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SOME OBSERVATIONS ON THE CHARACTERIZATION OF STRUCTURAL DAMPING

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This paper deals with the characterization of damping in dynamical structural systems. In particular, the problem of how the modal damping ratios change with different boundary conditions is addressed. It is shown that only Rayleigh-type damping is actually independent of boundary conditions and modal damping ratios can be easily converted from one boundary condition to another. This condition applies independently to continuous, discrete and discretized systems.

1. INTRODUCTION

Damping often plays an important role in the prediction of the dynamical response of engineering structures. However, since damping arises from many sources, it is difficult to describe analytically and in a simple and comprehensive way the complex physical phenomena which determine the energy dissipation that ultimately manifests itself in the form of attenuation of the structural response [1]. This is so much true that, so far, a general mathematical model for the description of damping in the dynamical behaviour of structures does not exist as it does for the description of other important mechanical properties such as mass, stiffness and strength. The structural engineer is, therefore, restricted in the description of the damping mechanism to the prescription of the modal damping ratios which may be evaluated by conducting experiments on real structures. These damping ratios, on the other hand, are based on a viscous mechanism of energy dissipation that has no rivals in terms of mathematical simplicity and physical efficiency [2, 3].

Although it has been proved in the literature that the actual mechanism of energy dissipation in real structures is closer to the so-called hysteretic damping than to the viscous damping, the latter, properly tuned, proves to be, by far, more efficient and reliable.

The prescription of damping in terms of modal damping ratios is of simple implementation in linear structures. In the non-linear dynamical behaviour of engineering structures, the effects of linear damping are often overshadowed by the energy dissipation due to the non-linear mechanical behaviour, but methods for the synthesis of the damping matrix from the modal damping ratios have been available in the literature for some time [2, 3].

Some problems, however, arise when prescribing modal damping ratios in the linear or non-linear dynamic analysis of engineering structures. It is, in general, impractical to conduct experiments on any analyzed structure, especially if this is still in the design stage. Usually, reference is made to experiments carried out on similar structures. However, structures are rarely identical and may often differ considerably in boundary or support conditions. The difference in the boundary conditions often results in a considerable difference in the periods or frequencies of vibration. One of the objects of the present paper is as to how the difference in frequencies affects the modal damping ratios.

Another important instance arises in dynamical soil–structure interaction. Damping in a soil–structure interacting system receives contributions from both the soil and the structure. The modal damping ratios reflect this circumstance in a complicated manner as has been shown in the literature [4–6]. The definition of the damping matrix of soil–structure interacting system may be achieved by the superposition of the damping matrix of the superstructure and the damping matrix of the foundation and the soil. However the superstructure must be treated in this instance as a free structure, i.e., a structure without supports. Clearly, experimental results on the modal damping ratios of buildings and other engineering structures with the foundations or supports removed do not exist. Modal damping ratios of similar buildings or structures founded on firm soil or rock are, however, generally available.

Does a relationship exist between the modal damping ratios of a building structure fixed at the base and the modal damping ratios of the same structure without supports? An answer to this question is the main object of the present paper.

In the theoretical development of this work, reference will often be made to mass-proportional, stiffness-proportional, Rayleigh and Caughey damping. These terms are now standard in the literature on structural dynamics. However, the reader not familiar with these concepts is referred to the standard textbooks such as Clough and Penzien [2, chapter 12] and Chopra [3, chapter 11]. For a comprehensive definition of classical damping, the reader is referred to the original work by Caughey and O'Kelly [7].

Finally, ample use is made in the following developments of the Cartesian tensor calculus and of the summation convention. For the readers not aware of these analytical tools, a concise and effective introduction may be found in the elegant booklet on Continuum Mechanics by Spencer [8].

2. THE INFLUENCE OF THE BOUNDARY CONDITIONS

In order to establish the dependence of the modal damping ratios on the boundary conditions, a simple continuous model is considered. A uniform Bernoulli beam in axial vibration constitutes a very simple model which can be studied in three different end conditions: (a) free ends; (b) one end fixed and the other end free; (c) fixed ends.

A uniform shear beam, with the same end conditions as above, exhibits the same dimensionless frequencies and modes of vibration as the corresponding beam in axial vibration. The shear beam, however, is often considered representative of more complex structural systems such as frame resisting buildings. The considered beam has length L, uniform distributed mass m, normal elasticity modulus E and area of the cross-section A.

The dimensionless frequencies $p_k = \omega_k/\sqrt{EA/mL^2}$ and the corresponding modes of vibration of the above continuous model with the end conditions (a), (b) and (c) are reported in Table 1. The modes of vibration have been referred to a dimensionless abscissa $\eta = x/L$ and their amplitudes B_k have been chosen in order to provide unit modal masses.

It is worth noting that in the case of the beam with free ends the first mode of vibration is a rigid body translation and corresponds to the zero frequency.

The modal damping ratios will depend on the type of viscous damping considered. In Table 2, the modal damping ratios for the three end conditions are reported for the two

TABLE 1

Dimensionless frequencies $p_k(k = 1, \ldots, \infty)$ and modes of vibration $\boldsymbol{\varphi}_k(k = 1, \ldots, \infty)$ of uniform shear beams and/or axially vibrating Bernoulli beams†

End conditions	Free–free	Fixed–free	Fixed–fixed
Frequencies p_k	$(k-1)\pi$	$(2k-1)\pi/2$	$k\pi$
Modal shapes $\boldsymbol{\varphi}_k$	$B_k \cos(k-1)\pi\eta$	$B_k \sin(2k-1)\pi\eta/2$	$B_k \sin k\pi\eta$

†$B_1 = \sqrt{1/m}$, $B_k = \sqrt{2/m}$ for $k > 1$ for unrestrained beams; $B_k = \sqrt{2/m}\ \forall k$ for restrained beams.

TABLE 2

Normalized modal damping ratios of uniform shear beams and/or axially vibrating Bernoulli beams for the two cases of mass-proportional and stiffness-proportional damping

End conditions	Free–free	Fixed–free	Fixed–fixed
Mass-proportional damping ξ_m	$1/(2(\mathrm{k}-1)\pi)$	$1/[(2k-1)\pi]$	$1/(2k\pi)$
Stiffness-proportional damping ξ_s	$(k-1)\pi/2$	$(2k-1)\pi/4$	$k\pi$

cases of mass- and stiffness-proportional damping. In the two previous damping models, the viscous damping constant c has been, respectively, expressed by $c_m = a_m m$ and $c_s = a_s E$.

The standard mass- and stiffness-proportional damping ratios ζ_{m_k} and ζ_{s_k} have been normalized resulting in the corresponding ratios $\xi_{m_k} = \zeta_{m_k}(m/a_m)\sqrt{EA/mL^2}$ and $\xi_{s_k} = \zeta_{s_k}/(a_s\sqrt{EA/mL^2})$ which may be fully expressed in terms of the integer number k.

From the results of Table 2, it appears at first sight that the modal damping ratios are highly dependent on the end conditions. A closer look, however, depicts a quite different situation. While the modal damping ratios are proportional to the reciprocal of the frequency for the mass-proportional damping, the same are proportional to the frequency for the stiffness-proportional damping. It is easy to realize that the damping ratios for different end conditions are in the same ratio as the frequencies.

