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MULTI-MODE TRIMMING OF IMPERFECT THIN RINGS USING MASSES AT PRE-SELECTED LOCATIONS

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This paper presents a practical method for trimming the natural frequencies of an initially imperfect ring to simultaneously eliminate certain of the frequency splits present. Compared with previous work, the novel feature of this method is that the trimming masses are positioned at pre-selected locations on the ring. The basis for the proposed method is the concept of equivalent imperfection mass, which allows any imperfect ring to be considered as a perfect ring with equivalent imperfection masses attached. By considering this trimming problem it is deduced that it is possible to trim N pairs of modes simultaneously by removing (a minimum of) $2N$ trimming masses at particular locations around the ring. By positioning the trimming masses at pre-selected locations, it is shown that a simple set of trimming masses can be calculated easily, and from this set an infinite number of solution sets can be found. Methods for generating these sets are outlined for the trimming of both a single and a dual pair of modes. In practice, it is likely that the trimming masses will be spaced regularly. For this special case, it is found that it is not possible to trim all single- and dual-mode pairs with any arrangement of masses. Validation of the derived simple solution set and the proposed procedure to generate further sets is achieved by studying a number of theoretical examples.

1. INTRODUCTION

In many practical engineering applications, for example bells [1], automotive or aircraft tyres [2] and circular gyroscopes [3, 4], the natural frequencies and mode shapes of nominally axisymmetric structures are of interest. In principle, for a given nodal configuration, the modes of vibration for these structures occur in degenerate pairs that have equal natural frequencies, are spatially orthogonal and have indeterminate angular positions. In practice, due to manufacturing tolerances and material non-uniformity, it is not possible to manufacture a "perfect" ring. This has the consequence that the previously equal natural frequencies become split and the indeterminacy of the modes is removed, fixing the modes within the structure. These qualitative effects of the imperfections on the vibrational modes have been well understood since the study performed by Tobias [5] in 1957.

The literature contains a variety of papers that quantitatively analyze the influence of particular types of imperfection on the vibration characteristics of axisymmetric structures. Laura *et al.* [6] considered the effect of circumferential variations in wall thickness on the axisymmetric modes of a ring using both a Rayleigh–Ritz and a finite element analysis. Tonin and Bies [7] used a Rayleigh–Ritz analysis to consider the influence of circumferential variations in the wall thickness of an eccentric cylinder. More recently, Hwang *et al.* [8–10] used the Rayleigh–Ritz analysis to consider general profile variations of a ring, whilst Eley *et al.* [11] considered the influence of anisotropy on the vibration of circular crystalline silicon rings.

The Rayleigh–Ritz analysis has the advantage over other methods in that relatively simple expressions for the natural frequencies and mode orientations can be obtained. This method is based on the assumption that the imperfections of the structure are small such that the mode shapes are unchanged by the presence of any imperfection. This method has been used by Fox [12, 13] for initially perfect circular rings with attached masses and springs and was compared with the receptance method, utilizing the modal expansion method, used by Allaei *et al.* [2]. Other works investigating the effect of added imperfections include those by Charnley and Perrin who investigated the addition of regularly spaced point masses to a circular ring using group theory [14] and a perturbation analysis [15].

In some applications, for example gyroscopic devices, it is necessary to reduce the frequency splits to the order of 0·01% to maintain strong resonant coupling between a given pair of modes. For this reason, some literature [12, 13, 16, 17] considers the inverse problem in which it is necessary to modify the structure to negate the influence of imperfection.

Rourke *et al.* [16, 17] reported the results of an investigation that extended and generalized the procedure and results given by Fox [12] to the simultaneous trimming of more than one pair of modes using more than one trimming mass. This utilized the concept of an "equivalent imperfection mass" that Fox [12] introduced, and extended it to include a set of "equivalent imperfection masses". Thus, as Fox [12] showed that it was possible to consider the inverse (trimming) problem by removing the "equivalent imperfection mass" and so trim one mode of the ring, Rourke *et al.* [17] showed that it was possible to trim a set of modes of the ring by removing a set of "equivalent imperfection masses". A general method for simultaneously trimming N pairs of modes using N trimming masses was outlined, with emphasis given to the special cases of single- and dual-mode trimming as well as the general multi-mode case. Central to the trimming procedure was the determination of the size and angular position around the circumference of the ring of the N trimming masses. In a practical situation, the need to locate the angular positions of the trimming masses for each trimming procedure may introduce unnecessary errors into the trimming of imperfect rings. This forms the main motivation for the current work.

The principal aim of this paper is to report an alternative procedure for eliminating the frequency splits from an imperfect ring. This is a further extension of the work by Fox [12], in which the possible errors that may be introduced by the procedure outlined in reference [17] are reduced by predetermining the angular positions of the trimming masses. The magnitudes (only) of the trimming masses then remain to be determined. Section 2 outlines the general equations from which these trimming masses can be determined. Specific solutions for the frequency trimming of a single mode and a dual pair of modes are given in sections 3 and 4 respectively. These sections include an extension to the equations to provide the ability to trim the modes to a "target" natural frequency. In addition, they investigate the special case of regularly spaced trimming masses and deduce rules for the existence of trimming solutions. Numerical examples demonstrating the developed methods are presented in section 5.

2. THE TRIMMING PROBLEM

Throughout this paper one considers an imperfect ring for which the natural frequencies of the pair of orthogonal in-plane modes having n_j nodal diameters are ω_{n_j1} and ω_{n_j2}. Furthermore, it is assumed that the radial (w) and tangential displacements (u) of the ring in

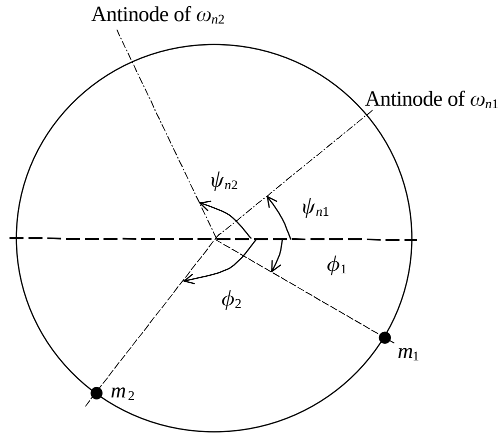

Figure 1. The imperfection masses and the generated imperfections.

these two pairs of modes are identical to those of a perfect ring and are given by

$$\begin{aligned} w_{n_j1} &= W_{n_j1} \cos n_j(\phi - \psi_{n_j1}) \exp(\mathrm{i}\omega_{n_j1} t), \quad u_{n_j1} = U_{n_j1} \sin n_j(\phi - \psi_{n_j1}) \exp(\mathrm{i}\omega_{n_j1} t), \\ w_{n_j2} &= W_{n_j2} \cos n_j(\phi - \psi_{n_j2}) \exp(\mathrm{i}\omega_{n_j2} t), \quad u_{n_j2} = U_{n_j2} \sin n_j(\phi - \psi_{n_j2}) \exp(\mathrm{i}\omega_{n_j2} t), \end{aligned} \tag{1–4}$$

where the mode orientations $\psi_{n_j1} = \psi_{n_j2} - \pi/2n_j$. This assumption has been tested in reference [17] using a Rayleigh–Ritz procedure, and was shown to be reasonable provided that the degree of imperfection was sufficiently small. It was also assumed that the effect of shear deformation and rotary inertia, which would have an effect on the mode shapes and natural frequencies of the in-plane modes, could be neglected. As the radial thickness of the ring increases with respect to its radius, there will be deviations in the mode shapes of the in-plane modes and their natural frequencies. However, for thin rings equations (1–4) are valid.

The first stage of the trimming problem is to calculate the magnitudes and the angular positions of masses which, when added to a perfect ring, produce the described "imperfect" natural frequencies and mode positions (see Figure 1). The concept of the equivalent imperfection mass was developed originally by Fox [12], and has been used previously by the authors [16, 17]. The main reason for its success is that, in conjunction with the Rayleigh–Ritz approach, it enables analytical expressions for the split natural frequencies to be obtained in terms of the orientations and magnitudes of the added masses. Following this approach it can be shown (see references [12, 17] for details) that the orientation, ψ_{n_j}, of a specific $n_j\theta$th mode and its split frequencies, ω_{n_j1} and ω_{n_j2}, are given by

$$\tan 2n_j\psi_{n_j} = \textstyle\sum_i m_i \sin 2n_j\phi_i / \sum_i m_i \cos 2n_j\phi_i, \tag{5}$$

$$\omega_{n_j1}^2 = \omega_{0n_j}^2 \left(\frac{1 + \alpha_{n_j}^2}{(1 + \alpha_{n_j}^2) + \sum_i m_i[(1 + \alpha_{n_j}^2) - (1 - \alpha_{n_j}^2) \cos 2n_j(\phi_i - \psi_{n_j})]/M_0} \right), \tag{6}$$

$$\omega_{n_j2}^2 = \omega_{0n_j}^2 \left(\frac{1 + \alpha_{n_j}^2}{(1 + \alpha_{n_j}^2) + \sum_i m_i[(1 + \alpha_{n_j}^2) + (1 - \alpha_{n_j}^2) \cos 2n_j(\phi_i - \psi_{n_j})]/M_0} \right), \tag{7}$$

respectively, where m_i and ϕ_i denote the magnitude and angular location of the ith added mass ($i = 1, 2, \ldots, N$), α_{n_j} is the amplitude ratio W_{n_j1}/U_{n_j1}, ω_{0n} is the natural frequency of the original perfect ring, and M_0 is the mass of the perfect ring.

Equations (5–7) provide a means of determining the frequency splits and mode orientations resulting from the addition of imperfection masses at particular locations around the circumference of an initially perfect ring. In what follows the inverse (trimming) problem is considered in which the aim is to determine the magnitude and angular location of the masses that need to be removed from the ring in order to eliminate splits between particular pairs of natural frequencies. For this purpose, it is convenient to express the mass of the perfect ring in terms of the mass of the imperfect ring M, which will be known at the outset of the trimming analysis, and the added masses, i.e.

$$M_0 = M - \sum_i m_i. \tag{8}$$

Substituting equation (8) into equations (6) and (7) yields

$$\omega_{n_j 1}^2 = \omega_{0n_j}^2 \left(\frac{(1+\alpha_{n_j}^2)(M - \sum_i m_i)}{M(1+\alpha_{n_j}^2) - (1-\alpha_{n_j}^2)\sum_i m_i \cos 2n_j(\phi_i - \psi_{n_j})} \right), \tag{9}$$

$$\omega_{n_j 2}^2 = \omega_{0n_j}^2 \left(\frac{(1+\alpha_{n_j}^2)(M - \sum_i m_i)}{M(1+\alpha_{n_j}^2) + (1-\alpha_{n_j}^2)\sum_i m_i \cos 2n_j(\phi_i - \psi_{n_j})} \right). \tag{10}$$

These equations relate the natural frequencies of the imperfect ring to the trimmed natural frequency ω_{0n_j}. Given that ω_{0n_j} will not be known at the outset of the trimming analysis, it is convenient to eliminate this term between equations (9) and (10) by dividing the two equations. Following this procedure, it can be shown that the relationship is obtained as

$$\sum_i m_i \cos 2n_j(\phi_i - \psi_{n_j}) = M\lambda_{n_j}, \tag{11}$$

where

$$\lambda_{n_j} = (\omega_{n_j 1}^2 - \omega_{n_j 2}^2)(1+\alpha_{n_j}^2)/(\omega_{n_j 1}^2 + \omega_{n_j 2}^2)(1-\alpha_{n_j}^2). \tag{12}$$

Before discussing the solution of the trimming problem using masses at pre-selected locations it is worthwhile noting that it is a simple task to calculate the trimmed natural frequencies by substituting equation (11) into either equation (9) or equation (10) such that

$$\omega_{0n_j}^2 = 2M\omega_{n_j 1}^2 \omega_{n_j 2}^2/(M - \textstyle\sum_i m_i)(\omega_{n_j 1}^2 + \omega_{n_j 2}^2). \tag{13}$$

3. TRIMMING USING MASSES AT PRE-SELECTED LOCATIONS

To trim the imperfect ring using trimming masses at pre-selected locations it is only necessary to determine the magnitude (m_i) of the trimming masses that satisfy both equations (5) and (11) for all of the modes considered. In practical applications of trimming, it is likely that the pre-selected locations (ϕ_i) will be chosen to be uniformly spaced around the circumference of the ring. This situation will form the basis for most of the examples considered in later sections, whilst for the purposes of analysis, the pre-selected locations will be kept as general as possible.

