNSTEADY FLOWS

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The limit cycle flutter of a two-dimensional wing with non-linear pitching stiffness is investigated. For modelling the aerodynamic forces of the wing steady linear and non-linear models as well as an unsteady model were used. The flutter speed was calculated using the harmonic balance method and by predicting Hopf bifurcation. Analytical solutions based on the centre manifold theory and normal forms were obtained as were results given by the harmonic balance method. The analytical solutions were compared with those obtained by numerical integration. The results show that the harmonic balance method can forecast flutter speed with a good accuracy while analytical solutions based on centre manifold theorem are accurate only in a small neighbourhood of the bifurcation point. The oscillation of the airfoil after flutter for two different models, linear and non-linear pitching stiffness were compared with each other and the flutter speeds for two linear steady and an unsteady aerodynamic model calculated. The obtained results show that flutter analysis based on the linear steady model is conservative only for the ratios of plunge frequency to pitch frequency lower than 1.

1. INTRODUCTION

Dynamical analysis of non-linear wing structures has been the subject of recent investigations. By introducing non-linear stiffness, Zhao and Yang [1] showed that the oscillation of the wing might become chaotic for certain positions of the elastic axis. Liu and Zhao [2] studied the bifurcation in the oscillation of an airfoil with pitch non-linearity. They used a steady aerodynamic model and showed that results obtained by the harmonic balance method are in good agreement with other methods. Yang [3] studied the limit cycle flutter of two models of a wing/store combination, a two-dimensional wing and a delta wing. In his study a linear model for the elastic behaviour of the wing in pitch was assumed. He showed that in some cases the features of the limit cycle flutter obtained by numerical integration can be predicted by the equivalent linearization method based on the second order asymptotic solution of the KBM method.

A two-dimensional flexible airfoil with a freeplay non-linearity in pitch in the subsonic flow was analyzed by Kim and Lee [4]. They used a doublet lattice method to obtain unsteady aerodynamic forces and showed that flexibility effect on flutter speed is significant when the frequency ratio of the pitch to plunge is > 1.

The present work uses a two-dimensional airfoil in incompressible flow with non-linear stiffness and linear viscous damping. In most investigations researchers have used a linear steady aerodynamic model to define the aerodynamic forces applied to the wing. Here,

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however, steady linear and non-linear and also unsteady aerodynamic models have been assumed, and the effect of these models on the flutter speed has been studied.

2. EQUATIONS OF MOTION

A schematic representation of a two-dimensional airfoil section is presented in Figure 1 with the following parameters: $\mu = 12{\cdot}8$, $a = -0{\cdot}41$, $b = 0{\cdot}118$, $c = 0{\cdot}2$, $\omega_h = 34{\cdot}6$, $\omega_\alpha = 88$, $r_\alpha^2 = 0{\cdot}3$, $x_\alpha = 0{\cdot}15$ where $\mu = m/\pi\rho b^2$ is the airfoil air-mass ratio, ρ is the air density, m is the mass of the wing per span length, b is the semi-chord length, ab is the distance of the elastic axis E from the mid-chord length, $(0{\cdot}5 + a)b$ is the distance of E from the aerodynamic centre A, $x_\alpha b$ is the distance of the center of gravity G from E and is positive when G is located aft of E towards the trailing edge, $r_\alpha b$ is the radius of gyration of the airfoil with respect to the elastic axis, and ω_h, ω_α are the eigenfrequencies of the constrained one-degree-of-freedom system associated with the linear plunging and the pitching springs respectively. $C = c_\alpha/\pi\rho b^4\omega_\alpha = c_h/\pi\rho b^2\omega_\alpha$ is the non-dimensional damping coefficient where c_α, c_h are the coefficients of damping in pitch and plunge respectively.

In terms of non-dimensional time $\tau = t\omega_\alpha$ and non-dimensional plunge displacement $H = h/b$ the equations of motion are:

$$\begin{aligned}
&\mu\left(\frac{\mathrm{d}^2H}{\mathrm{d}\tau^2}\right) + \mu\, x_\alpha\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2}\right) + C\left(\frac{\mathrm{d}H}{\mathrm{d}\tau}\right) + \mu\left(\frac{\omega_h}{\omega_\alpha}\right)^2 H = -\frac{Q_L}{\pi\rho b^3\omega_\alpha^2},\\
&\mu\, x_\alpha\left(\frac{\mathrm{d}^2H}{\mathrm{d}\tau^2}\right) + \mu\, r_\alpha^2\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2}\right) + C\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right) + \mu\, r_\alpha^2\alpha = \frac{Q_\alpha}{\pi\rho b^4\omega_\alpha^2},
\end{aligned} \tag{1}$$

where Q_L, Q_α are aerodynamic lift and aerodynamic moment of lift around the elastic axis.

By introducing a cubic pitching stiffness term, the governing equations are obtained as:

$$\begin{aligned}
&12{\cdot}8H'' + 1{\cdot}92\alpha'' + 0{\cdot}2H'' + 1{\cdot}97H = -\frac{Q_L}{\pi\,\rho\, b^3\,\omega_\alpha^2},\\
&1{\cdot}92H'' + 3{\cdot}84\alpha'' + 0{\cdot}2\alpha' + 3{\cdot}84\alpha + \frac{e}{\pi\rho b^4\omega_\alpha^2}\alpha^3 = \frac{Q_\alpha}{\pi\rho b^4\omega_\alpha^2},
\end{aligned} \tag{2}$$

where e is a non-linear stiffness factor and as in reference [2], its numerical value is 20.