Therefore, the knowledge of the change in frequency allows for the prediction of the change in the damping ratios. The same can be concluded for the more general case of the Rayleigh damping law. Once the general dependence of the damping ratios on the frequency has been established, this applies for every end conditions. The knowledge of the frequency allows for the prediction of the damping ratio.

3. TENSOR REPRESENTATION OF MASS, DAMPING AND STIFFNESS OPERATORS

The mass, damping and stiffness properties of a linear structure may be expressed in tensor form in a way that becomes independent of their representation. For instance, the mass, damping and stiffness operators may be written as

$$\mathbf{M} = m(\eta)\,\boldsymbol{\varphi}_k(\eta) \otimes m(\eta)\,\boldsymbol{\varphi}_k(\eta) = \sum_{k=1}^{\infty} \mathbf{M}^k, \tag{1}$$

$$\mathbf{K} = \omega_{(k)}^2 m(\eta)\boldsymbol{\varphi}_k(\eta) \otimes m(\eta)\boldsymbol{\varphi}_k(\eta) = \sum_{k=1}^{\infty} \mathbf{K}^k, \tag{2}$$

$$\mathbf{D} = 2\xi_{(k)}\omega_{(k)} m(\eta)\boldsymbol{\varphi}_k(\eta) \otimes m(\eta)\boldsymbol{\varphi}_k(\eta) = \sum_{k=1}^{\infty} \mathbf{D}^k, \tag{3}$$

where $m(\eta)$ is the mass distribution, ($\boldsymbol{\varphi}_k(\eta)$, $k = 1, 2, \dots$) are the natural modes of vibration, ξ_k is the modal damping ratio, ω_k is the kth natural frequency and the symbol $\otimes$ stands for the dyadic or tensor product [8]. It may be worth noting that the summation with respect to k in the above expressions implies the decomposition of the above operators in terms of their modal contributions $\mathbf{M}^k$, $\mathbf{D}^k$, $\mathbf{K}^k$. In this way, any damping ratio can be prescribed for each mode of vibration. The mass, damping and stiffness matrices can be easily obtained from the above tensor expressions.

For instance, in principal co-ordinates, the mass, damping and stiffness coefficients can be calculated as

$$\boldsymbol{M}_{pq} = \boldsymbol{\varphi}_p(\eta)\cdot\mathbf{M}\cdot\boldsymbol{\varphi}_q(\eta) = [\boldsymbol{\varphi}_p(\eta)\cdot m(\eta)\boldsymbol{\varphi}_k(\eta)][m(\eta)\boldsymbol{\varphi}_k(\eta)\cdot\boldsymbol{\varphi}_q(\eta)], \tag{4}$$

$$\boldsymbol{K}_{pq} = \boldsymbol{\varphi}_p(\eta)\cdot\mathbf{K}\cdot\boldsymbol{\varphi}_q(\eta) = \omega_{(k)}^2[\boldsymbol{\varphi}_p(\eta)\cdot m(\eta)\boldsymbol{\varphi}_k(\eta)]\,[m(\eta)\boldsymbol{\varphi}_k(\eta)\cdot\boldsymbol{\varphi}_q(\eta)], \tag{5}$$

$$\boldsymbol{D}_{pq} = \boldsymbol{\varphi}_p(\eta)\cdot\mathbf{D}\cdot\boldsymbol{\varphi}_q(\eta) = 2\xi_{(k)}\omega_{(k)}[\boldsymbol{\varphi}_p(\eta)\cdot m(\eta)\boldsymbol{\varphi}_k(\eta)]\,[m(\eta)\boldsymbol{\varphi}_k(\eta)\cdot\boldsymbol{\varphi}_q(\eta)], \tag{6}$$

where the inner product $[\boldsymbol{\varphi}_i(\eta)\cdot m(\eta)\boldsymbol{\varphi}_j(\eta)]$ can be expressed as

$$[\boldsymbol{\varphi}_i(\eta)\cdot m(\eta)\boldsymbol{\varphi}_j(\eta)] = \int_0^1 m(\eta)\boldsymbol{\varphi}_i(\eta)\boldsymbol{\varphi}_j(\eta)\,\mathrm{d}\eta = [m(\eta)\boldsymbol{\varphi}_i(\eta)\cdot\boldsymbol{\varphi}_j(\eta)]. \tag{7}$$

By accounting for the orthonormality conditions of the natural modes of vibration

$$[\boldsymbol{\varphi}_i(\eta)\cdot m(\eta)\boldsymbol{\varphi}_j(\eta)] = [m(\eta)\boldsymbol{\varphi}_i(\eta)\cdot\boldsymbol{\varphi}_j(\eta)] = \delta_{ij}, \tag{8}$$

where δ_{ij} is the Kronecker symbol, the above mass, damping and stiffness coefficients become

$$\boldsymbol{M}_{pq} = \delta_{pk}\delta_{kq} = \sum_{k=1}^{\infty}\boldsymbol{M}_{pq}^k, \qquad \boldsymbol{K}_{pq} = \omega_{(k)}^2\delta_{pk}\delta_{kq} = \sum_{k=1}^{\infty}\boldsymbol{K}_{pq}^k,$$

$$\boldsymbol{D}_{pq} = 2\xi_{(k)}\omega_{(k)}\delta_{pk}\delta_{kq} = \sum_{k=1}^{\infty}\boldsymbol{D}_{pq}^k. \tag{9–11}$$

This shows, as should have been expected, that in principal co-ordinates, the modal mass, damping and stiffness matrices have only one non-zero term relative to the corresponding natural mode. The full mass, damping and stiffness matrices are obviously diagonal with each non-zero term corresponding to a given natural mode.

For continuous systems, as in the present case, these matrices are obviously infinitely dimensional. If any other basis is chosen, then the modal matrices will generally be full and so will be the complete ones, i.e., the matrices resulting from the summation of the modal ones.

Special bases may be obtained from finite element discretizations of the beam. Such bases generally lead to banded finite dimensional matrices.

4. GENERAL RELATIONS BETWEEN THE DYNAMICAL PARAMETERS OF BEAMS WITH DIFFERENT END CONDITIONS

Let $\boldsymbol{\varphi}_k(\eta)$, ω_k be the eigenfunctions and eigenfrequencies of a beam with free ends, $\bar{\boldsymbol{\varphi}}_k(\eta)$, $\bar{\omega}_k$ the eigenfunctions and eigenfrequencies of the same beam with one end fixed and finally $\bar{\bar{\boldsymbol{\varphi}}}_k(\eta)$, $\bar{\bar{\omega}}_k$ the same characteristics for the beam with both ends fixed.