To facilitate the simultaneous solution of equations (5) and (11) it is sensible to re-write equation (5) in order to isolate the mass terms, such that

$$\sum_i m_i \sin 2n_j(\phi_i - \psi_{n_j}) = 0, \tag{14}$$

where standard trigonometric identities have been used in its formation.

So far, the number of pairs of modes to be trimmed has not been discussed. In what follows, it is assumed that it is required to trim J pairs of modes using N trimming masses. Denoting the J pairs of modes by $n_1, n_2, \ldots, n_J$ (where $n_j \neq n_k$ when $j \neq k$), the trimming problem reduces to finding the values for m_i that satisfy equations (11) and (14), where $j = 1, 2, \ldots, J$. Since the only variables in these equations are the magnitude and number of the trimming masses, it can be deduced that is necessary to have a minimum of twice as many trimming masses as modes to be trimmed, i.e., $N \geqslant 2J$.

In general the solutions to equations (11) and (14) do not guarantee that the trimmed natural frequency will take a specific value. However, in some applications it may be useful to trim the ring to a "target" natural frequency ω_{0n_j}. This situation will be denoted "target trimming" throughout what follows and can be achieved by solving equation (13) simultaneously with equations (11) and (14). This operation is simplified by rearranging equation (13) such that

$$\sum_{i=1}^{N} m_i = M\left(1 - \frac{2\omega_{n_j1}^2\omega_{n_j2}^2}{\omega_{0n_j}^2(\omega_{n_j1}^2 + \omega_{n_j2}^2)}\right) = M\delta_{n_j}. \tag{15}$$

This equation contains a simple sum of trimming masses, which has a direct effect on the ability to set "target" natural frequencies. It is possible to target the natural frequency of only one $n_j\theta$ mode onto a particular natural frequency ω_{0n_j}. This is because the value of δ_{n_j} is then fixed and the natural frequencies, ω_{0n_k}, of the other $n_k\theta$ modes, where $j \neq k$, are predetermined by the frequency ω_{0n_j}. Thus, equation (15) can only extend the trimming equations once and so the number of trimming masses required in that case is $N \geqslant 2J + 1$.

The solutions to equations (11) and (14), or equations (11), (14) and (15) are trivial if the number of trimming masses is equal to the number of equations that are to be solved. However, if there are more trimming masses to be applied than there are equations to be solved, an infinite number of solutions exist. Methods for generating workable solutions will be considered in sections 4 and 5 for the case of single- and dual-mode pairs.

4. SINGLE-MODE FREQUENCY TRIMMING

In this section, the trimming of a single mode will be considered using masses positioned at pre-selected locations. Initially, the so-called "simple trimming" situation will be considered in which the trimmed natural frequency is not targeted. Following this, the "target trimming" situation will be considered. Finally, the existence of invalid solutions for the special situation when the masses are equally spaced around the circumference of the ring will be investigated.

4.1. SIMPLE TRIMMING

From the analysis presented in section 3 it was shown that to trim the $n_j\theta$ mode, it is necessary to find the values of m_i that satisfy the equations

$$\sum_i m_i \cos 2n_j(\phi_i - \psi_{n_j}) = M\lambda_{n_j}, \quad \sum_i m_i \sin 2n_j(\phi_i - \psi_{n_j}) = 0, \tag{16, 17}$$

where the symbols are defined in section 3.

It can be seen easily from equation (17) that

$$m_1 = -\sum_{i=2}^{N} m_i \sin 2n_j(\phi_i - \psi_{n_j}) / \sin 2n_j(\phi_1 - \psi_{n_j}). \tag{18}$$

Substituting equation (18) into equation (16), it can be shown that

$$\sum_{i=2}^{N} m_i \sin 2n_j(\phi_1 - \phi_i) = M\lambda_{n_j} \sin 2n_j(\phi_1 - \psi_{n_j}). \tag{19}$$

Equations (18) and (19) are the two equations from which all of the simple single-mode trimming solutions can be generated. It can be seen that if $N = 2$, which is the trivial case, the solution to equation (19) is obvious. However, if $N > 2$ there are an infinite number of solutions to equation (19), and hence there are an infinite number of solutions to the trimming problem. One possible solution that can be taken directly from equation (19) and which can be used to generate multiple possible solution sets is to let

$$m_i = M\lambda_{n_j} \sin 2n_j(\phi_1 - \psi_{n_j})/(N-1) \sin 2n_j(\phi_1 - \phi_i), \quad i > 1. \tag{20}$$

From the solution set that comprises equations (18) and (20), multiple solution sets can be generated until the preferred set is formed. For example, the solutions can be modified to contain only positive or negative masses so that the trimming procedure consists of either the removal or addition of mass. Two such methods of achieving this are presented here.

The first method is to set any unwanted positive or negative trimming masses, which are solutions to equation (20), equal to zero and then reduce the term (N − 1) in the same equation by the number of masses that have been set to zero.

The second method involves the addition of an extra mass, M_k, to one of the trimming masses, m_k, where $k > 1$. Of course, this would introduce an incorrect solution to equation (19) that could then be brought back into equilibrium by the addition of a second mass, M_l, to another one of the trimming masses, m_l, where $l > 1$ and $k \neq l$. Using equation (20) it can be shown that the relationship between these two masses is

$$M_l = -M_k \sin 2n_j(\phi_1 - \phi_k)/\sin 2n_j(\phi_1 - \phi_l). \tag{21}$$

The effect that these two masses have on m_1 (see equation (18)) is to increase the trimming mass by M_1. Using equation (18) it can be shown that the size of this mass with respect to the extra mass M_k is

$$M_1 = \frac{M_k(\sin 2n_j(\phi_1 - \phi_k) \sin 2n_j(\phi_l - \psi_{n_j}) - \sin 2n_j(\phi_1 - \phi_l) \sin 2n_j(\phi_k - \psi_{n_j}))}{\sin 2n_j(\phi_1 - \psi_{n_j}) \sin 2n_j(\phi_1 - \phi_l)}. \tag{22}$$

Thus, knowing the initial modification to the trimming mass m_k, it is possible to predict the effect that this will have on mass m_1. This can also be performed in reverse. If a mass m_1 is too large or too small it is possible to specify a modification to m_1 and then use equation (22), in conjunction with equation (21), to calculate what values of M_k and M_l are required to keep the system in equilibrium.

4.2. TARGET TRIMMMING

To trim to a target frequency for the single-mode case it is necessary to determine solutions to equations (16), (17) and

$$\sum_{i=1}^{N} m_i = M\left(1 - \frac{2\omega_{n_j1}^2 \omega_{n_j2}^2}{\omega_{0n_j}^2(\omega_{n_j1}^2 + \omega_{n_j2}^2)}\right) = M\delta_{n_j}, \tag{23}$$

i.e., equation (15). Given the similarities with the "simple trimming" situation, the solution for the first mass, m_1, will be given by equation (18). Using equation (19), a solution for the second mass, m_2, can be found such that:

$$m_2 = [M\lambda_{n_j} \sin 2n_j(\phi_1 - \psi_{n_j}) - \textstyle\sum_{i=3}^{N} m_i \sin 2n_j(\phi_1 - \phi_i)] \sin 2n_j(\phi_1 - \phi_2). \qquad (24)$$

An equation for the remaining masses, m_i, can then be found by substituting equations (18) and (24) into equation (23) to give

$$\sum_{i=3}^{N} m_i \kappa_{2i} = M\left(\delta_{n_j} - \lambda_{n_j} \frac{\sin 2n_j(\phi_1 - \psi_{n_j}) - \sin 2n_j(\phi_2 - \psi_{n_j})}{\sin 2n_j(\phi_1 - \phi_2)}\right), \qquad (25)$$

where

$$\kappa_{ab} = \frac{\begin{pmatrix} \sin 2n_j(\phi_1 - \phi_a)(\sin 2n_j(\phi_1 - \psi_{n_j}) - \sin 2n_j(\phi_b - \psi_{n_j})) \\ - \sin 2n_j(\phi_1 - \phi_b)(\sin 2n_j(\phi_1 - \psi_{n_j}) - \sin 2n_j(\phi_a - \psi_{n_j})) \end{pmatrix}}{\sin 2n_j(\phi_1 - \phi_2) \sin 2n_j(\phi_1 - \psi_{n_j})}. \qquad (26)$$

Once again, there are an infinite number of solutions to equation (25) if $N > 3$. In a similar manner as for the simple case, one possible solution to equation (25) is

$$m_i = \frac{M}{(N-2)\kappa_{2i}}\left(\delta_{n_j} - \lambda_{n_j} \frac{\sin 2n_j(\phi_1 - \psi_{n_j}) - \sin 2n_j(\phi_2 - \psi_{n_j})}{\sin 2n_j(\phi_1 - \phi_2)}\right), \quad i > 2. \qquad (27)$$

As for the "simple trimming" case, the basic set of trimming masses can be derived from equation (27). Other sets can then be formed either by the negation of unwanted trimming masses or by the modification of the existing masses. This was illustrated by equations (21) and (22) for the simple case. In a similar fashion to the development of these equations it can be shown that these relationships must be satisfied:

$$M_1 = -M_k \frac{\kappa_{k2} \sin 2n_j(\phi_l - \psi_{n_j}) + \kappa_{2l} \sin 2n_j(\phi_k - \psi_{n_j}) + \kappa_{lk} \sin 2n_j(\phi_2 - \psi_{n_j})}{\kappa_{2l} \sin 2n_j(\phi_1 - \psi_{n_j})}, \qquad (28)$$

$$M_2 = M_k\, \kappa_{lk}/\kappa_{2l}, \quad M_l = M_k\, \kappa_{k2}/\kappa_{2l}. \qquad (29, 30)$$

These equations show the required relationship between the modifications to the trimming masses, which have been based on an addition of M_k to m_k and an addition of M_l to m_l. Equation (30) shows the relationship between these modifications, and equations (28) and (29) show what effect these modifications have on masses m_1 and m_2. Reversing equations (28) and (29), and using equation (30), it is possible to calculate the values of M_k and M_l that produce a specific modification to m_1 or m_2.

The validity of equations (24–30) will be tested later in section 6.

4.3. INVALID SOLUTIONS

The solutions described so far have been for general, non-specific angular positions. The most likely practical use of this method, however, will be to position the trimming masses at regularly spaced intervals around the circumference of the ring. For the case of N regularly spaced trimming masses, the location of the ith trimming mass is given by $\phi_i = 2\pi i/N$.

As an example, the arrangement for $N = 7$ is shown in Figure 2. In what follows, it will be shown that the number of regularly spaced masses is critical when trimming specific modes of vibration.