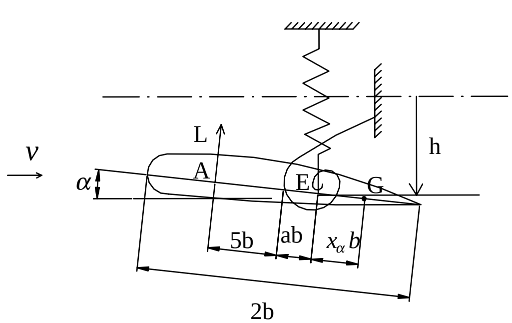

Figure 1. Sketch of a two-dimensional airfoil.

3. AERODYNAMIC MODEL

3.1. STEADY FLOW WITH A LINEAR MODEL

To obtain the aerodynamic lift and moment of the airfoil in a steady flow, the data from wind tunnel tests for a section NACA 23024 was used and a linear regression was assumed for angles of attack (α) between -10 and $5°$ [5].

By employing a linear model for steady flow, equations (2) become:

$$12{\cdot}8H'' + 1{\cdot}92\alpha'' + 0{\cdot}2H' + 1{\cdot}97H = -\frac{1}{\pi}\left(\frac{v}{b\omega_\alpha}\right)^2(5{\cdot}5682\alpha + 0{\cdot}0942),$$

$$1{\cdot}92H'' + 3{\cdot}84\alpha'' + 0.2\alpha' + 3{\cdot}84\alpha + 4{\cdot}24\alpha^3 = \frac{1}{\pi}\left(\frac{v}{b\omega_\alpha}\right)^2(1{\cdot}4845\alpha + 0{\cdot}0084). \tag{3}$$

where v is the flow velocity.

3.2. STEADY FLOW WITH A NON-LINEAR MODEL

Polynomials of degrees 5 and 3 were fitted, respectively, for the aerodynamic lift and aerodynamic moment using the experimental data available for NACA 23024. To model the aerodynamic forces of the wing for angles between -10 and $20°$ a fifth order polynomial was used. For this model the governing equations of motion are:

$$12{\cdot}8H'' + 1{\cdot}92\alpha'' + 0{\cdot}2H' + 1{\cdot}97H = -\frac{1}{\pi}\left(\frac{v}{b\omega_\alpha}\right)^2[-453{\cdot}6\alpha^5 + 22{\cdot}4\alpha^3 + 5{\cdot}42\alpha + 0{\cdot}096]$$

$$1{\cdot}92H'' + 3{\cdot}84\alpha'' + 0{\cdot}2\alpha' + 3{\cdot}84\alpha + 4{\cdot}24\alpha^3 = \frac{1}{\pi}\left(\frac{v}{b\omega_\alpha}\right)^2[-40{\cdot}824\alpha^5 - 3{\cdot}184\alpha^3 - 1{\cdot}932\alpha^2 + 1{\cdot}4058\alpha + 0{\cdot}00864]. \tag{4}$$

3.3. UNSTEADY MODEL

To obtain the aerodynamic forces of an oscillating airfoil usually a simple harmonic oscillation of infinitesimal amplitude is superposed on the airfoil. Since arbitrary motions of an airfoil can be decomposed into harmonic components by means of Fourier analysis, harmonic oscillation forms the basis of general airfoil theory in unsteady flow. In this study the results obtained by Theodorsen [6] for an airfoil with harmonic oscillation have been used.

By employing an unsteady model for the aerodynamic forces, equations (2) become:

$$13{\cdot}8H'' + 2{\cdot}33\alpha'' + (0{\cdot}2 + 0{\cdot}1926v)H' + 1{\cdot}9787H + 0{\cdot}2715v\alpha' + 0{\cdot}01854v^2\alpha = 0$$

$$2{\cdot}33H'' + 4{\cdot}1331\alpha'' + (0{\cdot}2 + 0{\cdot}0718v)\alpha' + (3{\cdot}84 - 0{\cdot}001669v^2)\alpha + 4{\cdot}24\alpha^3 - 0{\cdot}01733vH' = 0. \tag{5}$$

4. HARMONIC BALANCE METHOD

To find the flutter speed by a first order harmonic balance method the stiffness term $k_0\alpha + e\alpha^3$ in equation (3) is replaced by an equivalent stiffness term $k_\alpha\alpha$. Consequently, the flutter determinant for steady flow with a linear model can be written as:

$$\begin{vmatrix} -12{\cdot}8\Omega^2 + 0{\cdot}2\mathrm{i}\Omega + 1{\cdot}9787 & -1{\cdot}92\Omega^2 + 0{\cdot}01643 v_f^2 \\ -1{\cdot}92\Omega^2 & -3{\cdot}84\Omega^2 + 0{\cdot}2\mathrm{i}\Omega + k_\alpha - 0{\cdot}00438\, v_f^2 \end{vmatrix} = 0, \qquad (6)$$

where $\Omega = \omega_f/\omega_\alpha$ is the non-dimensional flutter frequency and ω_f is flutter frequency and v_f is flutter speed. The equivalent linear stiffness is

$$k_\alpha = k_0 + \frac{3}{4} e A^2, \qquad (7)$$

where A is the amplitude of α in the limit cycle. By separating real and imaginary parts of equation (6) the following equations can be obtained:

$$\begin{aligned} &45{\cdot}46\Omega^4 + 1{\cdot}9787k_\alpha - 12{\cdot}8\Omega^2 k_\alpha + 0{\cdot}0876\Omega^2 v_f^2 \\ &\quad - 0{\cdot}00867 v_f^2 - 7{\cdot}6382\Omega^2 = 0, \\ &0{\cdot}3957\Omega - 3{\cdot}328\Omega^3 + 0{\cdot}2\Omega\, k_\alpha - 0{\cdot}000876\Omega\, v_f^2 = 0. \end{aligned} \qquad (8)$$