It is obvious that the orthonormal basis ($\bar{\bar{\boldsymbol{\varphi}}}_k(\eta)$, $k = 1, \ldots, \infty$), spans a subspace of the space spanned by the orthonormal basis ($\bar{\boldsymbol{\varphi}}_k(\eta)$, $k = 1, \ldots, \infty$) which in turn is a subspace of the space spanned by the basis ($\boldsymbol{\varphi}_k(\eta), k = 1, \ldots, \infty$). If ($\bar{\bar{\mathbf{M}}}, \bar{\bar{\mathbf{D}}}, \bar{\bar{\mathbf{K}}}$), ($\bar{\mathbf{M}}, \bar{\mathbf{D}}, \bar{\mathbf{K}}$), ($\mathbf{M}, \mathbf{D}, \mathbf{K}$) are the mass, damping and stiffness operators for the three previously mentioned beams, this means that the components of an operator defined in a given subspace may be calculated in terms of the operators defined in any of the larger vector spaces.[†,‡] For instance, in principal co-ordinates, it is

$$\bar{\bar{M}}_{ij} = \bar{\bar{\boldsymbol{\varphi}}}_i(\eta)\cdot\bar{\bar{\mathbf{M}}}\cdot\bar{\bar{\boldsymbol{\varphi}}}_j(\eta) = \bar{\bar{\boldsymbol{\varphi}}}_i\cdot(\eta)\bar{\mathbf{M}}\cdot\bar{\bar{\boldsymbol{\varphi}}}_j(\eta) = \bar{\bar{\boldsymbol{\varphi}}}_i(\eta)\cdot\mathbf{M}\bar{\bar{\boldsymbol{\varphi}}}_j\cdot(\eta), \tag{12}$$

$$\bar{M}_{ij} = \bar{\boldsymbol{\varphi}}_i(\eta)\cdot\bar{\mathbf{M}}\cdot\bar{\boldsymbol{\varphi}}_j(\eta) = \bar{\boldsymbol{\varphi}}_i(\eta)\cdot\mathbf{M}\cdot\bar{\boldsymbol{\varphi}}_j(\eta), \qquad M_{ij} = \boldsymbol{\varphi}_i(\eta)\cdot\mathbf{M}\cdot\boldsymbol{\varphi}_j(\eta) \tag{13, 14}$$

and the same relations can be written for the damping and the stiffness operators. By using, in the previous equations, the proper expressions for the mass operators, one finds the identities:

$$\bar{\bar{M}}_{ij} = \bar{\bar{\delta}}_{ij} = \bar{\bar{\beta}}_{ik}\bar{\bar{\beta}}_{jk} = \bar{\bar{\alpha}}_{ik}\bar{\bar{\alpha}}_{jk}, \qquad \bar{M}_{ij} = \bar{\delta}_{ij} = \bar{\alpha}_{ik}\bar{\alpha}_{jk}, \qquad M_{ij} = \delta_{ij}, \tag{15–17}$$

where

$$\bar{\alpha}_{ik} = [m\boldsymbol{\varphi}_k\cdot\bar{\boldsymbol{\varphi}}_i] = \int_0^1 m(\eta)\bar{\boldsymbol{\varphi}}_i(\eta)\boldsymbol{\varphi}_k(\eta)\,\mathrm{d}\eta, \qquad \bar{\bar{\alpha}}_{ik} = [m\boldsymbol{\varphi}_k\cdot\bar{\bar{\boldsymbol{\varphi}}}_i] = \int_0^1 m(\eta)\bar{\bar{\boldsymbol{\varphi}}}_i(\eta)\boldsymbol{\varphi}_k(\eta)\,\mathrm{d}\eta, \tag{18, 19}$$

$$\bar{\bar{\beta}}_{ik} = [m\bar{\boldsymbol{\varphi}}_k\cdot\bar{\bar{\boldsymbol{\varphi}}}_i] = \int_0^1 m(\eta)\bar{\bar{\boldsymbol{\varphi}}}_i(\eta)\boldsymbol{\varphi}_k(\eta)\,\mathrm{d}\eta. \tag{20}$$

For the stiffness operator, the following may be found

$$\bar{\bar{K}}_{ij} = \bar{\bar{\omega}}^2_{(k)}\bar{\bar{\delta}}_{ik}\bar{\bar{\delta}}_{kj} = \bar{\omega}^2_{(k)}\bar{\bar{\beta}}_{ij}\bar{\bar{\beta}}_{jk} = \omega^2_{(k)}\bar{\bar{\alpha}}_{ik}\bar{\bar{\alpha}}_{jk}, \tag{21}$$

$$\bar{K}_{ij} = \bar{\omega}^2_{(k)}\bar{\delta}_{ik}\bar{\delta}_{kj} = \omega^2_{(k)}\bar{\alpha}_{ik}\bar{\alpha}_{jk}, \qquad K_{ij} = \omega^2_{(k)}\delta_{ik}\delta_{kj}. \tag{22, 23}$$

As expected, all the above matrices are diagonal with the non-zero terms taking the expressions:

$$\bar{\bar{K}}_{(p)(p)} = \bar{\bar{\omega}}^2_{(p)} = \bar{\omega}^2_{(k)}\bar{\bar{\beta}}_{(p)k}\bar{\bar{\beta}}_{(p)k} = \omega^2_{(k)}\bar{\bar{\alpha}}_{(p)k}\bar{\bar{\alpha}}_{(p)k}, \tag{24}$$

$$\bar{K}_{(p)(p)} = \bar{\omega}^2_{(p)} = \omega^2_{(k)}\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k}, \qquad K_{(p)(p)} = \omega^2_{(p)}. \tag{25, 26}$$

[†] While this is true for the mass, damping and stiffness operators **M**, **K** and **D**, it is not true for functions of operators such as powers like $\mathbf{M}^n$, $\mathbf{K}^n$, $\mathbf{D}^n$ and roots like $\mathbf{M}^{1/n}$, $\mathbf{K}^{1/n}$, $\mathbf{D}^{1/n}$. See Appendix A for a proof.

[‡] A formal proof of this statement is reported in Appendix B in the case of finite-dimensional vector spaces.

Similar relationships hold for the modal damping ratios

$$\bar{\bar{\boldsymbol{D}}}_{(p)(p)} = 2\bar{\bar{\xi}}_{(p)}\bar{\bar{\omega}}_{(p)} = 2\bar{\xi}_{(k)}\bar{\omega}_{(k)}\bar{\bar{\beta}}_{(p)k}\bar{\bar{\beta}}_{(p)k} = 2\xi_{(k)}\omega_{(k)}\bar{\bar{\alpha}}_{(p)k}\bar{\bar{\alpha}}_{(p)k}, \tag{27}$$

$$\bar{\boldsymbol{D}}_{(p)(p)} = 2\bar{\xi}_{(p)}\bar{\omega}_{(p)} = 2\xi_{(k)}\omega_{(k)}\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k}, \qquad \boldsymbol{D}_{(p)(p)} = 2\xi_{(p)}\omega_{(p)}. \tag{28, 29}$$

5. VERIFICATION OF RESULTS

The results found so far can be checked against those available for uniform beams in the two cases of mass- and stiffness-proportional damping. These results may be collected in the statement: "*given a uniform beam in axial or shear vibration, if the damping is mass-proportional (stiffness-proportional) for one end condition it will be mass-proportional (stiffness-proportional) for any other end condition*".
Let us suppose

$$\xi_{(k)}\omega_{(k)} = c_m \text{ (mass-proportional damping)},$$

$$\xi_{(k)} = c_s\omega_{(k)} \text{ (stiffness-proportional damping)}. \tag{30, 31}$$

From equalities (27) and (28), it follows

$$\bar{\xi}_{(p)}\bar{\omega}_{(p)} = c_m\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} = c_m, \qquad \bar{\bar{\xi}}_{(p)}\bar{\bar{\omega}}_{(p)} = c_m\bar{\bar{\alpha}}_{(p)k}\bar{\bar{\alpha}}_{(p)k} = c_m, \tag{32, 33}$$

for mass-proportional damping, and

$$\bar{\xi}_{(p)}\bar{\omega}_{(p)} = c_s\omega^2_{(k)}\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} = c_s\bar{\omega}^2_{(p)}, \quad \bar{\bar{\xi}}_{(p)}\bar{\bar{\omega}}_{(p)} = c_s\omega^2_{(k)}\bar{\bar{\alpha}}_{(p)k}\bar{\bar{\alpha}}_{(p)k} = c_s\bar{\bar{\omega}}^2_{(p)}, \tag{34, 35}$$

for stiffness-proportional damping.