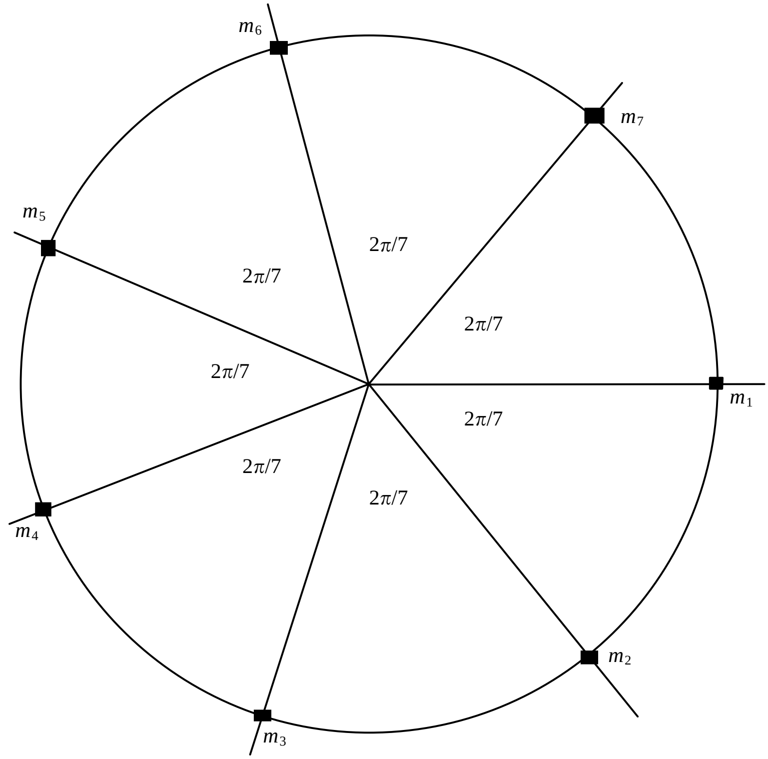

Figure 2. Angular positions of $N = 7$ trimming masses.

Substituting $\phi_i = 2\pi i/N$ into equation (19), it is easily shown that each term in the summation on the left side is zero when $N = 4n/k$, where n is the mode of vibration to be trimmed and k is an integer. In this situation, equation (19) is only satisfied when $\lambda_n = 0$ (i.e., the modes are already trimmed) or $\psi_n = \phi_1$ (i.e., the mode orientation is already lined up with the pre-selected position of the first trimming mass). However, in this latter situation it would not be possible to determine the magnitude of the first trimming mass from equation (18), and for this reason the rule can be deduced that there are no valid trimming solutions when $N = 4n/k$.

For single-mode frequency trimming, the minimum number of masses that can be used is two in order to provide a solution to equation (19). Applying the above rule eliminates the possibility of using two masses. In fact, the rule also automatically eliminates the use of four masses. These are the only combinations of trimming masses that allow no trimming of any modes. The other arrangements of trimming masses will be impractical for certain modes but not for all modes. The viability of specific combinations of modes of vibration and trimming masses for single-mode trimming for $n_j \leqslant 12$ and $N \leqslant 20$ is summarized in Table 1. The shaded squares mark the combinations that do not provide valid solutions. For $N > 20$, it will always be possible to find valid solutions for the first four modes of vibration.

5. DUAL-MODE FREQUENCY TRIMMING

In this section, the trimming of two pairs of modes will be considered using masses positioned at pre-selected locations. As in section 4, the situations of "simple" and "target"

TABLE 1

Combinations of N trimming masses that have valid solutions for the $n_j\theta$ mode

n_j \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	■		▒				▒												
3	■	▒	▒		▒						▒								
4	■		▒				▒								▒				
5	■		▒	▒					▒										▒
6	■	▒	▒		▒		▒				▒								
7	■		▒			▒							▒						
8	■		▒				▒								▒				
9	■	▒	▒		▒			▒			▒						▒		
10	■		▒	▒			▒		▒										▒
11	■		▒							▒									
12	■	▒	▒		▒		▒				▒				▒				

trimming will be considered, as will the existence of invalid solutions when equally spaced masses are used.

5.1. SIMPLE TRIMMING

For the case of dual-mode trimming, the problem is more complicated. Instead of two equations, it is necessary to solve four equations for the unknown mass values. Considering the $n_1\theta$ and $n_2\theta$ modes, the four equations can be formed from equations (16) and (17) and are

$$\sum_{i=1}^{N} m_i \sin 2n_1(\phi_i - \psi_{n_1}) = 0, \quad \sum_{i=1}^{N} m_i \cos 2n_1(\phi_i - \psi_{n_1}) = M\lambda_{n_1}, \tag{31, 32}$$

$$\sum_{i=1}^{N} m_i \sin 2n_2(\phi_i - \psi_{n_2}) = 0, \quad \sum_{i=1}^{N} m_i \cos 2n_2(\phi_i - \psi_{n_2}) = M\lambda_{n_2}. \tag{33, 34}$$

Equations (31–34) can be manipulated in a similar way as has been done for the single-mode frequency trimming to find solutions for trimming masses m_1, m_2, m_3 and m_i. A complete derivation is included in Appendix A but for clarity only the final four equations are shown:

$$m_1 = -\sum_{i=2}^{N} m_i \frac{\sin 2n_1(\phi_i - \psi_{n_1})}{\sin 2n_1(\phi_1 - \psi_{n_1})}, \quad m_2 = -\sum_{i=3}^{N} m_i \frac{\zeta_i}{\zeta_2}, \quad m_3 = \frac{M\lambda_{n_2}\zeta_2}{\chi_{13}} - \sum_{i=4}^{N} m_i \frac{\chi_{1i}}{\chi_{13}}, \tag{35–37}$$

$$\sum_{i=4}^{N} m_i(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}) = M\zeta_2(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23}), \tag{38}$$

where

$$\zeta_i = \sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_i - \psi_{n_2}) - \sin 2n_1(\phi_i - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2}), \tag{39}$$

$$\chi_{1i} = \sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_2 - \phi_i) + \sin 2n_1(\phi_2 - \psi_{n_1}) \sin 2n_2(\phi_i - \phi_1) + \sin 2n_1(\phi_i - \psi_{n_1}) \sin 2n_2(\phi_1 - \phi_2), \tag{40}$$

$$\chi_{2i} = \sin 2n_2(\phi_1 - \psi_{n_2}) \sin 2n_1(\phi_2 - \phi_i) + \sin 2n_2(\phi_2 - \psi_{n_2}) \sin 2n_1(\phi_i - \phi_1) + \sin 2n_2(\phi_i - \psi_{n_2}) \sin 2n_1(\phi_1 - \phi_2). \tag{41}$$

As with the single-mode trimming problem, there are an infinite number of solutions to equations (35–38) when $N > 4$, and one possible solution set consists of the solutions

$$m_i = M\zeta_2(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23})/(N-3)(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}), \quad i > 3. \tag{42}$$

This solution set can also be modified to consist of only positive or negative trimming masses, as will be shown in a later numerical example. One such method is to set any unwanted positive or negative trimming masses, which have been formed from equation (42), equal to zero and then reduce the term $(N-3)$ in equation (42) by the number of masses that have been set to zero.

As with the single-mode frequency trimming problem, another method involves the addition of an extra mass, M_k, to one of the trimming masses, m_k, where $k > 3$. This would introduce an incorrect solution to equation (38), which could then be brought back into equilibrium by the addition of a second mass, M_l, to another one of the trimming masses, m_l, where $l > 3$ and $k \neq l$. The relationship between these two masses is

$$M_l = -M_k((\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k})/(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l})). \tag{43}$$

The effect that masses M_k and M_l have on m_1, m_2 and m_3 is to increase each of those trimming masses by M_1, M_2 and M_3 respectively. The sizes of these masses with respect to the extra mass M_k are

$$M_1 = \frac{M_k\begin{pmatrix}(\chi_{1l}\chi_{2k} - \chi_{1k}\chi_{2l})(\zeta_2 \sin 2n_1(\phi_3 - \psi_{n_1}) - \zeta_3 \sin 2n_1(\phi_2 - \psi_{n_1})) \\ + (\chi_{13}\chi_{2l} - \chi_{1l}\chi_{23})(\zeta_2 \sin 2n_1(\phi_k - \psi_{n_1}) - \zeta_k \sin 2n_1(\phi_2 - \psi_{n_1})) \\ + (\chi_{1k}\chi_{23} - \chi_{13}\chi_{2k})(\zeta_2 \sin 2n_1(\phi_l - \psi_{n_1}) - \zeta_l \sin 2n_1(\phi_2 - \psi_{n_1}))\end{pmatrix}}{\zeta_2(\chi_{1l}\chi_{23} - \chi_{13}\chi_{2l}) \sin 2n_1(\phi_1 - \psi_{n_1})}, \tag{44}$$

$$M_2 = \frac{M_k(\zeta_3(\chi_{1l}\chi_{2k} - \chi_{1k}\chi_{2l}) + \zeta_k(\chi_{13}\chi_{2l} - \chi_{1l}\chi_{23}) + \zeta_l(\chi_{1k}\chi_{23} - \chi_{13}\chi_{2k}))}{\zeta_2(\chi_{1l}\chi_{23} - \chi_{13}\chi_{2l})}, \tag{45}$$

$$M_3 = M_k((\chi_{1k}\chi_{2l} - \chi_{1l}\chi_{2k})/(\chi_{1l}\chi_{23} - \chi_{13}\chi_{2l})). \tag{46}$$

Full details of the derivation of equations (44–46) can be found in Appendix B. Thus, knowing the initial modification to trimming mass m_k it is possible to predict the effect that this will have on masses m_1, m_2 and m_3. This can also be performed in reverse. If a mass is too large or too small at a specific angular position it is possible to specify a modification to either m_1, m_2 or m_3 and then use the respective equations (44), (45) or (46), with equation (43), to calculate what values of M_k and M_l are required to keep the system in equilibrium. The validity of equations (35–46) will be tested later in a numerical example.

5.2. TARGET TRIMMING

Extending "simple trimming" to "target trimming" requires the simultaneous solution of equations (15) and (35–38). As has already been explained (see discussion following equation (15)), only one form of equation (15) can be solved and in what follows the $n_1\theta$ mode is considered.

The solutions for the first three masses, m_1, m_2 and m_3, will remain unchanged as equations (35), (36) and (37) respectively. Expanding equation (38), a solution for the fourth mass, m_4, can be found. The solutions for the remaining masses, m_i, can then be found by substituting equations (35–37) and (47) into equation (15). Hence

$$m_4 = \frac{M\zeta_2(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23}) - \sum_{i=5}^{N} m_i(\chi_{23}\chi_{1i} - \chi_{2i}\chi_{13})}{(\chi_{23}\chi_{14} - \chi_{24}\chi_{13})}, \tag{47}$$

$$\sum_{i=5}^{N} m_i\sigma_{i4} = M\zeta_2 \begin{pmatrix} (\delta_{n_1}\chi_{13}\sin 2n_1(\phi_1 - \psi_{n_1}) - \lambda_{n_2}\xi_3)(\chi_{23}\chi_{14} - \chi_{24}\chi_{13}) \\ -(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23})(\xi_4\chi_{13} - \xi_3\chi_{14}) \end{pmatrix}, \tag{48}$$

where

$$\sigma_{ab} = (\xi_a\chi_{13} - \xi_3\chi_{1a})(\chi_{23}\chi_{1b} - \chi_{2b}\chi_{13}) - (\xi_b\chi_{13} - \xi_3\chi_{1b})(\chi_{23}\chi_{1a} - \chi_{2a}\chi_{13}), \tag{49}$$

$$\xi_i = \zeta_2(\sin 2n_1(\phi_1 - \psi_{n_1}) - \sin 2n_1(\phi_i - \psi_{n_1})) - \zeta_i(\sin 2n_1(\phi_1 - \psi_{n_1}) - \sin 2n_1(\phi_2 - \psi_{n_1})). \tag{50}$$

Once again, a basic solution set to equation (48), from which other solutions can be formed by negation or modification, can be expressed as

$$m_i = \frac{M\zeta_2}{(N-4)\sigma_{i4}} \begin{pmatrix} (\delta_{n_1}\chi_{13}\sin 2n_1(\phi_1 - \psi_{n_1}) - \lambda_{n_2}\xi_3)(\chi_{23}\chi_{14} - \chi_{24}\chi_{13}) \\ -(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23})(\xi_4\chi_{13} - \xi_3\chi_{14}) \end{pmatrix}, \quad i > 4. \tag{51}$$