To find the flutter speed, A was assumed to be zero in equation (7). The analysis is valid for airspeeds lower than the static divergence speed v_D [2], which can be found from equation (3) to be $\sqrt{k_0 \pi b^2 \omega_\alpha^2 / 1{\cdot}4845}$ where k_0 is 3·84. For this value of k_0 the static divergence speed is 29·6 m/s. The amplitude of the limit cycle after flutter for different speeds found by solving the equations (8) together is shown in Figure 2. For other aerodynamic models the flutter speed using the harmonic balance method was calculated (Table 1). The equivalent linear terms $\frac{5}{8} A^4 \alpha, 0$ were used for nonlinear terms α^5, α^2 in the non-linear aerodynamic model.

For $v < v_f = 17{\cdot}19$ m/s the static equilibrium point is a stable focus while for $v > v_f$ it becomes unstable and a limit cycle develops. The amplitude of this limit cycle will grow as v increases (Figure 2).

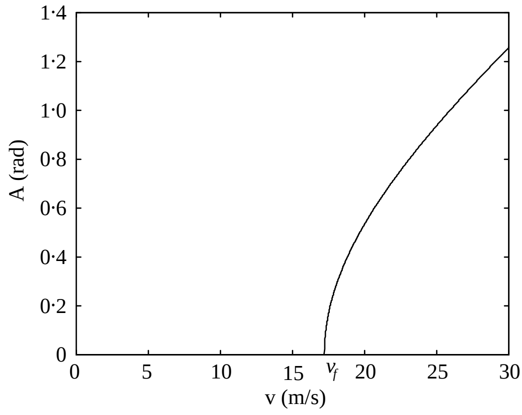

Figure 2. Amplitude of α in the limit cycle with variation of flow velocity (obtained from linear steady model).

TABLE 1

Flutter speed and flutter frequency using harmonic balance

Aerodynamic model	Linear steady model	Non-linear steady model	Unsteady model
v_f(m/s)	17·19	17·54	19·38
v_D(m/s)	29·6	30·42	47·96
Ω	0·521	0·522	0·823
$\omega_{f(\mathrm{rad/s})}$	45·848	45·936	72·424

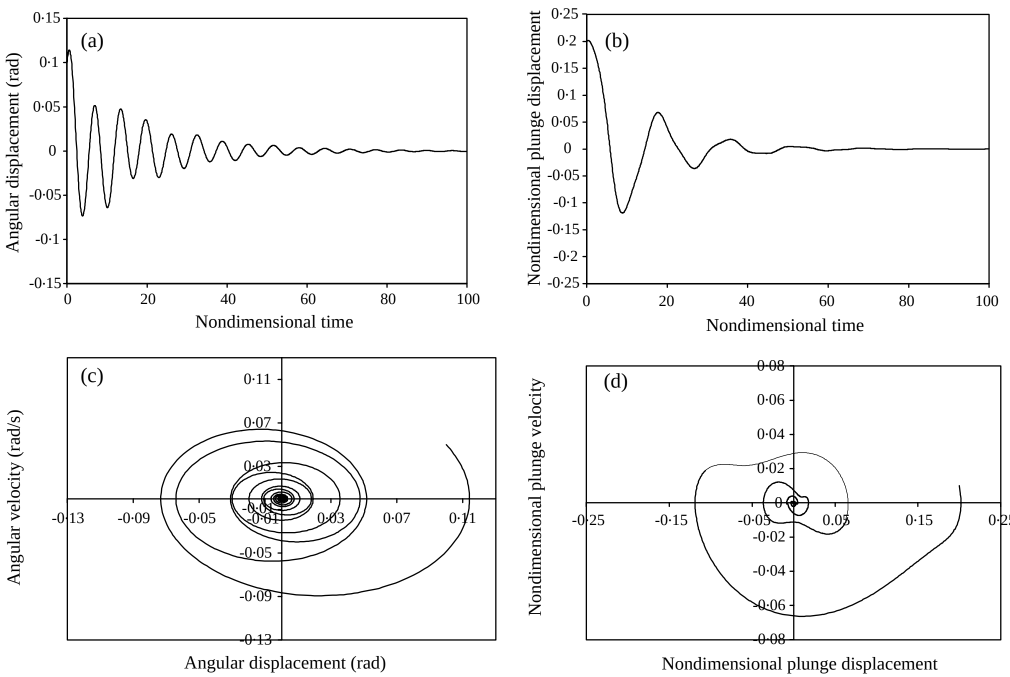

Figure 3. Time histories and phase diagrams of the airfoil response in unsteady flow ($v = 8$ m/s).

5. NUMERICAL INTEGRATION

A fifth order Runge–Kutta subroutine was used as numerical integration method and the oscillation of the airfoil with unsteady aerodynamic model for some velocities before and after flutter was studied (Figures 3–8).