From the previous equalities, it follows that

$$\xi_{(p)}\omega_{(p)} = \bar{\xi}_{(p)}\bar{\omega}_{(p)} = \bar{\bar{\xi}}_{(p)}\bar{\bar{\omega}}_{(p)} = c_m, \qquad \bar{\bar{\xi}}_{(p)}/\bar{\bar{\omega}}_{(p)} = \bar{\xi}_{(p)}/\bar{\omega}_{(p)} = \xi_{(p)}/\omega_{(p)} = c_s. \tag{36, 37}$$

These results allow for the generalization of the previous statement. *For any beam satisfying the standard orthogonality conditions*,

$$\int_0^1 m(\eta)\,\boldsymbol{\varphi}_i(\eta)\,\boldsymbol{\varphi}_j(\eta)\,\mathrm{d}\eta = \delta_{ij} \quad (i, j = 1, 2, \ldots, \infty) \tag{38}$$

if the damping is mass-(stiffness-) proportional for one end condition it will be mass- (stiffness-) proportional for any other end condition.

Therefore, the previous results not only apply to non-uniform beams but also to torsional and flexural vibrations. The extension of the statement to Rayleigh damping is trivial and will not be pursued here.

6. FINITE-DIMENSIONAL AND DISCRETIZED SYSTEMS

Discretized systems are finite dimensional as inherently are many physical models of real structures.

Mass, damping and stiffness operators for discrete systems take the forms

$$\mathbf{M} = \mathbf{M}\cdot\boldsymbol{\varphi}_k\otimes\boldsymbol{\varphi}_k\cdot\mathbf{M}, \quad \mathbf{K} = \omega^2_{(k)}\mathbf{M}\cdot\boldsymbol{\varphi}_k\otimes\boldsymbol{\varphi}_k\cdot\mathbf{M}, \quad \mathbf{D} = 2\xi_{(k)}\omega_{(k)}\mathbf{M}\cdot\boldsymbol{\varphi}_k\otimes\boldsymbol{\varphi}_k\cdot\mathbf{M}. \tag{39–41}$$

where $\omega_{(k)}$ and $\boldsymbol{\varphi}_k$ are the eigenfrequencies and eigenvectors (or modes of vibration) of the finite-dimensional system.

The orthonormality condition usually holds for such systems:

$$\boldsymbol{\varphi}_i \cdot \mathbf{M} \cdot \boldsymbol{\varphi}_j = \delta_{ij} \quad (i, j = 1, 2, \ldots, N). \tag{42}$$

Let ($\boldsymbol{\varphi}_k$, $k = 1, 2, \ldots, N$) be the complete set of eigenvectors of an unrestrained (i.e., free) discrete system and ($\omega_k, k = 1, 2, \ldots, N$) the associated set of eigenfrequencies. Let ($\bar{\boldsymbol{\varphi}}_k, k = 1, 2, \ldots, M$) with $M < N$ be the complete set of eigenvectors of the same system considered before with the addition of some restrains. Let ($\bar{\omega}_k, k = 1, 2, \ldots, M$) be the corresponding set of eigenfrequencies. The orthonormal basis ($\boldsymbol{\varphi}_k$, $k = 1, 2, \ldots, N$) spans an N-dimensional vector space $\mathbf{V}_u$ while the orthonormal basis ($\bar{\boldsymbol{\varphi}}_k, k = 1, 2, \ldots, M$) spans an M-dimensional vector space $\mathbf{V}_s$ which is a subspace of $\mathbf{V}_u$, i.e.,

$$\mathbf{V}_s \subset \mathbf{V}_u. \tag{43}$$

The suffixes s and u stand here for supported and unsupported systems respectively. The fact that $\mathbf{V}_s$ is a subspace of $\mathbf{V}_u$ allows for the transposition of the results obtained for infinite-dimensional systems to finite-dimensional ones. In particular, the components of the mass, damping and stiffness operators in principal co-ordinates take the following expressions:

$$\boldsymbol{M}_{ij} = \boldsymbol{\varphi}_i \cdot \mathbf{M} \cdot \boldsymbol{\varphi}_j = \delta_{ik}\delta_{kj}, \quad \boldsymbol{K}_{ij} = \boldsymbol{\varphi}_i \cdot \mathbf{K} \cdot \boldsymbol{\varphi}_j = \omega_{(k)}^2 \delta_{ik}\delta_{kj}, \quad \boldsymbol{D}_{ij} = \boldsymbol{\varphi}_i \cdot \mathbf{D} \cdot \boldsymbol{\varphi}_j = 2\xi_{(k)}\omega_{(k)}\delta_{ik}\delta_{kj}. \tag{44–46}$$

7. RELATIONS BETWEEN DYNAMICAL CHARACTERISTICS OF SUPPORTED AND UNSUPPORTED SYSTEMS

Since $\mathbf{V}_s$ is a proper subspace of $\mathbf{V}_u$, the dynamical characteristics of the supported system may be obtained from the operators of the unsupported system, i.e.,

$$\bar{\boldsymbol{M}}_{ij} = \bar{\boldsymbol{\varphi}}_i \cdot \mathbf{M} \cdot \bar{\boldsymbol{\varphi}}_j = \bar{\alpha}_{ik}\bar{\alpha}_{jk} = \delta_{ij}, \qquad \bar{\boldsymbol{K}}_{ij} = \bar{\boldsymbol{\varphi}}_i \cdot \mathbf{K} \cdot \bar{\boldsymbol{\varphi}}_j = \omega_{(k)}^2 \bar{\alpha}_{ik}\bar{\alpha}_{jk} = \bar{\omega}_{(k)}^2 \delta_{ik}, \delta_{kj}, \tag{47, 48}$$

$$\bar{\boldsymbol{D}}_{ij} = \bar{\boldsymbol{\varphi}}_i \cdot \mathbf{D} \cdot \bar{\boldsymbol{\varphi}}_j = 2\xi_{(k)}\omega_{(k)}\bar{\alpha}_{ik}\bar{\alpha}_{jk} = 2\bar{\xi}_{(k)}\bar{\omega}_{(k)}\delta_{ik}, \delta_{kj}, \tag{49}$$

where

$$\bar{\alpha}_{ik} = \bar{\boldsymbol{\varphi}}_i \cdot \mathbf{M} \cdot \boldsymbol{\varphi}_k \quad (i = 1, \ldots, M; k = 1, \ldots, N). \tag{50}$$

The relationships which provide the characteristics of the supported system in terms of those of the unsupported one may be derived from the more general relationships represented by equations (48) and (49) i.e.,

$$\bar{\omega}_{(p)}^2 = \omega_{(k)}^2 \bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} \quad (k = 1, \ldots, N; \quad p = 1, \ldots, M), \tag{51}$$

$$\bar{\xi}_{(p)}\bar{\omega}_{(p)} = \xi_{(k)}\omega_{(k)}\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} \quad (k = 1, \ldots, N; p = 1, \ldots, M). \tag{52}$$

With these relationships at hand the results already established for infinite-dimensional systems may be extended to discrete ones. In particular, the following statement is true.