For dual-mode trimming, the equations that determine the relationship between the modifications to masses m_1, m_2, m_3, m_4 and m_l and the modification to mass m_k are

$$M_1 = \frac{M_k \begin{pmatrix} \sigma_{k4}\begin{pmatrix} \chi_{13}(\zeta_l \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_l - \psi_{n_1})) \\ -\chi_{1l}(\zeta_3 \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_3 - \psi_{n_1})) \end{pmatrix} \\ + \sigma_{4l}\begin{pmatrix} \chi_{13}(\zeta_k \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_k - \psi_{n_1})) \\ -\chi_{1k}(\zeta_3 \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_3 - \psi_{n_1})) \end{pmatrix} \\ + \sigma_{lk}\begin{pmatrix} \chi_{13}(\zeta_4 \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_4 - \psi_{n_1})) \\ -\chi_{14}(\zeta_3 \sin 2n_1(\phi_2 - \psi_{n_1}) - \zeta_2 \sin 2n_1(\phi_3 - \psi_{n_1})) \end{pmatrix} \end{pmatrix}}{\sigma_{4l}\chi_{13}\zeta_2 \sin 2n_1(\phi_1 - \psi_{n_1})}, \tag{52}$$

$$M_2 = M_k \frac{\sigma_{k4}(\chi_{1l}\zeta_3 - \chi_{13}\zeta_l) + \sigma_{4l}(\chi_{1k}\zeta_3 - \chi_{13}\zeta_k) + \sigma_{lk}(\chi_{14}\zeta_3 - \chi_{13}\zeta_4)}{\sigma_{4l}\chi_{13}\zeta_2}. \tag{53}$$

$$M_3 = -M_k(\sigma_{k4}\chi_{1l} + \sigma_{4l}\chi_{1k} + \sigma_{lk}\chi_{14})/\sigma_{4l}\chi_{13}, \quad M_4 = M_k\,\sigma_{lk}/\sigma_{4l}, \quad M_l = M_k\,\sigma_{k4}/\sigma_{4l}. \tag{54–56}$$

As in previous situations, these equations can be used to modify the masses in such a way that one mass is increased or reduced by a predetermined amount.

The validity of equations (47–56) will be tested later in section 6.

5.3. INVALID SOLUTIONS

The solutions considered so far in this section have been for general, non-specific angular positions. As described for single-mode trimming in the previous section, the most likely use of the proposed method will be to position the trimming masses at regularly spaced intervals such that $\phi_i = 2\pi i/N$. For this situation, it is worthwhile considering the implications for the trimming of two pairs of modes.

For dual-mode trimming it is shown in Appendix C that there are two situations when no valid trimming masses are available for the situation of regularly spaced trimming masses. The first is when either mode number $n_j = Nk/4$, where k is an integer, and is identical to that described for single-mode trimming. The second is when the relationship between the mode numbers is such that $n_2 \pm n_1 = Nk/2$, where k is an integer. These rules have been used to investigate the number of masses capable of trimming a specific pair of modes.

In a similar fashion to single-mode trimming, the minimum number of masses that can be used is four in order to provide a solution to equation (38). Again, this is not permitted by the first rule for determining invalid solutions. Also, due to the additional effect of the second rule, it is not possible to use only six or eight trimming masses to eliminate the frequency splits from any pair of vibrational modes.

The combined effect of the two rules on the trimming of the two vibrational modes is summarized in Tables 2–4. It is difficult to represent the solutions in a single table, as was done for single-mode trimming. There are more variables to be considered and so more tables are necessary. In this way, it will be more convenient to predetermine some of the variables and then use the tables to determine the other variables. For example, Table 2 summarizes the valid solutions for all the possible combinations of the first six vibrational

TABLE 2

Combinations of N trimming masses that have valid solutions for the $n_1\theta$ and $n_2\theta$ modes

N	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n_1=2 n_2=3	■	■	■		■		■		■								
n_1=2 n_2=4	■		■		■				■				■				
n_1=2 n_2=5	■	■	■	■	■		■				■						■
n_1=2 n_2=6	■		■		■				■				■				
n_1=2 n_2=7	■	■	■	■	■	■	■				■				■		
n_1=3 n_2=4	■		■	■	■				■		■		■				
n_1=3 n_2=5	■	■	■		■		■		■				■				■
n_1=3 n_2=6	■		■		■	■			■						■		
n_1=3 n_2=7	■		■	■	■		■		■		■						■
n_1=4 n_2=5	■	■	■		■	■	■						■		■		■
n_1=4 n_2=6	■	■	■		■		■		■				■				■
n_1=4 n_2=7	■		■	■	■			■			■		■				
n_1=5 n_2=6	■	■	■		■		■	■	■								■
n_1=5 n_2=7	■		■	■	■		■		■		■						■
n_1=6 n_2=7	■		■	■	■				■	■	■						

TABLE 3

Combinations of $n_1\theta$ and $n_2\theta$ modes that have valid solutions for $N = 11$ trimming masses

n_2 \ n_1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	■							■		■		■							■		■	
3		■					■			■			■					■			■	
4			■			■				■				■			■				■	
5				■	■					■					■	■					■	
6				■	■					■					■	■					■	
7			■			■				■				■			■				■	
8		■					■			■			■					■			■	
9	■							■		■		■							■		■	
10									■	■	■									■	■	■
11	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■
12									■	■	■									■	■	■
13	■							■		■		■							■		■	
14		■					■			■			■					■			■	
15			■			■				■				■			■				■	
16				■	■					■					■	■					■	
17				■	■					■					■	■					■	
18			■			■				■				■			■				■	
19		■					■			■			■					■			■	
20	■							■		■		■							■		■	
21									■	■	■									■	■	■
22	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■
23									■	■	■									■	■	■

TABLE 4

Combinations of N trimming masses with the $n_2\theta$ mode that have valid solutions for $n_1 = 2$

n_2 \ N	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	■	■	■		■		■		■								
4	■		■		■				■				■				
5	■	■	■	■	■		■				■						■
6	■		■		■				■				■				
7	■	■	■	■	■	■	■				■				■		
8	■	■	■		■		■		■				■				■
9	■		■	■	■	■		■	■		■				■		
10	■	■	■		■		■		■				■				■
11	■		■		■	■		■		■					■		
12	■	■	■	■	■		■		■		■		■				■
13	■	■	■		■		■	■		■		■					
14	■		■	■	■				■		■		■				
15	■	■	■		■		■		■	■		■		■			■
16	■		■	■	■	■			■		■		■		■		
17	■	■	■		■		■					■		■		■	
18	■	■	■		■	■	■		■				■		■		■
19	■		■	■	■						■			■		■	
20	■	■	■		■	■	■	■	■				■		■		■

modes. These modes are most likely to be trimmed in a practical situation. Thus, this table will be of most use for most situations. Alternatively, it may be desired to use a certain number of trimming masses or a particular vibrational mode and then determine the best combination of other terms. Table 3 shows a typical summary of valid solutions for a specific number of masses, in this case $N = 11$. The pattern that can be observed will be continued for higher vibrational modes. Finally, Table 4 shows a typical summary for a single vibrational mode, which in this case is the 2θ mode, for use with a second mode and a range of trimming masses.

6. NUMERICAL EXAMPLES

The numerical examples will be considered in four distinct parts. Sections 6.1 and 6.2 consider "simple" and "target" trimming for a single mode of vibration, respectively, while sections 6.3 and 6.4 consider "simple" and "target" trimming for the dual-mode case respectively. Each of these examples will consider an imperfect ring formed by the addition, to an initially perfect ring, of three imperfection masses of 0·1, 0·2 and 0·3 kg at 0, 20 and 70° respectively. In each case, the mass of the perfect ring considered is 7·3984 kg. The dimensions of the ring are given in Appendix D along with the frequency splits and orientations of the first four modes for the imperfect ring studied.

6.1. SIMPLE TRIMMING OF A SINGLE MODE

Consider the problem of trimming a single mode using several evenly spaced masses. From Table 1, a valid combination of trimming masses and mode number can be found. The 2θ mode has been chosen as this is the most common mode that is driven in a gyroscopic system. The results for $N = 5$, 6 and 7 trimming masses have been considered and are recorded in Table 5. In each case, the angular position of the first mass, ϕ_1, is shifted by $2\pi/N$ to produce N different sets of masses. For $N = 7$, this is shown by examples (i)–(vii). In each example the angular positions of the other masses are as defined by $\phi_i = 2\pi i/N$. Each combination of masses eliminates the frequency split of the 2θ mode and typically produces a different natural frequency.

If N is even then this leads to a degree of symmetry and repetition as can be seen from the $N = 6$ solutions. The fourth mass was infinite for $N = 6$, so was negated and the denominator of equation (20) modified to account for this change. The problem arose due to the difference between ϕ_1 and ϕ_4 being equal to π and so the denominator of equation (20) became zero.

The other combinations of masses in Table 5 have been included to demonstrate the flexibility of the method. For each value of N, it can be seen that the mass can be purely added to or removed from the structure, see (viii) and (ix), Table 5, for $N = 7$. The masses that do not have the correct sign have been set to zero and the denominator of equation (20) has been modified to reflect this. Due to the nature of equations (18) and (20) it is not possible to set m_1 equal to zero. Instead the value of ϕ_1 has to be shifted by $2\pi l/N$, where l is an integer, until the value of m_1 has the sign required.

For many combinations of masses, it can be seen that a single mass exists that is significantly larger than the other masses. This larger mass may have a significant effect on the mode shapes of the structure [17]. For this reason, it is more practical to reduce the size of this mass. This can be done by applying equations (21) and (22). The other masses included in the $N = 7$ portion of Table 5 illustrate the validity of equations (21) and (22). The sets of masses (x)–(xiv) have been generated from the (viii) set of masses.

Set (x) has been generated by the addition of 0·1 kg to mass m_2. Substitution of the relevant values into equations (21) and (22) produces the relationships $m_4 = -0{\cdot}445 m_2$ and

TABLE 5

Valid solutions for $n_j = 2$ for the "simple" case of single-mode frequency trimming

	N	ϕ_1 (rad)	m_1 (kg)	m_2 (kg)	m_3 (kg)	m_4 (kg)	m_5 (kg)	m_6 (kg)	m_7 (kg)	ω_{01} (Hz)
(i)	7	0	−0·187	−0·038	0·021	−0·017	0·017	−0·021	0·038	35·793
(ii)	7	2π/7	0·126	0·065	−0·036	0·029	−0·029	0·036	−0·065	35·098
(iii)	7	4π/7	−0·039	−0·080	0·044	−0·036	0·036	−0·044	0·080	35·460
(iv)	7	6π/7	−0·054	0·078	−0·043	0·035	−0·035	0·043	−0·078	35·494
(v)	7	8π/7	0·138	−0·062	0·034	−0·027	0·027	−0·034	0·062	35·072
(vi)	7	10π/7	−0·193	0·033	−0·018	0·014	−0·014	0·018	−0·033	35·808
(vii)	7	12π/7	0·211	0·003	−0·002	0·001	−0·001	0·002	−0·003	34·915
(viii)	7	0	−0·236	−0·076	0	−0·034	0	−0·042	0	36·262
(ix)	7	2π/7	0·211	0·130	0	0·058	0	0·072	0	34·373
(x)	7	0	−0·156	0·024	0	−0·078	0	−0·042	0	35·943
(xi)	7	0	−0·112	0·024	0	−0·034	0	−0·097	0	35·866
(xii)	7	0	−0·136	−0·076	0	0·066	0	−0·167	0	36·084
(xiii)	7	0	−0·152	−0·026	0	−0·011	0	−0·097	0	36·023
(xiv)	7	0	−0·108	0	0	0	0	−0·126	0	35·902
	6	0	−0·187	−0·028	0·028	0	−0·028	0·028	N/A	35·793
	6	π/3	0·008	0·061	−0·061	0	0·061	−0·061	N/A	35·355
	6	2π/3	0·179	−0·032	0·032	0	−0·032	0·032	N/A	34·984
	6	π	−0·187	−0·028	0·028	0	−0·028	0·028	N/A	35·793
	6	4π/3	0·008	0·061	−0·061	0	0·061	−0·061	N/A	35·355
	6	5π/3	0·179	−0·032	0·032	0	−0·032	0·032	N/A	34·984
	6	0	−0·244	−0·057	0	0	−0·057	0	N/A	36·190
	6	π/3	0·130	0·122	0	0	0·122	0	N/A	34·574
	5	0	−0·187	−0·026	−0·042	0·042	0·026	N/A	N/A	35·793
	5	2π/5	−0·151	0·039	0·063	−0·063	−0·039	N/A	N/A	35·712
	5	4π/5	0·093	0·050	0·081	−0·081	−0·050	N/A	N/A	35·168
	5	6π/5	0·209	−0·008	−0·013	0·013	0·008	N/A	N/A	34·919
	5	8π/5	0·036	−0·055	−0·089	0·089	0·055	N/A	N/A	35·293
	5	0	−0·239	−0·052	−0·084	0	0	N/A	N/A	36·230
	5	4π/5	0·193	0·100	0·161	0	0	N/A	N/A	34·410

Note: N/A marks the masses that are unnecessary as the mass number, k of m_k, is greater than the value of N.