For step-by-step integration, equations (5) should be written as a first order differential equation. The state variables are defined as:

$$\mathbf{X} = (x_1, x_2, x_3, x_4)^{\mathrm{T}} = (\alpha, \alpha', H, H')^{\mathrm{T}}, \tag{9}$$

and the state space form of equations (5) is

$$\mathbf{X}' = \mathbf{AX} + f(\mathbf{X}), \tag{10}$$

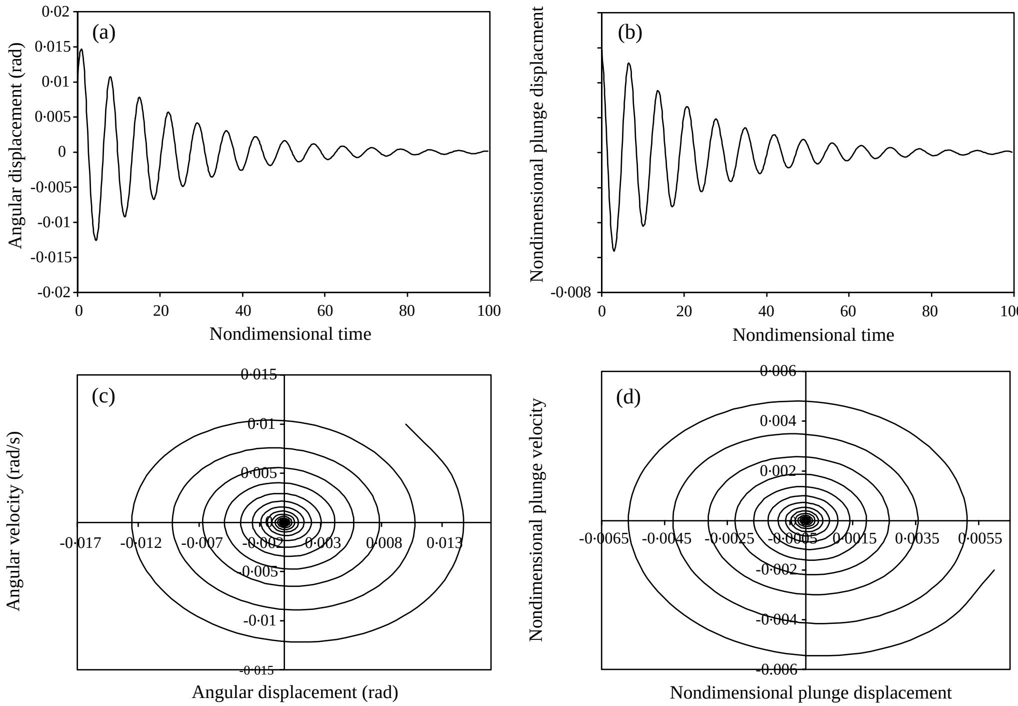

Figure 4. Time histories and phase diagrams of the airfoil response in unsteady flow ($v = 15$ m/s).

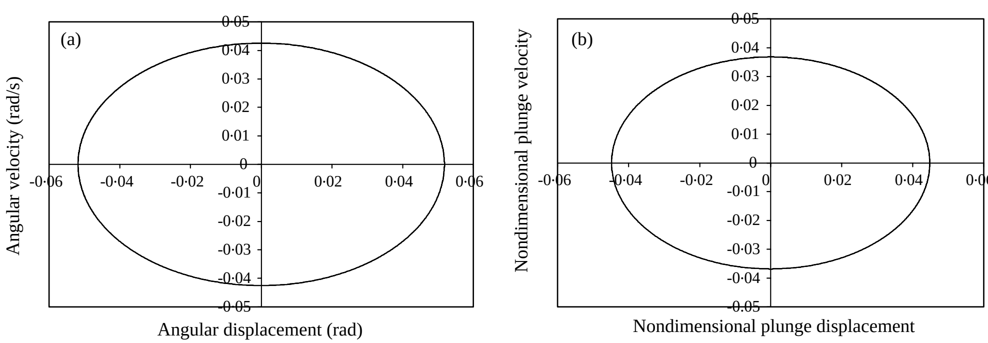

Figure 5. Limit cycle appeared in phase diagram after flutter ($v = 19{\cdot}5$ m/s).

where

$$
\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1{\cdot}0268 + 0{\cdot}0012v^2 & -0{\cdot}0535 - 0{\cdot}0069v & 0{\cdot}0895 & 0{\cdot}009 + 0{\cdot}0133v \\ 0 & 0 & 0 & 1 \\ 0{\cdot}1734 - 0{\cdot}0015v^2 & -0{\cdot}0185v + 0{\cdot}009 & -0{\cdot}1586 & -0{\cdot}016 - 0{\cdot}0162v \end{bmatrix},
$$

$$
f(\mathbf{X}) = (0 \quad -1{\cdot}1338 \quad 0 \quad 0{\cdot}1914)^{\mathrm{T}} x_1^3 .
$$