Let the damping in a discrete unrestrained structure belong to one of the classes: mass-proportional, stiffness-proportional, Rayleigh-type, then the damping in any structure *obtained from the previous one by removing some of its degrees of freedom will be of the same class.*

For more general damping laws, the modal damping ratios of the restrained system may be obtained from those of the unrestrained system by using the general relation already established. This, however, is not always convenient because, even if only a few modal damping ratios are required for the restrained system, the complete set of dynamical characteristics of the unrestrained system is required for their evaluation

8. GENERAL DAMPING LAWS

In the previous paragraphs, it has been shown that the modal damping ratios of structural systems with Rayleigh-type damping laws are independent of the boundary conditions. In other words, in order to specify the modal damping ratios, only the frequencies of vibration and the two Rayleigh coefficients are needed. For instance, if the Rayleigh coefficients have been determined for an unrestrained system, then the same coefficients apply to any restrained system obtained from the previous unrestrained one. Therefore, it is trivial to evaluate the damping ratios for any boundary condition once the Rayleigh coefficients are known.

In this paragraph, the following question is addressed: *Does a more general damping law than the Rayleigh one, independent of the boundary conditions exist*? If such a law were to exist, then it should satisfy some basic requirements which are derived below.

For any restrained and a corresponding unrestrained system, the relationship

$$2\bar{\xi}_{(p)}\bar{\omega}_{(p)} = 2\xi_{(k)}\omega_{(k)}\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} \tag{53}$$

holds. If the damping ratios are independent of the boundary conditions, then they must be specified only in terms of the frequencies of vibration. In other words, a function of the frequency ω must exist such that

$$2\xi_k\omega_k = \Im(\omega_k) \quad \forall\omega_k, \qquad 2\bar{\xi}_k\bar{\omega}_k = \Im(\bar{\omega}_k) \quad \forall\bar{\omega}_k. \tag{54}$$

With such positions, equation (53) becomes

$$\Im(\bar{\omega}_p) = \Im(\omega_k)\,\bar{\alpha}_{(p)k}\bar{\alpha}_{(p)k} \quad \forall\bar{\omega}_p. \tag{55}$$

To answer the previous question, the most general form of the function $\Im(\omega_k)$ must be found. By confining oneself to the most popular case where the damping can be specified in terms of arbitrary modal damping ratios, the most general damping law can be expressed by the Caughey law

$$\Im(\omega_k) = a_l\omega_k^{2l}. \tag{56}$$

This law contains, as particular cases, the mass- and stiffness-proportional damping laws and consequently, the Rayleigh damping law.

By a proper choice of the coefficients a_l ($l = 1, 2, \ldots$), any combination of modal damping ratios can be prescribed. For the damping law to be independent of the boundary conditions, equation (55) must be satisfied, that is

$$a_l\bar{\omega}_p^{2l} = a_l\alpha_{pk}^2\omega_k^{2l}. \tag{57}$$

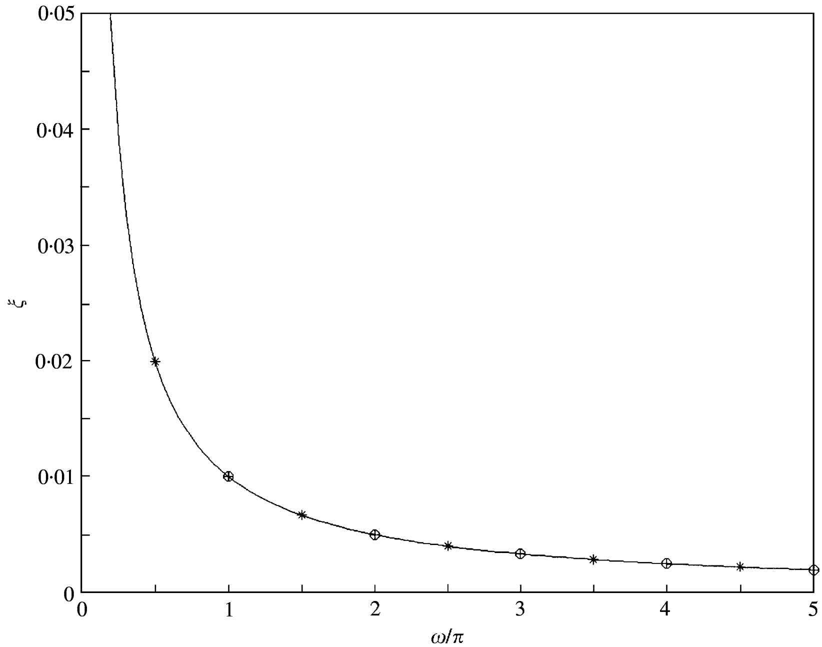

Figure 1. Modal damping ratios for the three considered end conditions for mass-proportional damping. Key for structure: ○, free–free; ∗, fixed–free; +, fixed–fixed.

By the principle of polynomial equalities, the previous relationship can only be satisfied if

$$\bar{\omega}_p^{2l} = \alpha_{pk}^2 \omega_k^{2l} \quad \forall l. \tag{58}$$

The above equalities are certainly satisfied for $l = 0$ and 1 but cannot be satisfied for any other l. Therefore, for the identity (57) to be true, it must be

$$a_l = 0 \quad \forall l > 1. \tag{59}$$

Therefore, one can state: *Rayleigh damping is the most general law which is independent of the boundary conditions.*

9. NUMERICAL APPLICATIONS

9.1. CONTINUOUS SYSTEMS

9.1.1. *Mass- and stiffness-proportional damping*

In what follows, some numerical applications are reported in order to illustrate in graphical form the main results obtained in the paper. First of all, the results reported in Table 2 are represented. In the graph of Figure 1 the modal damping ratios for a uniform beam with the end conditions considered in Table 2 are reported for the case of mass-proportional damping and axial-, torsional- or shear-type vibrations. The different damping ratios corresponding to different end conditions are distinguished with a circle (free–free), a star (fixed–free) and a cross (fixed–fixed).