$m_1 = 0{\cdot}802m_2$. The value of m_4 has been increased according to this proportion and it is observed that m_1 increases according to this proportion also. Thus, equations (21) and (22) are verified.

This may not be the optimal choice of modifications, though. Sets (xi)–(xiii) show other possible combinations, all of which have been generated from set (viii). Set (xi) modifies masses m_1, m_2 and m_6 by an addition of 0·1 kg to m_2 and equations (21) and (22) reveal that $m_6 = -0{\cdot}555\,m_2$ and $m_1 = 1{\cdot}247\,m_2$. Similarly, set (xii) modifies masses m_1, m_4 and m_6 by an addition of 0.1kg to m_4 and equations (21) and (22) reveal that $m_6 = -1{\cdot}247\,m_4$ and $m_1 = m_4$.

Finally, sets (xiii) and (xiv) have been formed by the modification of all four masses. To form set (xiii), a mass of 0·05 kg has been added to m_2, a mass of $(-0{\cdot}445 \times 0{\cdot}05)$ kg has been removed from m_4 and a mass of $(-0{\cdot}555 \times 0{\cdot}1)$ kg has been added to m_5. The effect of these changes is to add a mass of 0·084 kg to m_1. The overall effect is to significantly reduce the size of the largest mass, m_1, whilst maintaining the signs of the other masses. In fact, of the other trimming masses only m_6 has increased in size. The other two masses have decreased in size.

The logical conclusion to this can be to reduce the number of trimming masses down to a minimum of two. Adding masses of 0·076 and 0·034 kg to m_2 and m_4, respectively, eliminates these masses in set (viii). Modifying m_1 and m_6 in accordance with equations (21) and (22) produces set (xiv). This has produced two trimming masses that are similar in size and roughly half of the size of the largest original trimming mass. In fact, this combination of masses has the smallest overall trimming mass to be applied to the ring, compared to all other combinations of trimming masses considered.

Thus, only a small number of masses need to be applied to a circular structure at easily determined positions to eliminate the frequency split of a single mode.

6.2. TARGET TRIMMING OF A SINGLE MODE

For the "target trimming" case of the single mode, solutions have been split into three sections to illustrate different consequences of equations (24–30), see Table 6. In each section, trimming of the 2θ mode has been performed and the number of trimming masses used has been kept as $N = 7$, although in certain cases some of these masses have been negated. As for "simple trimming" the angular position of the ith trimming mass is assumed to be given by $\phi_i = 2\pi i/N$.

The first set of solutions has been included to show that a variety of different sets of trimming masses can still be produced by varying the angular position of the first trimming mass. This has had no effect on the frequency that the frequency splits were focussed on and identical results have been produced in each case, as expected. The last five solutions in this set have been included to illustrate the need for care in choosing the target natural frequency. From Appendix D it can be seen that the frequencies for the imperfect ring are 35·096 and 36·656 Hz. Thus, the first set of solutions is targeted at a frequency that is greater than one of the imperfect frequencies but less than the other. The consequence of this is that it has not been possible to find a solution set that consists solely of addition or removal of trimming masses, even using the minimum number of masses. Thus, it looks unlikely that it will be possible to trim an imperfect structure to a frequency that lies between the existing frequency split by a mass addition or mass removal only. Instead, a combination of addition and removal of trimming masses will be required.

However, the second set of solutions shows that for frequencies outside of a particular frequency range it is possible to use either mass addition or reduction. It is possible to only remove mass from the structure if the desired frequency is higher than an upper "threshold" frequency and it is possible to only add mass to the structure if the desired frequency is lower than a lower "threshold" frequency. For target frequencies between these two frequencies it is necessary to use a combination of methods. In this example, the "threshold" frequencies are the central pair of frequencies of the second set of solutions. These frequencies are 34·9071 and 35·9023 Hz, and have been found by trial and error. It can also be seen that the threshold frequencies are one of the combinations of angular positions for which the number of necessary trimming masses reduces to two.

It is still possible to use a combination of addition and removal of trimming masses for frequencies outside the threshold frequencies, as can be seen from the other results. It can also be seen that it is theoretically possible to focus the frequency splits onto any target frequency. However, the further that the target frequency is from the split frequencies the larger the trimming masses become. These trimming masses will be more likely to have a significant effect on the mode shapes of the structure. The original assumption, as proposed in reference [7], was that the trimming masses were not large enough for this to happen, so it is only feasible to trim the structure to a value close to the imperfect split frequencies.

TABLE 6

Valid solutions for $n_j = 2$ for the "target" frequency case of single-mode frequency trimming

ϕ_1 (rad)	m_1 (kg)	m_2 (kg)	m_3 (kg)	m_4 (kg)	m_5 (kg)	m_6 (kg)	m_7 (kg)	ω_{01} (Hz)
0	−0·114	0·051	0·030	−0·037	0·020	−0·037	0·030	35·5
2π/7	0·020	−0·067	−0·049	0·061	−0·034	0·061	−0·049	35·5
4π/7	−0·029	−0·037	0·047	−0·058	0·033	−0·058	0·047	35·5
6π/7	−0·074	0·027	−0·047	0·059	−0·033	0·059	−0·047	35·5
8π/7	0·056	−0·118	0·027	−0·033	0·018	−0·033	0·027	35·5
10π/7	−0·134	0·079	−0·012	0·015	−0·009	0·015	−0·012	35·5
12π/7	0·078	−0·132	−0·016	0·020	−0·011	0·020	−0·016	35·5
0	−0·244	0·039	0·148	0	0	0	0	35·5
0	−0·060	0·186	0	−0·184	0	0	0	35·5
0	−0·162	0·002	0	0	0·102	0	0	35·5
0	0·022	0·105	0	0	0	−0·184	0	35·5
0	−0·125	−0·079	0	0	0	0	0·148	35·5
0	3·880	4·044	0·454	−0·566	0·314	−0·566	0·454	25
0	2·906	3·070	0·757	0	0·524	0	0·757	25
0	1·879	3·864	2·271	0	0	0	0	25
0	1·459	1·623	0·197	−0·245	0·136	−0·245	0·197	30
0	1·037	1·202	0·328	0	0·227	0	0·328	30
0	0·592	1·545	0·984	0	0	0	0	30
0	0·241	0·405	0·067	−0·084	0·046	−0·084	0·067	34
0	0·097	0·261	0·112	0	0·078	0	0·112	34
0	0·131	0·295	0	0	0·232	0	0	34
8π/7	0·0098	0	0·2049	0	0	0	0	34·9071
0	−0·108	0	0	0	0	−0·126	0	35·9023
0	−0·222	−0·058	0·018	−0·022	0·012	−0·022	0·018	36
0	−0·164	0	0	−0·056	0	−0·056	0	36
0	−0·948	−0·784	−0·059	0·074	−0·041	0·074	−0·059	40
0	−0·821	−0·657	−0·099	0	−0·068	0	−0·099	40
0	−0·687	−0·760	−0·296	0	0	0	0	40
0	−1·598	−1·433	−0·128	0·160	−0·089	0·160	−0·128	45
0	−1·323	−1·158	−0·214	0	−0·148	0	−0·214	45
0	−1·033	−1·382	−0·641	0	0	0	0	45
0	−0·222	−0·058	0·018	−0·022	0·012	−0·022	0·018	36
0	−0·097	0·042	−0·082	−0·147	0·012	−0·022	0·018	36
0	−0·167	−0·082	−0·082	−0·022	0·082	−0·022	0·018	36
0	−0·042	−0·013	−0·082	−0·022	0·012	−0·147	0·018	36
0	−0·142	−0·138	−0·082	−0·022	0·012	−0·022	0·118	36
0	−0·347	−0·158	0·118	0·102	0·012	−0·022	0·018	36
0	0·003	−0·158	−0·387	−0·022	0·293	−0·022	0·018	36
0	−0·627	−0·158	0·243	−0·022	0·012	0·258	0·018	36
0	−0·122	−0·158	−0·107	−0·022	0·012	−0·022	0·143	36
0	−0·322	−0·138	0·098	0·078	0·012	−0·022	0·018	36
0	−0·322	−0·012	0·198	−0·022	−0·112	−0·022	0·018	36
0	−0·322	−0·082	0·073	−0·022	0·012	0·047	0·018	36
0	−0·322	0·042	0·143	−0·022	0·012	−0·022	−0·107	36

Finally, the third set of solutions has been included to validate equations (28–30). The first set of trimming masses is the basic set of solutions as generated by equations (18), (24) and (27). The remainder of the solutions have been modified by the addition of extra mass to m_3 and one of m_4–m_7. The first four sets of trimming masses involve the removal of 0·1 kg from m_3 with the addition of extra mass, as specified by equation (30), to each of m_4–m_7 in turn. The next four sets involve the addition of the masses to m_3 and one of m_4–m_7 in turn so as to require a removal of 0·1 kg from m_2 and the last four sets create the same required removal from m_1.

Thus, equations (24–30) have been validated and it has been shown that there is a range of trimming masses that can be used to focus the frequency splits onto a pre-selected target frequency.

6.3. SIMPLE TRIMMING FOR DUAL MODES

Now consider the problem of trimming two modes simultaneously, specifically the 2θ and the 3θ modes. As with the single-mode example, seven trimming masses have been considered and some solutions are recorded in Table 7. The first seven examples show the possible solutions that can be found by shifting the angular position of the first mass by $2\pi/N$. The angular positions of the other masses are as specified earlier.

The next two examples, (viii) and (ix), demonstrate that the trimming problem can again be reduced to either the addition or removal of trimming masses to the ring. This problem can then be reduced to the minimum four-mass system by the negation of the extra trimming masses, as illustrated by examples (x) and (xi).

Finally, the validity of equations (43–46) is tested by examples (xii–xviii). Examples (xii), (xiii) and (xiv) have been generated from example (ix) by the addition of 0·1 kg to m_3, m_2 and m_1 respectively. In these examples $k = 5$ and $l = 7$. The modified masses m_5 and m_7 have been calculated using equations (43–46) and then masses m_1–m_3 have been calculated from equations (35–37). Comparing examples (ix) and (xii–xiv) shows that the required modifications to m_3, m_2 and m_1 have been produced so validating equations (43–46).