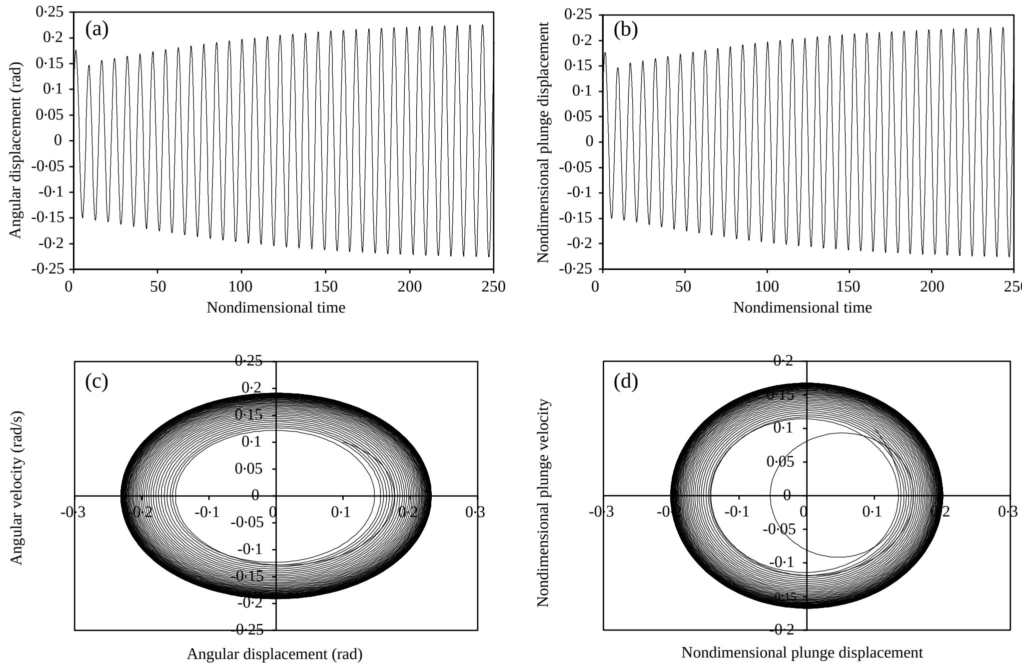

Figure 6. Time histories and phase diagrams of the airfoil response in unsteady flow with initial conditions in the limit cycle ($v = 20$ m/s).

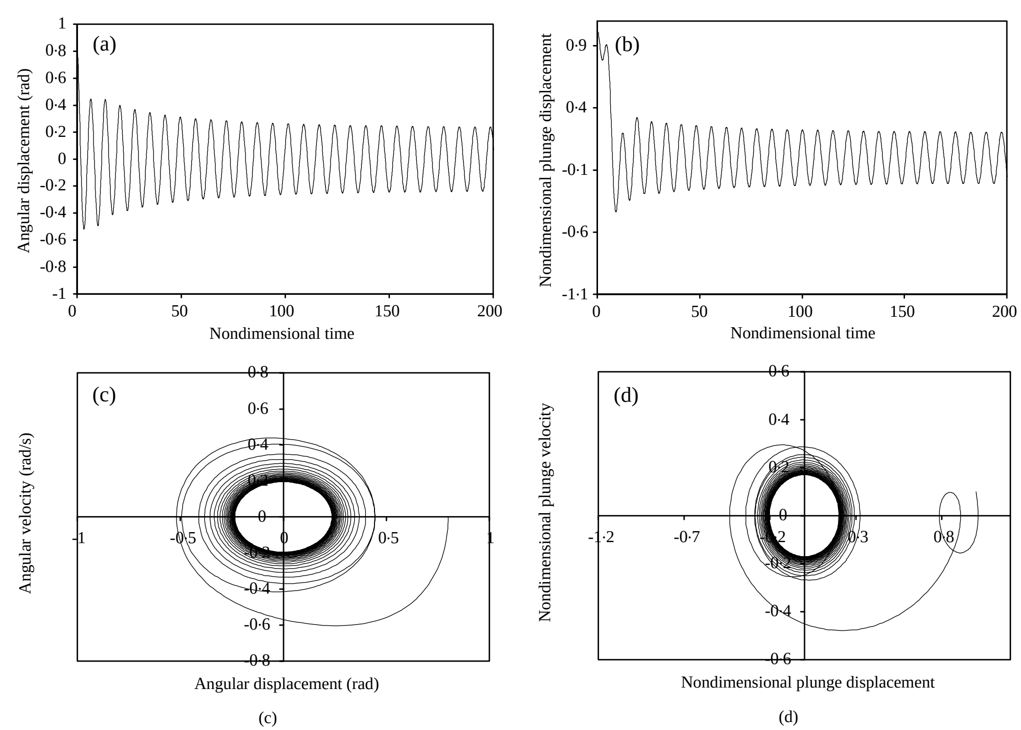

Figure 7. Time histories and phase diagrams of the airfoil response in unsteady flow with initial conditions out of the limit cycle ($v = 20$ m/s).