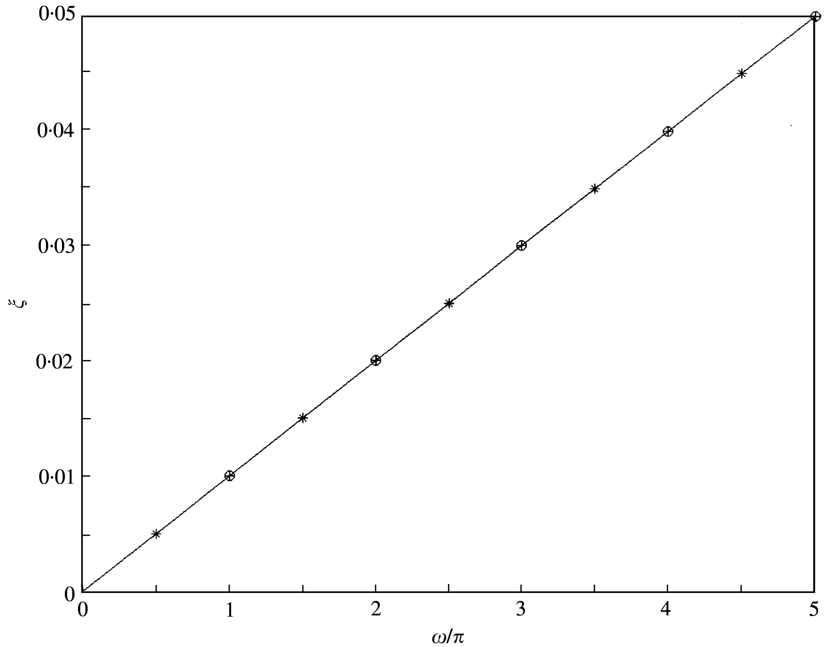

Figure 2. Modal damping ratios for the three considered end conditions for stiffness-proportional damping. Key for structure: ○, free–free; ∗, fixed–free; +, fixed–fixed.

It is interesting to note that circles and crosses are superimposed. This occurs because the $(n + 1)$th frequency of the free–free beam coincides with the nth frequency of the fixed–fixed beam and so do the corresponding modal damping ratios. The graph has been constructed by setting the damping ratio in the first mode of the fixed–fixed beam to 10%.

The corresponding graph for stiffness-proportional damping is reported in Figure 2. Once again the graph has been constructed by setting the damping ratio in the first mode of the fixed–fixed uniform beam to 10%.

These graphs have been derived on the same assumptions that have produced the results reported in Table 2, i.e., distributed damping coefficients $c_m = a_m m$ and $c_s = a_s E$. In the case of shear or torsional vibrations, obviously the normal elasticity modulus E must be replaced by the shear modulus G, the extensional rigidity EA by the shear rigidity kGA or torsional rigidity GJ, where k and J are, respectively, the shear correction and the torsional rigidity factors of the cross-section. The important point that must be kept in mind is that given a beam with prescribed mechanical characteristics (specific mass m, rigidity EA or GA or GJ, c_m and c_s) the damping ratio is related only to the natural frequencies and is not in other ways affected by the end conditions.

9.1.2. *Rayleigh damping*

In the paper, it has been shown that the previously mentioned property of mass- and stiffness-proportional damping extends also to Rayleigh-type damping and to any kind of vibration (i.e., Bernoulli and Euler–Bernoulli beams in flexural vibration). The results are also valid for non-uniform beams.

In order to show the significance of the results obtained, a graph showing the modal damping ratios for uniform beams in axial, shear or torsional vibration is reported in Figure 3

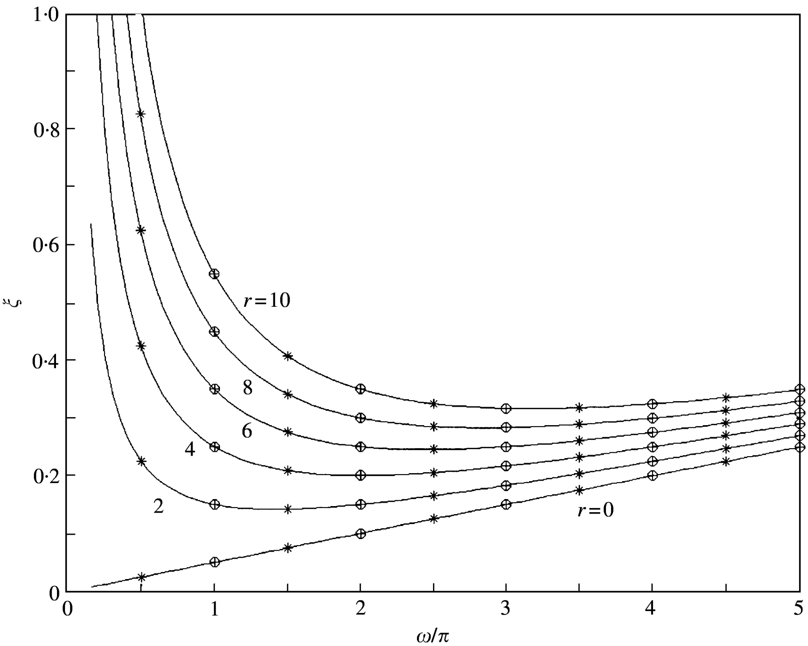

Figure 3. Modal damping ratios for the three considered end conditions for different Rayleigh damping laws characterized by the dimensionless parameter r. Key for structure: ○, free–free; ∗, fixed–free; +, fixed–fixed.

for a Rayleigh-type damping mechanism. The graph has been constructed under the condition that the damping ratio in the first mode of the fixed–fixed beam is equal to 10% when mass and stiffness contribute equally to the effective damping ratio. The graph reports several curves each corresponding to a value of the ratio $r = a_m/(a_s\bar{\bar{\omega}}_1^2)$ between the mass contribution and the stiffness contribution to the damping ratio in the first mode of the fixed–fixed uniform beam. In this way, for $r = 0$ the condition for stiffness-proportional damping is realized, while for large values of r, the condition for mass-proportional damping is approached.

9.1.3. *Caughey damping*

It has been previously shown that the Rayleigh-type damping is the most general damping law that allows for the damping ratios to be carried on from one end condition to another only on the basis of the knowledge of the frequencies of vibration for the two conditions.

In order to show that this is really the case, a damping law of the Caughey type distinct from the Rayleigh law is considered and it is shown how the damping ratios cannot be reconstructed only on the basis of the frequency ratios in the different end conditions. For the sake of simplicity, consider the damping law $\Im(\omega_k) = 2\xi_k\omega_k = a_2\omega_k^4$ and choose a_2 in such a way that the damping ratio in the third mode of a free–free axially vibrating uniform beam is equal to 10%. The modal damping ratios corresponding the first few modes are reported in Figure 4 for the free–free beam by circle symbols. The corresponding damping ratios for the fixed–free beam calculated by means of formula (28), are reported by star symbols in the same figure. The calculations have been conducted by truncating the summation after the first 50 terms. However, calculations performed by including more

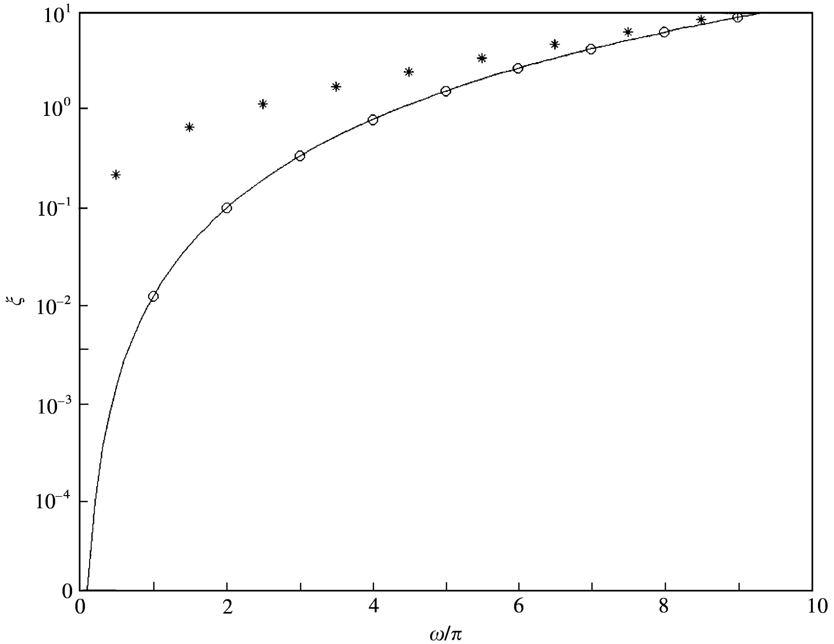

Figure 4. Modal damping ratios for a free–free beam with a Caughey damping law. Corresponding values for a fixed–free beam. Key for structure: ○, free–free; ∗ fixed–free.

terms and other calculations performed by including all terms in finite-dimensional systems have confirmed the behaviour depicted in the graph of Figure 4.