The solutions for (xiii) and (xiv) are identical however, so a further example has been included to show that this is not necessarily true. Using example (xv) as a template and with

TABLE 7

Valid solutions for $n_1 = 2$ and $n_2 = 3$ for the "simple" case of dual-mode frequency trimming

	ϕ_1 (rad)	m_1 (kg)	m_2 (kg)	m_3 (kg)	m_4 (kg)	m_5 (kg)	m_6 (kg)	m_7 (kg)	ω_{01} (Hz)	ω_{02} (Hz)
(i)	0	−0·206	0·196	0·285	0·009	−0·015	0·015	−0·009	34·781	98·375
(ii)	2π/7	0·141	0·253	0·183	0·044	−0·080	0·080	−0·044	34·160	96·620
(iii)	4π/7	0·314	0·169	−0·066	−0·073	0·131	−0·131	0·073	34·487	97·543
(iv)	6π/7	0·224	0·025	−0·349	−0·017	0·031	−0·031	0·017	35·595	100·679
(v)	8π/7	0·081	−0·309	−0·209	−0·041	0·075	−0·075	0·041	36·380	102·900
(vi)	10π/7	−0·305	−0·169	−0·117	0·015	−0·026	0·026	−0·015	36·756	103·963
(vii)	12π/7	−0·249	−0·165	0·273	0·064	−0·115	0·115	−0·064	35·688	100·942
(viii)	2π/7	0·202	0·342	0·244	0·089	0	0·160	0	33·282	94·136
(ix)	10π/7	−0·325	−0·198	−0·137	0	−0·052	0	−0·029	37·135	105·034
(x)	10π/7	−0·288	−0·162	−0·166	0	0	0	−0·058	36·965	104·553
(xi)	10π/7	−0·361	−0·234	−0·107	0	−0·105	0	0	37·307	105·521
(xii)	10π/7	−0·449	−0·323	−0·037	0	−0·233	0	0·071	37·737	106·736
(xiii)	10π/7	−0·225	−0·098	−0·217	0	0·092	0	−0·109	36·673	103·726
(xiv)	10π/7	−0·225	−0·098	−0·217	0	0·092	0	−0·109	36·673	103·726
(xv)	10π/7	−0·309	−0·205	−0·120	0·029	−0·052	0	0	36·923	104·435
(xvi)	10π/7	−0·713	−0·430	−0·020	−0·196	−0·457	0	0	40·237	113·807
(xvii)	10π/7	−0·128	−0·105	−0·165	0·129	0·128	0	0	35·691	100·948
(xviii)	10π/7	−0·209	−0·150	−0·145	0·085	0·048	0	0	36·224	102·457

TABLE 8

Valid solutions for $n_1 = 2$ and $n_2 = 3$ for the "simple" case of dual-mode frequency trimming neglecting fixed relationship between the angular positions

m_1 (kg) ϕ_1 (rad)	m_2 (kg) ϕ_2 (rad)	m_3 (kg) ϕ_3 (rad)	m_4 (kg) ϕ_4 (rad)	m_5 (kg) ϕ_5 (rad)	m_6 (kg) ϕ_6 (rad)	m_7 (kg) ϕ_7 (rad)	ω_{01} (Hz)	ω_{02} (Hz)
−0·165	−0·249	0·273	−0·115	0·115	−0·064	0·064	35·688	100·942
0	12π/7	2π/7	6π/7	8π/7	10π/7	4π/7	–	–
−0·443	−0·148	−0·284	−0·292	0·130	−0·072	0·104	37·830	107·000
0	8π/7	12π/7	6π/7	2π/7	10π/7	4π/7	–	–
−0·750	−0·382	−0·486	−0·584	0	−0·144	0	42·078	119·014
0	8π/7	12π/7	6π/7	–	10π/7	–	–	–
−0·208	−0·311	−0·111	0·064	−0·178	0·079	−0·044	37·055	104·807
0	10π/7	2π/7	6π/7	8π/7	4π/7	12π/7	–	–
−0·331	−0·454	−0·299	0	−0·357	0	−0·088	39·331	111·245
0	10π/7	2π/7	–	8π/7	–	12π/7	–	–
−0·181	−0·304	−0·028	0	−0·086	0	−0·155	37·171	105·135
0	10π/7	2π/7	–	8π/7	–	12π/7	–	–
−1·332	−1·535	−1·413	−0·663	0·234	−0·156	0·146	55·218	156·181
0	0·5	1	1·5	2	2·5	3	–	–
−0·255	−0·115	0·075	0·106	0·173	0·119	0·072	34·993	98·974
0	1	2	3	4	5	6	–	–
−0·105	0·217	0·170	0·013	−0·022	0·020	−0·021	34·786	98·391
0	0·8	1·8	2·9	3·3	4·7	5·5	–	–

$k = 5$ and $l = 7$, a mass of 0·1 kg was once again added to m_3, m_2 and m_1 in turn. Examples (xvi–xviii) are the resulting sets of masses. Once again equations (43–46) are validated. However, in these examples the modifications to the trimming masses differ.

A further set of solutions can also be generated by varying the angular positions of the second and third trimming masses. The examples considered so far have fixed the angular positions of each trimming mass by considering the angular position of the previous trimming mass. The only reason for this was to simplify the generation of the solutions. It can be seen from Table 8 that valid solutions can also be determined from angular positions that have not been fixed in such a way. The final three examples in Table 8 have been included to show that a random choice of angular positions of the trimming masses will also generate valid solutions as theory predicts.

6.4. TARGET TRIMMING FOR DUAL MODES

Now consider the "target trimming" case. As for the "simple" case, the modes that are being trimmed are the 2θ and the 3θ modes using $N = 7$ trimming masses. The frequency of the 2θ mode is pre-selected as the target frequency for the trimming procedure.

The solutions presented in Table 9 have been arranged in the same style as the solutions to the "target trimming" case for a single mode as shown in Table 6. The first set of solutions again shows that a range of trimming masses can be generated to eliminate the frequency splits of both modes and focus the split of the second mode on to a target frequency.

The second set of solutions again shows that it is theoretically possible to trim the ring to any frequency, although the trimming masses become infeasible for large differences between the frequency splits and the target frequencies. In this example, it can be seen that

TABLE 9

Valid solutions for $n_1 = 2$ and $n_2 = 3$ for the "target" frequency case of dual-mode frequency trimming for seven trimming masses

ϕ_1 (rad)	m_1 (kg)	m_2 (kg)	m_3 (kg)	m_4 (kg)	m_5 (kg)	m_6 (kg)	m_7 (kg)	ω_{01} (Hz)	ω_{02} (Hz)
0	−0·191	0·109	0·192	0·020	−0·044	−0·100	−0·044	35·5	100·409
2π/7	0·259	0·038	−0·060	0·146	−0·104	−0·233	−0·104	35·5	100·409
4π/7	0·068	0·192	0·004	−0·293	−0·007	−0·015	−0·007	35·5	100·409
6π/7	0·171	0·065	−0·298	−0·065	0·017	0·037	0·017	35·5	100·409
8π/7	−0·036	−0·155	−0·028	−0·144	0·072	0·161	0·072	35·5	100·409
10π/7	−0·240	−0·079	−0·036	0·074	0·052	0·118	0·052	35·5	100·409
12π/7	−0·044	−0·258	0·139	0·246	−0·033	−0·074	−0·033	35·5	100·409
0	−0·311	0·118	0·308	−0·040	−0·133	0	0	35·5	100·409
0	−0·012	−0·015	0·068	0·200	0	−0·299	0	35·5	100·409
0	−0·252	0·225	0·201	−0·099	0	0	−0·133	35·5	100·409
0	0·098	1·860	1·943	0·310	0·895	2·012	0·895	25	70·711
4π/7	0·357	1·943	1·755	−0·003	0·933	2·097	0·933	25	70·711
2π/7	0·373	0·728	0·629	0·260	0·267	0·599	0·267	30	84·853
10π/7	−0·126	0·611	0·654	0·188	0·423	0·949	0·423	30	84·853
2π/7	0·303	0·309	0·211	0·191	0·042	0·094	0·042	33	93·338
4π/7	0·113	0·463	0·275	−0·248	0·139	0·312	0·139	33	93·338
2π/7	0·291	0·231	0·133	0·178	0	0	0	33·6645	95·218
8π/7	−0·058	−0·288	−0·162	−0·166	0	0	0	36·9651	104·553
8π/7	−0·059	−0·291	−0·165	−0·166	−0·002	−0·004	−0·002	37	104·652
2π/7	0·236	−0·099	−0·197	0·123	−0·177	−0·398	−0·177	37	104·652
0	−0·252	−0·256	−0·173	−0·040	−0·241	−0·541	−0·241	40	113·137
2π/7	0·198	−0·328	−0·426	0·085	−0·300	−0·674	−0·300	40	113·137
0	−0·299	−0·541	−0·458	−0·087	−0·393	−0·884	−0·393	45	127·279
2π/7	0·151	−0·612	−0·711	0·038	−0·453	−1·017	−0·453	45	127·279
0	−0·199	0·062	0·145	0·013	−0·070	−0·157	−0·070	36	101·823
0	−0·424	0·162	0·325	−0·168	−0·170	0·068	−0·070	36	101·823
0	−0·244	−0·018	0·225	0·057	−0·170	−0·157	0·030	36	101·823
0	−0·324	0·117	0·245	−0·087	−0·125	−0·032	−0·070	36	101·823
0	−0·099	0·242	−0·035	−0·087	0·155	−0·157	−0·295	36	101·823
0	−0·075	0·006	0·045	0·112	−0·014	−0·282	−0·070	36	101·823
0	−0·144	0·162	0·045	−0·043	0·055	−0·157	−0·195	36	101·823
0	0·025	−0·038	−0·035	0·193	0·030	−0·382	−0·070	36	101·823
0	−0·255	−0·038	0·245	0·068	−0·195	−0·157	0·055	36	101·823
0	−0·299	0·106	0·225	−0·068	−0·114	−0·057	−0·070	36	101·823
0	−0·299	−0·118	0·325	0·113	−0·295	−0·157	0·155	36	101·823

solutions can be found that involve seven trimming masses with the same nature, either addition or removal. Solutions that involve a combination of the two trimming methods can be found but these are not necessary. It is possible that finding all of the masses with the same nature could be due to the fact that only three terms have been generated by equation (51) or it could be symptomatic of this particular example. Considering $N = 9$, which has five terms being generated by equation (51), it can be seen from Table 10 that once again solutions containing a combination of masses of different natures have become typical.

It can also be seen from Table 9 that a pair of threshold frequencies have been found, just as they were for the single-mode frequency trimming. The difference between these frequencies is larger than for single-mode frequency trimming but, as in that case, the minimum number of masses necessary to trim the ring has been reduced by one at the threshold frequencies. There are also differences between the threshold frequencies for dual-mode frequency trimming for different values of N, as can be seen by comparing Tables 9 and 10.

However, the threshold frequencies recorded may not be the exact threshold frequencies possible. They are the exact frequencies for the trimming masses that have been fixed in position by the angular position of the first trimming mass. It is possible that the threshold frequencies could be found closer to the frequency splits by considering a random selection of angular positions. The angle between neighbouring masses will remain as $2\pi/N$ but the angular position of the ith mass will not be determined by incrementally increasing the angular position of the $(i-1)$th mass. As the threshold frequencies have been found by trial and error, this has not been done yet. To find the threshold frequencies for the angular positions determined by incremental increases involved comparing 21 sets of trimming masses for different target frequencies. To find the threshold frequencies for every possible combination of angular position would involve comparing 2520 sets.

Finally, the third set validates equations (52–56). The first solution has been generated from equations (35–37), (47) and (51). Then, the following solutions alternate between additions to masses m_5 and m_6 and masses m_5 and m_7 in order to create specific modifications to masses m_5, m_4, m_3, m_2 and m_1 in turn. The modification, as can be seen from the table, is to remove a mass of 0·1 kg from the trimming mass considered.

Thus, equations (47–56) have been validated. It is possible to perform dual-mode trimming with a specific target frequency for one of the modes.