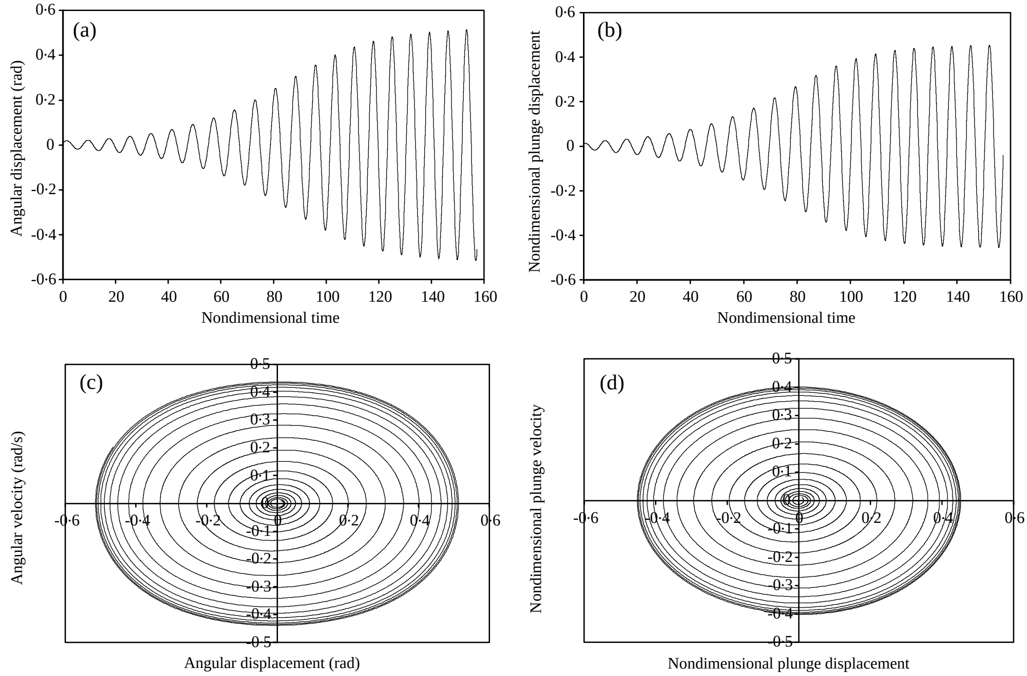

Figure 8. Time histories and phase diagrams of the airfoil response in unsteady flow ($v = 22$ m/s).

TABLE 2

Amplitude of limit cycle for various flow speeds (unsteady model)

v(m/s)	Amplitude of α	
	Numerical integration	Harmonic balance
19·5	0·0516	0·1196
19·8	0·1812	0·2113
20	0·2308	0·2551
21	0·3989	0·4128
22	0·5203	0·5299
24	0·7146	0·7189
26	0·8793	0·8794
30	1·1688	1·1616
34	1·4316	1·4179

To calculate the flutter speed by numerical integration the velocity of the flow was increased from 19 m/s until the limit cycle in the phase plane appeared. The approximate value of the flutter speed was seen to be 19·468 m/s. The amplitude of α in the limit cycle for different flow velocities after flutter is recorded in Table 2.

6. THE METHOD OF NORMAL FORM AND CENTRE MANIFOLD THEORY

For system (10) from numerical integration it was seen that bifurcation occurs at $v = v_B = 19{\cdot}468$. Let $u = v - v_B$ be the new control parameter, then at $u = 0$ the four eigenvalues of the linearized system are:

$$\lambda_{1,2} = \pm 0{\cdot}8216\mathrm{i}, \quad \lambda_{3,4} = -0{\cdot}2615 \pm 0{\cdot}3372\mathrm{i}. \tag{11}$$

To obtain the Jordan form of matrix **A** the transformation $\mathbf{X} = \mathbf{PY}$ was used, where **P** is a (4×4) matrix formed by real and imaginary parts of the eigenvectors, and $\mathbf{Y} = (y_1 \quad y_2 \quad y_3 \quad y_4)^{\mathrm{T}}$. By employing this transformation equation (10) is reduced to

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{bmatrix} 0 & -0{\cdot}8216 & 0 & 0 \\ 0{\cdot}8216 & 0 & 0 & 0 \\ 0 & 0 & -0{\cdot}2615 & -0{\cdot}3372 \\ 0 & 0 & 0{\cdot}3372 & -0{\cdot}2615 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$+ u \begin{bmatrix} -0{\cdot}0102 & 0{\cdot}0302 & -0{\cdot}0084 & 0{\cdot}0162 \\ 0{\cdot}0018 & 0{\cdot}0373 & -0{\cdot}0076 & 0{\cdot}0188 \\ 0{\cdot}0035 & 0{\cdot}0858 & -0{\cdot}0176 & 0{\cdot}0432 \\ -0{\cdot}0109 & -0{\cdot}0659 & 0{\cdot}0119 & -0{\cdot}0326 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$+ u^2 \begin{bmatrix} -0{\cdot}000147 & 0{\cdot}000959 & -0{\cdot}000178 & 0{\cdot}000243 \\ -0{\cdot}000171 & 0{\cdot}001118 & -0{\cdot}000208 & 0{\cdot}000284 \\ 0{\cdot}000394 & 0{\cdot}002574 & -0{\cdot}000478 & 0{\cdot}000653 \\ 0{\cdot}000297 & -0{\cdot}00194 & 0{\cdot}000361 & 0{\cdot}000493 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$+ \begin{Bmatrix} 1{\cdot}8909 \\ 0{\cdot}9185 \\ 2{\cdot}1836 \\ -0{\cdot}8764 \end{Bmatrix} (0{\cdot}0883 y_1 - 0{\cdot}5764 y_2 + 0{\cdot}1072 y_3 - 0{\cdot}1464 y_3)^3 \cdot \tag{12}$$