It is evident that the damping mechanism in the two beams, differing only by the end conditions, do not conform to the same Caughey law.

9.2. DISCRETE SYSTEMS

A better understanding of the theoretical results obtained in the paper may be gained by means of numerical applications conducted on a simple discrete system. To this purpose, a three-storey shear frame is considered. In a first instance, the frame is fixed at the base (Figure 5). It is assumed that a similar frame either rests on a layer of soft soil or is supported by a rubber-bearing seismic isolation system (Figure 6). For the sake of simplicity, the soil layer or the isolation system is represented by the physical model reported in Figure 7 and constituted by the equivalent stiffness k_b and the damping coefficient c_b. As clearly shown in the figures, the storey masses are denoted by m, the interstorey stiffnesses by k and the damping coefficients, when applicable, by c while the mass of the foundation mat has been denoted by m_b.

In engineering practice, the modal damping ratios for a building are determined from various testing procedures and sometimes from the analysis of the response to actual earthquakes. In order to analyze a building, represented here by the model of Figure 6, that is a soil–structure interacting system or a seismically isolated system, the damping in the structure and in the soil, or in the isolation bearings, needs to be properly specified. It is of no use to refer to experiments conducted on buildings or structures in similar conditions because the damping ratios would be affected by the coupling between structure and soil or

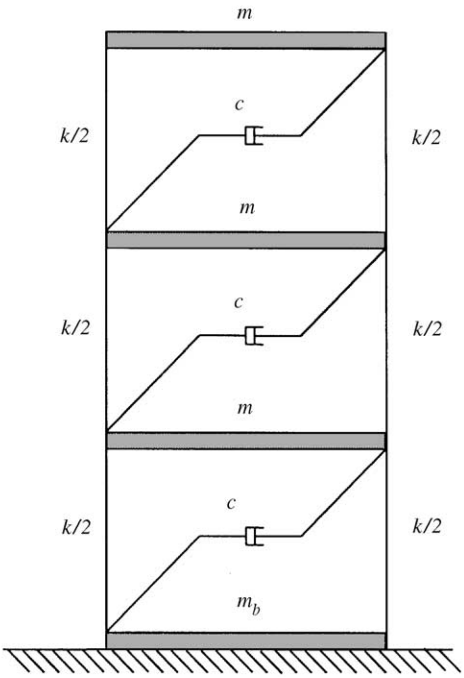

Figure 5. Restrained structural model or structural model on rigid soil.

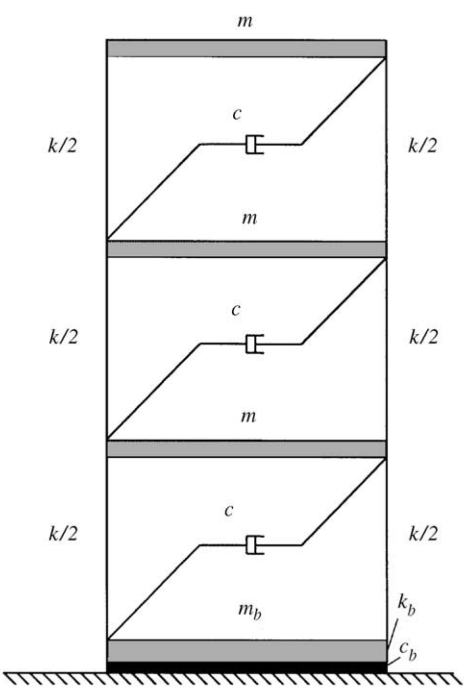

Figure 6. Structural model on a seismic isolation system or on a soft soil layer.

bearings. In order to properly describe the structural damping, the experiments should be conducted on the structure deprived of its supports but this is clearly an impossible endeavour. Therefore, the only alternative is to try to extend the results of tests on structures

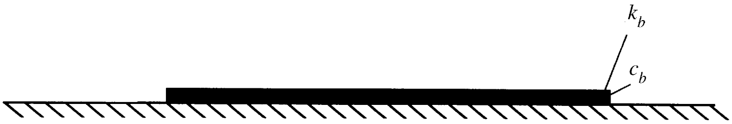

Figure 7. Seismic isolation bearing system or soft soil layer.

TABLE 3

Dynamic characteristics of the three considered systems: natural frequencies (rad/s), *periods* (s) *and normal modes of vibration*

	Restrained system			Free system			Coupled system		
n	ω_n	T_n	ϕ_n	ω_n	T_n	ϕ_n	ω_n	T_n	ϕ_n
1	12·568	0·5	0 0·328 0·591 0·737	0	∞	0·500 0·500 0·500 0·500	3·178	2	0·477 0·496 0·509 0·516
2	35·216	0·178	0 0·737 0·328 − 0·591	21·615	0·291	− 0·653 − 0·271 0·271 0·653	22·024	0·285	− 0·660 − 0·293 0·252 0·642
3	50·888	0·123	0 0·591 − 0·737 0·327	39·938	0·157	0·500 − 0·500 − 0·500 0·500	40·071	0·157	0·510 − 0·490 − 0·503 0·497
4				52·182	0·120	0·271 − 0·653 0·653 − 0·271	52·212	0·120	0·276 − 0·654 0·651 − 0·269

on rigid soil conditions, that may be conceivably considered fixed at the base, to the same structures with the supports removed. In the following, the results obtained in the preceding sections will be illustrated with reference to the considered discrete models in the significant cases: (1) mass-proportional damping; (2) stiffness-proportional damping; (3) Rayleigh damping; (4) arbitrary damping law.

9.2.1. *Characteristics of the models*

The mass and the stiffness matrices of the restrained model may be written in the form

$$\boldsymbol{M_R} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K_R} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{60}$$

The ratio k/m has been derived by setting the fundamental period of the restrained system to 0·5 s leading to $k/m = 797{\cdot}54$ (rad/s^2). The complete set of periods and frequencies is represented in Table 3. The model deprived of its supports is represented in Figure 8. The

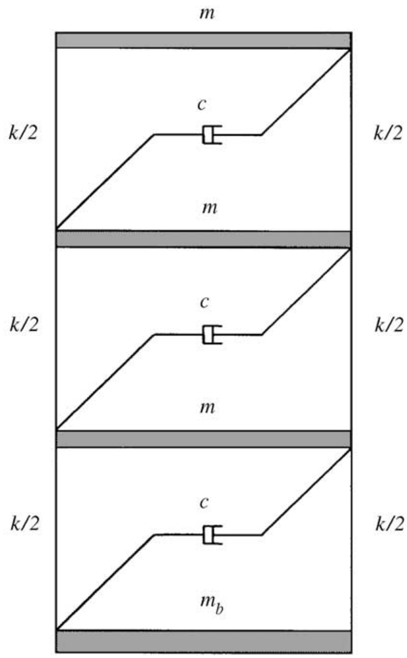

Figure 8. Free or unrestrained structural model.

mass and stiffness matrices for this free model may be written as

$$\boldsymbol{M_F} = m\begin{bmatrix} \eta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K_F} = k\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{61}$$

where $\eta = m_b/m$.