7. CONCLUSION

A general method for eliminating the frequency split of both a single mode and a pair of in-plane modes of a ring using a set of trimming masses located at pre-selected locations has been outlined. In some trimming processes, a particular trimmed natural frequency may be required. For this reason, the general method has been extended to include a "target trimming" method. The method can only be extended to allow for "target trimming" of one mode of vibration. Both the general method and the "target trimming" method have been extended further to consider a specific set of pre-selected locations for the trimming masses. In some trimming processes, it may be convenient for the trimming masses to be positioned at regular intervals around the ring's circumference and the trimming methods have been extended to account for these pre-selected locations. This choice of angular locations has some consequences for the choice of modes of vibration to be trimmed and the possible number of trimming masses that can be used. Rules governing the relationship between the modes and the number of trimming masses have been derived. Numerical examples have been given to demonstrate the applicability of the methods, which have revealed a pair of

TABLE 10

Valid solutions for $n_1 = 2$ and $n_2 = 3$ for the "target" frequency case of dual-mode frequency trimming for nine trimming masses

ϕ_1 (rad)	m_1 (kg)	m_2 (kg)	m_3 (kg)	m_4 (kg)	m_5 (kg)	m_6 (kg)	m_7 (kg)	m_8 (kg)	m_9 (kg)	ω_{01} (Hz)	ω_{02} (Hz)
0	0·626	1·325	0·825	0·301	0·034	−0·018	0·013	−0·018	0·034	30	84·853
2π/9	0·708	0·796	0·896	0·595	0·095	−0·051	0·037	−0·051	0·095	30	84·853
4π/9	0·091	0·914	1·314	0·727	0·056	−0·030	0·022	−0·030	0·056	30	84·853
6π/9	0·818	1·356	0·856	0·135	−0·032	0·017	−0·013	0·017	−0·032	30	84·853
8π/9	0·559	0·750	0·850	0·706	0·193	−0·102	0·076	−0·102	0·193	30	84·853
10π/9	0·179	0·969	1·369	0·686	−0·062	0·033	−0·024	0·033	−0·062	30	84·853
12π/9	0·801	1·298	0·798	0·103	0·091	−0·049	0·036	−0·049	0·091	30	84·853
14π/9	0·506	0·804	0·904	0·808	0·078	−0·042	0·031	−0·042	0·078	30	84·853
16π/9	1·218	0·041	1·340	0·513	0·111	−0·263	0·255	−0·263	0·111	30	84·853
8π/9	0·143	0	0·255	0·100	0	0	0	0·102	0	34·1161	96·495
0	−0·003	−0·500	0	−0·327	0	0	−0·070	0	0	37·5482	106·202
0	−0·168	−0·258	−0·758	−0·493	−0·050	0·027	−0·020	0·027	−0·050	40	113·137
2π/9	−0·087	−0·786	−0·686	−0·199	0·011	−0·006	0·004	−0·006	0·011	40	113·137
4π/9	−0·703	−0·668	−0·268	−0·067	−0·028	0·015	−0·011	0·015	−0·028	40	113·137
6π/9	0·023	−0·227	−0·727	−0·659	−0·116	0·062	−0·046	0·062	−0·116	40	113·137
8π/9	−0·235	−0·832	−0·732	−0·089	0·109	−0·058	0·043	−0·058	0·109	40	113·137
10π/9	−0·615	−0·613	−0·213	−0·109	−0·145	0·077	−0·057	0·077	−0·145	40	113·137
12π/9	0·007	−0·285	−0·785	−0·691	0·008	−0·004	0·003	−0·004	0·008	40	113·137
14π/9	−0·293	−0·778	−0·678	0·014	−0·005	0·003	−0·002	0·003	−0·005	40	113·137
16π/9	−0·917	0·431	−0·842	−0·406	−0·129	0·305	−0·297	0·305	−0·129	40	113·137

"threshold" frequencies that must be taken into consideration when trimming a physical ring. The "threshold" frequencies will be significant if the physical operation of trimming the modes of the ring can only be performed by the removal or addition of trimming masses.

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REFERENCE

1. LORD RAYLEIGH 1887 *The Theory of Sound*, Vol. 1, section 235. New York: Dover, second edition, 1945 reissue.
2. D. ALLAEI, W. SOEDEL and T. Y. YANG 1986 *Journal of Sound and Vibration* **111**, 9–27. Natural frequencies of rings that depart from perfect axial symmetry.
3. C. H. J. FOX 1988 *Proceedings of DGON Symposium on Gyro Technology*, *Stuttgart*, chapter 5. Vibrating cylinder rate gyro: theory of operation and error analysis.
4. I. HOPKIN 1997 *Proceedings of DGON Symposium on Gyro Technology*, *Stuttgart*, chapter 1. Performance and design of a silicon micro-machined gyro.
5. S. A. TOBIAS 1951 *Engineering* **172**, 409–410. A theory of imperfection for elastic bodies of revolution.
6. P. A. A. LAURA, C. P. FILIPICH, R. E. ROSSI and J. A. REYES 1988 *Applied Acoustics* **25**, 225–234. Vibrations of rings of variable cross section.
7. R. F. TONIN and D. A. BIES 1979 *Journal of Sound and Vibration* **62**, 165–180. Free vibration of a circular cylinder with variable thickness.
8. R. S. HWANG, C. H. J. FOX and S. MCWILLIAM 1999 *Journal of Sound and Vibration* **220**, 497–516. The in-plane vibration of thin rings with in-plane profile variations. Part I: general background and theoretical formulation.
9. C. H. J. FOX, R. S. HWANG and S. MCWILLIAM 1999 *Journal of Sound and Vibration* **220**, 517–539. The in-plane vibration of thin rings with in-plane profile variations. Part II: application to nominally circular rings.
10. R. S. HWANG, C. H. J. FOX and S. MCWILLIAM 1999 *Journal of Sound and Vibration* **228**, 683–699. Free vibrations of elliptical rings with circumferentially variable thickness.
11. R. ELEY, C. H. J. FOX and S. MCWILLIAM 1999 *Journal of Sound and Vibration* **228**, 11–35. Anisotropy effects on the vibration of circular rings made from crystalline silicon.
12. C. H. J. FOX 1990 *Journal of Sound and Vibration* **142**, 227–243. A simple theory for the analysis and correction of frequency splitting in slightly imperfect rings.
13. C. H. J. FOX 1997 *Proceedings of XV International Modal Analysis Conference*, *Tokyo*, 566–572. Mode trimming in nominally axi-symmetric structures.
14. R. PERRIN 1971 *Acustica* **25**, 69–72. Selection rules for the splitting of the degenerate pairs of natural frequencies of thin circular rings.
15. T. CHARNLEY and R. PERRIN 1973 *Acustica* **28**, 139–146. Perturbation studies with a thin circular ring.
16. A. K. ROURKE, S. MCWILLIAM and C. H. J. FOX 2001 *Proceedings of the Fourth International Conference on Modern Practice in Stress and Vibration Analysis*, *Nottingham*, 361–372. Dual-mode frequency trimming of imperfect rings.
17. A. K. ROURKE, S. MCWILLIAM and C. H. J. FOX 2001 *Journal of Sound and Vibration* **248**, 695–724. Multi-mode frequency trimming of imperfect rings.

APPENDIX A: DERIVATION OF THE DUAL-MODE TRIMMING EQUATIONS (35–38)

First consider equation (35). This is directly derived by expanding equation (31) into

$$\sum_{i=1}^{N} m_i \sin 2n_1(\phi_i - \psi_{n_1}) = m_1 \sin 2n_1(\phi_1 - \psi_{n_1}) + \sum_{i=2}^{N} m_i \sin 2n_1(\phi_i - \psi_{n_1}) = 0. \quad \text{(A1)}$$

To derive equation (36), substitute equation (35) into equation (33) to give

$$\sum_{i=2}^{N} m_i \sin 2n_2(\phi_i - \psi_{n_2}) - \left(\sum_{i=2}^{N} m_i \frac{\sin 2n_1(\phi_i - \psi_{n_1})}{\sin 2n_1(\phi_1 - \psi_{n_1})}\right) \sin 2n_2(\phi_1 - \psi_{n_2}) = 0. \tag{A2}$$

Equation (A2) can be rearranged into

$$\sum_{i=2}^{N} m_i(\sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_i - \psi_{n_2}) - \sin 2n_1(\phi_i - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2})) = 0. \tag{A3}$$

Equation (A3) can be expanded into equation (36) in the same manner equation (31) was expanded into equation (35):

$$\begin{aligned} m_2 &= \frac{-\sum_{i-3}^{N} m_i(\sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_i - \psi_{n_2}) - \sin 2n_1(\phi_i - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2}))}{\sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_2 - \psi_{n_2}) - \sin 2n_1(\phi_2 - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2})} \\ &= -\textstyle\sum_{i=3}^{N} m_i \zeta_i/\zeta_2. \end{aligned} \tag{A4}$$

Now substituting equation (35) into equation (34) gives

$$\sum_{i=2}^{N} m_i \begin{pmatrix} \sin 2n_1(\phi_1 - \psi_{n_1}) \cos 2n_2(\phi_i - \psi_{n_2}) \\ - \sin 2n_1(\phi_i - \psi_{n_1}) \cos 2n_2(\phi_1 - \psi_{n_2}) \end{pmatrix} = M\lambda_{n_2} \sin 2n_1(\phi_1 - \psi_{n_1}). \tag{A5}$$

Substituting equation (A4) into equation (A5) gives

$$\frac{-\sum_{i=3}^{N} m_i \begin{pmatrix} \sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_i - \psi_{n_2}) \\ - \sin 2n_1(\phi_i - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2}) \end{pmatrix} \begin{pmatrix} \sin 2n_1(\phi_1 - \psi_{n_1}) \cos 2n_2(\phi_2 - \psi_{n_2}) \\ - \sin 2n_1(\phi_2 - \psi_{n_1}) \cos 2n_2(\phi_1 - \psi_{n_2}) \end{pmatrix}}{\sin 2n_1(\phi_1 - \psi_{n_1}) \sin 2n_2(\phi_2 - \psi_{n_2}) - \sin 2n_1(\phi_2 - \psi_{n_1}) \sin 2n_2(\phi_1 - \psi_{n_2})}$$

$$+ \sum_{i=3}^{N} m_{\text{i}} \begin{pmatrix} \sin 2n_1(\phi_1 - \psi_{n_1}) \cos 2n_2(\phi_i - \psi_{n_2}) \\ - \sin 2n_1(\phi_i - \psi_{n_1}) \cos 2n_2(\phi_1 - \psi_{n_2}) \end{pmatrix} = M\lambda_{n_2} \sin 2n_1(\phi_1 - \psi_{n_1}). \tag{A6}$$

Expanding equation (A6) produces

$$\sin 2n_1(\phi_1 - \psi_{n_1}) \sum_{i=3}^{N} m_i \left(\begin{array}{l} \sin 2n_1(\phi_1 - \psi_{n_1}) \left\{ \begin{array}{l} \sin 2n_2(\phi_2 - \psi_{n_2}) \cos 2n_2(\phi_i - \psi_{n_2}) \\ - \sin 2n_2(\phi_i - \psi_{n_2}) \cos 2n_2(\phi_2 - \psi_{n_2}) \end{array} \right\} \\ + \sin 2n_1(\phi_2 - \psi_{n_1}) \left\{ \begin{array}{l} \sin 2n_2(\phi_i - \psi_{n_2}) \cos 2n_2(\phi_1 - \psi_{n_2}) \\ - \sin 2n_2(\phi_1 - \psi_{n_2}) \cos 2n_2(\phi_i - \psi_{n_2}) \end{array} \right\} \\ + \sin 2n_1(\phi_i - \psi_{n_1}) \left\{ \begin{array}{l} \sin 2n_2(\phi_1 - \psi_{n_2}) \cos 2n_2(\phi_2 - \psi_{n_2}) \\ - \sin 2n_2(\phi_2 - \psi_{n_2}) \cos 2n_2(\phi_1 - \psi_{n_2}) \end{array} \right\} \end{array} \right)$$

$$= M\lambda_{n_2} \sin 2n_1(\phi_1 - \psi_{n_1}) \left\{ \begin{array}{l} \sin 2n_2(\phi_1 - \psi_{n_2}) \cos 2n_2(\phi_2 - \psi_{n_2}) \\ - \sin 2n_2(\phi_2 - \psi_{n_2}) \cos 2n_2(\phi_1 - \psi_{n_2}) \end{array} \right\}. \tag{A7}$$

Using standard trigonometric identities and substituting identities ζ_2 and χ_{1i} into equation (A7), it can be seen that

$$\sum_{i=3}^{N} m_i \chi_{1i} = m_3 \chi_{23} + \sum_{i=4}^{N} m_i \chi_{1i} = M\lambda_{n_2} \zeta_2. \tag{A8}$$

Hence, equation (37) has been derived. It can be shown in a similar way that equation (32) can be manipulated into the form

$$\sum_{i=3}^{N} m_i \chi_{2i} = m_3 \chi_{13} + \sum_{i=4}^{N} m_i \chi_{2i} = -M\lambda_{n_1} \zeta_2. \tag{A9}$$

Combining equations (A8) and (A9) produces equation (38), from which equation (42) can also be generated.