According to the centre manifold theorem equation (12) has a centre manifold $y_3 = h_3(y_1, y_2, u)$, $y_4 = h_4(y_1, y_2, u)$ where functions h_3, h_4 can be approximated by polynomials of arbitrary degree in terms of y_1, y_2, u that satisfy the conditions

$$\frac{\partial h_i}{\partial y_j}(0,0,0) = 0, \quad h_i(0,0,0) = 0,$$

$$i = 3{,}4 \quad j = 1, 2.$$

In this study quadratic and cubic polynomials for h_3, h_4 were used. The quadratic polynomials are

$$\begin{aligned} h_3 &= -0{\cdot}1285573 y_1 u + 0{\cdot}0273443 y_2 u, \\ h_4 &= 0{\cdot}0436085 y_1 u - 0{\cdot}0799786 y_2 u \end{aligned} \tag{13}$$

and the cubic ones

$$\begin{aligned} h'_3 &= 0{\cdot}408947y_1^3 - 0{\cdot}034560y_2^3 + 0{\cdot}574003y_1y_2^2 - 0{\cdot}128557y_1u \\ &\quad + 0{\cdot}027344y_2u - 0{\cdot}104424y_1^2y_2 - 0{\cdot}000447y_1u^2 + 0{\cdot}005406y_2u^2, \\ h'_4 &= -0{\cdot}058314y_1^3 + 0{\cdot}184948y_2^3 - 0{\cdot}131264y_1y_2^2 + 0{\cdot}043608y_1u \\ &\quad - 0{\cdot}079978y_2u + 0{\cdot}185689y_1^2y_2 - 0{\cdot}005736y_1u^2 - 0{\cdot}000867y_2u^2. \end{aligned} \tag{14}$$

By substituting h_3, h_4 from equation (13) in equation (12) and according to the trivial equation $\dot{u} = 0$ the flow on the centre manifold is obtained as

$$\begin{aligned} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} &= \begin{bmatrix} 0 & -0{\cdot}8216 \\ 0{\cdot}8216 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \\ &\quad + u \begin{bmatrix} -0{\cdot}01025 & 0{\cdot}03017 & -0{\cdot}00846 & 0{\cdot}01617 \\ 0{\cdot}00186 & 0{\cdot}03733 & -0{\cdot}00761 & 0{\cdot}01880 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ h_3 \\ h_4 \end{Bmatrix} \\ &\quad + u^2 \begin{bmatrix} -0{\cdot}00015 & 0{\cdot}00095 & -0{\cdot}00018 & 0{\cdot}00024 \\ -0{\cdot}00017 & 0{\cdot}00111 & -0{\cdot}00020 & 0{\cdot}00028 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ h_3 \\ h_4 \end{Bmatrix} \\ &\quad + \begin{bmatrix} 1{\cdot}89094 \\ 0{\cdot}91855 \end{bmatrix} (0{\cdot}08832y_1 - 0{\cdot}57644y_2 + 0{\cdot}10722h_3 - 0{\cdot}14641h_4)^3. \end{aligned} \tag{15}$$

To find the periodic orbits after flutter the method of normal form was used. The normal form of equation (15) is written in the complex plane by the transformation:

$$\begin{aligned} y_1 &= \xi + \bar{\xi}, \\ y_2 &= \mathrm{i}(\xi - \bar{\xi}). \end{aligned} \tag{16}$$

Using the non-linear transformation $\xi = \eta + \varepsilon h(\eta, \bar{\eta}) + \cdots$ where

$$\dot{\eta} = \mathrm{i}\eta + \varepsilon g(\eta, \bar{\eta}) \tag{17}$$

and the equation $r\dot{r} = \mathrm{Re}(\dot{\eta}\bar{\eta})$, the differential equation (15) can be written in polar co-ordinates [7]

$$\begin{aligned} \dot{r} &= r^3(-0{\cdot}57512 \times 10^{-5}u^3 + 0{\cdot}00010u^2 - 0{\cdot}00037u - 0{\cdot}04622) \\ &\quad + r(0{\cdot}25706 \times 10^{-5}u^3 + 0{\cdot}00052u^2 + 0{\cdot}01354u). \end{aligned} \tag{18}$$

The non-zero equilibrium point r_0 of equation (18) is the amplitude of the limit cycle of the system in polar co-ordinates and the sign of $f'(r)$ determines the stability of the limit

TABLE 3

Amplitude of limit cycle for various flow velocities (unsteady model)

v(m/s)	Amplitude of α		
	Second degree polynomial	Third degree polynomial	Numerical integration
19·5	0·0564	0·0565	0·0516
19·8	0·1810	0·1830	0·1812
20	0·2284	0·2326	0·2308
21	0·3827	0·4041	0·3989
22	0·4863	0·5360	0·5203
24	0·6380	0·8105	0·7146
26	0·7538	—	0·8793
30	0·9348	—	1·1688
34	1·0806	0·9317	1·4316

TABLE 4

Flutter speed and flutter frequency obtained by Hopf bifurcation

	Linear steady model	Non-linear steady model	Unsteady model
v_f(m/s)	17·19	17·57	19·35
Ω	0·5214	0·5228	0·8235
$\omega_{f(\mathrm{rad/s})}$	45·883	46·006	72·468

cycle. By transforming back to the physical co-ordinate, α becomes

$$\alpha = r_0[(0{\cdot}088329 - 0{\cdot}020169u)\ \cos\varphi + (-0{\cdot}576441 + 0{\cdot}014642u)\ \sin\varphi] \tag{19}$$

and the amplitude of α in the limit cycle is

$$\alpha = r_0\sqrt{(0{\cdot}088329 - 0{\cdot}020169u)^2 + (-0{\cdot}576441 + 0{\cdot}014642u)^2}\,. \tag{20}$$

For cubic polynomials we have

$$\begin{aligned} \dot{r} &= 0{\cdot}000127r^9 + r^7(0{\cdot}45168\times10^{-5}u^2 - 0{\cdot}00013u + 0{\cdot}00092) \\ &+ r^5(0{\cdot}73991\times10^{-7}u^4 - 0{\cdot}32010\times10^{-5}u^3 + 0{\cdot}000053u^2 - 0{\cdot}00067u + 0{\cdot}0057) \\ &+ r^3(0{\cdot}90699\times10^{-9}u^6 - 0{\cdot}19488\times10^{-7}u^5 - 0{\cdot}92180\times10^{-7}u^4 - 0{\cdot}000014u^3 \\ &+ 0{\cdot}00045u^2 - 0{\cdot}00095u - 0{\cdot}04622) \\ &+ r(-0{\cdot}13446\times10^{-5}u^4 - 0{\cdot}00007u^3 + 0{\cdot}00052u^2 + 0{\cdot}01354u) = 0. \end{aligned} \tag{21}$$

The amplitude of the limit cycle for various flow velocities is recorded in Table 3.

7. HOPF BIFURCATION

The flutter speed and its frequency calculated by predicting Hopf bifurcation are recorded in Table 4. This was done through Bifpack program [8] that uses a trial and error scheme and predicts Hopf bifurcation.

Flutter speeds and frequencies obtained by Hopf bifurcation are equal to those calculated from harmonic balance method. However, the flutter frequency adjusts itself to the flutter speed such that equations (8) are always satisfied. The difference in flutter frequencies for different aerodynamic models as recorded in Tables 1 and 4 can be explained in the light of the aforementioned equation.

8. CONCLUSIONS

A two-dimensional airfoil with different aerodynamic models was studied and the dynamic responses for steady and unsteady aerodynamic models and for different flow speeds were investigated. The results obtained indicate that:

1. The flutter speed obtained from linear steady model and non-linear steady model are nearly equal to each other. However, the flutter speed calculated from an unsteady model is different to the steady model (Figure 9). For $\omega_h/\omega_\alpha < 1$ the flutter speed calculated from an unsteady model is greater than a steady model speed, but when $\omega_h/\omega_\alpha > 1$ a steady model shows a greater flutter speed than an unsteady model.
2. As Liu and Zhao [2] mentioned, the harmonic balance method can provide useful information about the flutter speed and the amplitude of limit cycle after flutter.
3. By comparing the results obtained from analytical methods and numerical integration it is seen that the results obtained by centre manifold theory and normal form are

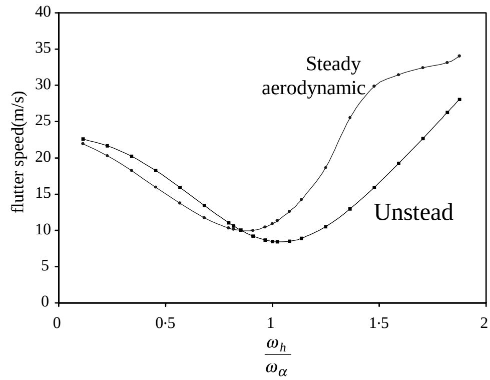

Figure 9. Flutter speed calculated for steady and unsteady aerodynamic models for different ratios of ω_h/ω_α.

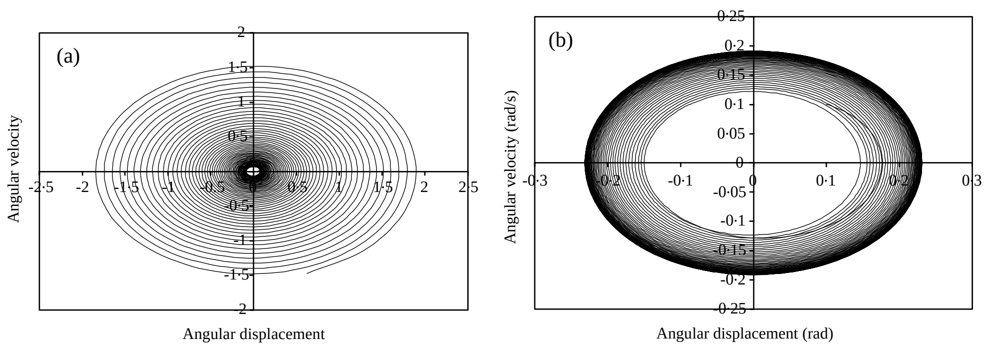

Figure 10. Phase diagrams for flow speed of 20 m/s. (a) Linear spring model; (b) non-linear spring model.

satisfactory only in the neighbourhood of the critical point. The degrees of polynomials used to approximate functions h_3, h_4 can increase the boundary of accuracy.

4. The existence of cubic non-linearity in the stiffness term did not affect the flutter speed, but the responses of the two systems with linear stiffness (Figure 10(a)) and non-linear stiffness (Figure 10(b)) after flutter is different from each other. The conclusion is correct for other types of non-linearity. Similar to the cubic non-linearity, an equivalent linear model in α can be used as follows:

$$k_0\alpha + k_1\alpha^3 + k_2\alpha^5 + \cdots = k_0\alpha + \tfrac{3}{4}k_1A^2\alpha + \tfrac{5}{8}k_2A^4\alpha + \cdots .$$

Since in calculating the flutter speed, A is assumed to be zero in the above equation, the type of non-linearity does not affect the flutter speed.

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