APPENDIX B: DERIVATION OF THE MODE TRIMMING EQUATIONS (43–46)

First consider equation (38). Assume that the masses m_4–m_N are in equilibrium in this equation. Now add a mass of M_k to trimming mass m_k and a mass of M_l to trimming mass m_l. This will modify equation (38) into the form

$$\begin{aligned}\sum_{i=4}^{k-1} m_i(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}) + (m_k + M_k)(\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k}) \\ + \sum_{i=k+1}^{l-1} m_i(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}) + (m_l + M_l)(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}) \\ + \sum_{i=l+1}^{k-1} m_i(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}) = M\zeta_2(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23}).\end{aligned} \tag{B1}$$

This can also be expressed as

$$\begin{aligned}\sum_{i=4}^{N} m_i(\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}) + M_k(\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k}) + M_l(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}) \\ = M\zeta_2(\lambda_{n_1}\chi_{13} + \lambda_{n_2}\chi_{23}).\end{aligned} \tag{B2}$$

As the equation was in equilibrium before the addition of M_k and M_l, it is possible to simplify equation (B2) to equation (43)

$$M_k(\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k}) + M_l(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}) = 0. \tag{B3}$$

In a similar way, replacing m_3 by $m_3 + M_3$, m_k by $m_k + M_k$ and m_l by $m_l + M_l$ in equation (37) reduces it to the form

$$M_3 = -M_k\,\chi_{1k}/\chi_{13} - M_l\,\chi_{1l}/\chi_{13}. \tag{B4}$$

Substituting equation (B3) into equation (B4) produces the equation, which can be simplified to form equation (46),

$$M_3 = M_k\,(\chi_{1l}(\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k}) - \chi_{1k}(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}))/\chi_{13}(\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}). \tag{B5}$$

Now making the relevant trimming mass modifications to equation (36) produces

$$M_2 = -M_3\,\zeta_3/\zeta_2 - M_k\,\zeta_k/\zeta_2 - M_l\,\zeta_l/\zeta_2. \tag{B6}$$

Substituting equations (43) and (46) into equation (B6) gives

$$M_2 = -M_k\left(\frac{\chi_{1k}\chi_{2l} - \chi_{1l}\chi_{2k}}{\chi_{1l}\chi_{23} - \chi_{13}\chi_{2l}}\right)\frac{\zeta_3}{\zeta_2} - M_k\frac{\zeta_k}{\zeta_2} + M_k\left(\frac{\chi_{23}\chi_{1k} - \chi_{13}\chi_{2k}}{\chi_{23}\chi_{1l} - \chi_{13}\chi_{2l}}\right)\frac{\zeta_l}{\zeta_2}. \tag{B7}$$

This is equation (45). Finally, equation (44) can be derived by performing the relevant trimming mass modifications to equation (35) and then substituting in the values of M_2, M_3 and M_l as given in equations (45), (46) and (43) respectively.

APPENDIX C: RULES FOR INVALID COMBINATIONS OF MASSES AND MODES

First consider the general sine terms of χ_{1i} and χ_{2i} for modes n_k, as shown in equations (40) and (41). These terms can be expressed as $\sin 2n_k(\phi_a - \psi_{n_k})$ and $\sin 2n_k(\phi_a - \phi_b)$. Substitution of the specific angular positions of the trimming masses into the first of these terms does not have an immediately obvious effect. Substitution into the second term however does have an obvious effect

$$\sin 2n_k(\phi_a - \phi_b) = \sin((4n_k\pi/N)(a-b)). \tag{C1}$$

Both a and b are integers and the minimum difference between them is 1. Thus, for any combination of a and b, if $n_k = N/4$ the value of equation (C1) will be 0. The direct effect of this is that if $n_1 = N/4$ then $\chi_{2i} = 0$ and if $n_2 = N/4$ then $\chi_{1i} = 0$. In both cases the denominator of equation (42) becomes zero so the trimming masses become indeterminable.

This is the first of the rules for predicting which combination of vibrational modes and trimming masses cannot produce valid solutions for dual-mode trimming.

The second condition for invalid solutions is not as immediately obvious. From equation (42) it is apparent that the trimming masses are dependent on the relationship $\chi_{23}\chi_{1i} - \chi_{13}\chi_{2i}$. If this relationship is zero then the trimming masses are indeterminable, so it is necessary to consider the χ_{1i} and χ_{2i} terms. Expanding the terms involving the mode orientation, ψ_n, produces the two relationships

$$\begin{aligned}\chi_{1i} = {} & \cos 2n_1\psi_{n1}\left\{\begin{matrix}\sin 2n_1\phi_1 \sin 2n_2(\phi_2-\phi_i) + \sin 2n_1\phi_2 \sin 2n_2(\phi_i-\phi_1)\\ + \sin 2n_1\phi_i \sin 2n_2(\phi_1-\phi_2)\end{matrix}\right\}\\ & - \sin 2n_1\psi_{n1}\left\{\begin{matrix}\cos 2n_1\phi_1 \sin 2n_2(\phi_2-\phi_i) + \cos 2n_1\phi_2 \sin 2n_2(\phi_i-\phi_1)\\ + \cos 2n_1\phi_i \sin 2n_2(\phi_1-\phi_2)\end{matrix}\right\},\end{aligned} \tag{C2}$$

$$\begin{aligned}\chi_{2i} = {} & \cos 2n_2\psi_{n2}\left\{\begin{matrix}\sin 2n_2\phi_1 \sin 2n_1(\phi_2-\phi_i) + \sin 2n_2\phi_2 \sin 2n_1(\phi_i-\phi_1)\\ + \sin 2n_2\phi_i \sin 2n_1(\phi_1-\phi_2)\end{matrix}\right\}\\ & - \sin 2n_2\psi_{n2}\left\{\begin{matrix}\cos 2n_2\phi_1 \sin 2n_1(\phi_2-\phi_i) + \cos 2n_2\phi_2 \sin 2n_1(\phi_i-\phi_1)\\ + \cos 2n_2\phi_i \sin 2n_1(\phi_1-\phi_2)\end{matrix}\right\}.\end{aligned} \tag{C3}$$

Substituting $n_2 = Nk/2 \mp n_1$, where k is an integer, into equation (C1) shows that

$$\sin 2n_2(\phi_a - \phi_b) = \sin(2k\pi \mp 4n_1\pi/N)(a-b) = \mp \sin 2n_1(\phi_a - \phi_b). \tag{C4}$$

Substituting equation (C4) into equation (C2) and performing a similar analysis on equation (C3) using $n_1 = Nk/2 \mp n_2$ produces the two equations

$$\begin{aligned}\chi_{1i} = {} & \mp \cos 2n_1\psi_{n1}\left\{\begin{matrix}\sin 2n_1\phi_1(\sin 2n_1\phi_2 \cos 2n_1\phi_i - \cos 2n_1\phi_2 \sin 2n_1\phi_i)\\ + \sin 2n_1\phi_2(\sin 2n_1\phi_i \cos 2n_1\phi_1 - \cos 2n_1\phi_i \sin 2n_1\phi_1)\\ + \sin 2n_1\phi_i(\sin 2n_1\phi_1 \cos 2n_1\phi_2 - \cos 2n_1\phi_1 \sin 2n_1\phi_2)\end{matrix}\right\}\\ & \pm \sin 2n_1\psi_{n1}\left\{\begin{matrix}\cos 2n_1\phi_1(\sin 2n_1\phi_2 \cos 2n_1\phi_i - \cos 2n_1\phi_2 \sin 2n_1\phi_i)\\ + \cos 2n_1\phi_2(\sin 2n_1\phi_i \cos 2n_1\phi_1 - \cos 2n_1\phi_i \sin 2n_1\phi_1)\\ + \cos 2n_1\phi_i(\sin 2n_1\phi_1 \cos 2n_1\phi_2 - \cos 2n_1\phi_1 \sin 2n_1\phi_2)\end{matrix}\right\},\end{aligned} \tag{C5}$$

$$\chi_{2i} = \mp \cos 2n_2\psi_{n2}\left\{\begin{array}{l}\sin 2n_2\phi_1(\sin 2n_2\phi_2 \cos 2n_2\phi_i - \cos 2n_2\phi_2 \sin 2n_2\phi_i) \\ + \sin 2n_2\phi_2(\sin 2n_2\phi_i \cos 2n_2\phi_1 - \cos 2n_2\phi_i \sin 2n_2\phi_1) \\ + \sin 2n_2\phi_i(\sin 2n_2\phi_1 \cos 2n_2\phi_2 - \cos 2n_2\phi_1 \sin 2n_2\phi_2)\end{array}\right\}$$

$$\pm \sin 2n_2\psi_{n2}\left\{\begin{array}{l}\cos 2n_2\phi_1(\sin 2n_2\phi_2 \cos 2n_2\phi_i - \cos 2n_2\phi_2 \sin 2n_2\phi_i) \\ + \cos 2n_2\phi_2(\sin 2n_2\phi_i \cos 2n_2\phi_1 - \cos 2n_2\phi_i \sin 2n_2\phi_1) \\ + \cos 2n_2\phi_i(\sin 2n_2\phi_1 \cos 2n_2\phi_2 - \cos 2n_2\phi_1 \sin 2n_2\phi_2)\end{array}\right\}. \quad \text{(C6)}$$

It can easily be seen from equations (C5) and (C6) that both the χ_{1i} and χ_{2i} terms are equal to zero. Thus, the trimming masses cannot be determined from equation (38) for this combination of modes of vibration.

APPENDIX D: DIMENSIONS AND MATERIAL PROPERTIES OF THE PERFECT RING

Mean radius = 0·3 m, density = 7850 kg/m^3, radial thickness = 0·005 m, Young's modulus = $2{\cdot}06 \times 10^{11}$ N/m^2, axial length = 0·1 m, the Poisson ratio = 0·3, mass of ring = 7·3984 kg.

TABLE D1

Natural frequencies and amplitude ratios

Nodal diameters, n_j	Natural frequency (Hz)	Amplitude ratio W/U
2	36·779	1·99992
3	104·026	2·99956
4	199·461	3·99877
5	322·571	4·99743

TABLE D2

Natural frequencies formed by the addition of 0·1, 0·2 *and* 0·3 *kg at* 0, 20 *and* 70°

Nodal diameters, n_j	Natural frequency (Hz)		Orientation (deg)	
2	35·096	35·656	− 6·95	38·05
3	97·832	102·423	11·82	41·82
4	188·02	195·89	23·16	0·66
5	305·71	314·97	− 4·14	13·86

APPENDIX E: NOMENCLATURE

a	ring cross-sectional area (= hL)
E	Young's modulus
h	ring radial thickness
L	ring axial length
R	ring mean radius
β	$(= h^2/(12R^2))$
ρ	density of ring material
ν	the Poisson ratio