The corresponding periods and frequencies are also represented in Table 3. The mass and the stiffness matrices for the model in Figure 6 are obtained by coupling the corresponding matrices for the unrestrained system of Figure 8 with the ones of the foundation system of Figure 7.

$$\boldsymbol{M_C} = m\begin{bmatrix} \eta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K_C} = k\begin{bmatrix} 1+\gamma & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{62}$$

where $\gamma = k_b/k$. The stiffness of the soil or bearing system has been evaluated by setting $\eta = 1$ and the fundamental period of the coupled model to 2·0 s leading to $\gamma = 0{\cdot}052$. The complete set of periods and frequencies is also represented in Table 3.

9.2.2. *Relationship between operators*

One of the main results of this work is that the components of an operator defined on a subspace of a given vector space may be derived in terms of the components of the corresponding operator defined in the full vector space. This result will be illustrated herein with reference to the three models previously introduced. In particular, it will be shown how the stiffness and the mass matrices of the restrained model can be obtained from the corresponding matrices of the free or of the coupled one.

9.2.2.1. *Geometrical co-ordinates.* A basis for the restrained vector space $\mathbf{V}_R$ is represented by the following vectors:

$$\mathbf{v}_{R1}^{\mathrm{T}} = [0 \quad 1 \quad 0 \quad 0], \qquad \mathbf{v}_{R2}^{\mathrm{T}} = [0 \quad 0 \quad 1 \quad 0], \qquad \mathbf{v}_{R3}^{\mathrm{T}} = [0 \quad 0 \quad 0 \quad 1], \tag{63}$$

while a basis for the free vector space $\mathbf{V}_F$ is represented by the set of vectors:

$$\mathbf{v}_{F0}^{\mathrm{T}} = [1 \quad 0 \quad 0 \quad 0], \qquad \mathbf{v}_{F1}^{\mathrm{T}} = [0 \quad 1 \quad 0 \quad 0], \qquad \mathbf{v}_{F2}^{\mathrm{T}} = [0 \quad 0 \quad 1 \quad 0],$$
$$\mathbf{v}_{F3}^{\mathrm{T}} = [0 \quad 0 \quad 0 \quad 1]. \tag{64}$$

The same set of vectors is also a basis for the vector space corresponding to the coupled model. It is a trivial exercise to show that the mass and stiffness matrices for the restrained model can be obtained from those of the free or coupled ones. In fact, it may be written as

$$\boldsymbol{K}_R = \mathbf{v}_R^{\mathrm{T}} \boldsymbol{K}_F \mathbf{v}_R = \mathbf{v}_R^{\mathrm{T}} \boldsymbol{K}_C \mathbf{v}_R, \qquad \boldsymbol{M}_R = \mathbf{v}_R^{\mathrm{T}} \boldsymbol{M}_F \mathbf{v}_R = \mathbf{v}_R^{\mathrm{T}} \boldsymbol{M}_C \mathbf{v}_R, \tag{65, 66}$$

or in numerical form

$$\boldsymbol{K}_R = k \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\gamma & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{67}$$

$$\boldsymbol{M}_R = m \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{68}$$

The results for $\boldsymbol{M}_F$ and $\boldsymbol{K}_F$ have not been reported because they are a particular case of that considered for $m_b = 0$ and $k_b = 0$. Obviously, the basis in the restrained vector space may be expressed in terms of the basis in the free or coupled vector spaces as

$$\mathbf{v}_{R_i} = \boldsymbol{\alpha}_{ij} \mathbf{v}_{F_j}. \tag{69}$$

This leads to the following expressions for the components of the restrained matrix in terms of the components of the free matrix

$$\boldsymbol{K}_{R_{ij}} = \boldsymbol{\alpha}_{ip} \alpha_{jq} \boldsymbol{K}_{F_{pq}}. \tag{70}$$

For the case at hand, the $\boldsymbol{\alpha}$ matrix takes the following form:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{71}$$

Therefore, the matrices for the restrained system may be obtained in the form

$$\boldsymbol{K_R} = \boldsymbol{\alpha K_F \alpha}^{\mathrm{T}} = \boldsymbol{\alpha K_C \alpha}^{\mathrm{T}}, \qquad \boldsymbol{M_R} = \boldsymbol{\alpha M_F \alpha}^{\mathrm{T}} = \boldsymbol{\alpha M_C \alpha}^{\mathrm{T}}. \tag{72, 73}$$

9.2.2.2. *Principal co-ordinates.* If instead of using geometrical or physical co-ordinates, principal co-ordinates are considered, the representations of the stiffness and mass operators occur in a diagonal form. Once more, the principal basis vectors, or a set of orthonormal eigenvectors, for the restrained vector space may be used on the operator defined in the larger space. In the case at hand, the eigenvectors for the three models previously considered are represented in Table 3. Obviously, each eigenvector in the subspace may be expressed in terms of the full set of eigenvectors in the larger space. Similar operations to those conducted in the case of geometrical co-ordinates lead to the results

$$\boldsymbol{\Omega}_R^2 = \boldsymbol{\Phi}_R^T \boldsymbol{K}_F \boldsymbol{\Phi}_R = \boldsymbol{\Phi}_R^T \boldsymbol{K}_C \boldsymbol{\Phi}_R, \qquad \boldsymbol{I}_R = \boldsymbol{\Phi}_R^T \boldsymbol{M}_F \boldsymbol{\Phi}_R = \boldsymbol{\Phi}_R^T \boldsymbol{M}_C \boldsymbol{\Phi}_R, \tag{74, 75}$$

where $\boldsymbol{\Phi}_R$ is the matrix of the eigenvectors of the restrained system and $\boldsymbol{I}_R$ is the identity matrix. Equation (74) can be expressed in numerical term as follows:

$$\boldsymbol{\Phi}_R^T \boldsymbol{K}_F \boldsymbol{\Phi}_R = k \begin{bmatrix} 0 & 0{\cdot}328 & 0{\cdot}591 & 0{\cdot}737 \\ 0 & 0{\cdot}737 & 0{\cdot}328 & -0{\cdot}591 \\ 0 & 0{\cdot}591 & -0{\cdot}737 & 0{\cdot}328 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0{\cdot}328 & 0{\cdot}737 & 0{\cdot}591 \\ 0{\cdot}591 & 0{\cdot}328 & -0{\cdot}737 \\ 0{\cdot}737 & -0{\cdot}591 & 0{\cdot}328 \end{bmatrix} = \begin{bmatrix} 157{\cdot}955 & 0 & 0 \\ 0 & 1240{\cdot}167 & 0 \\ 0 & 0 & 2589{\cdot}589 \end{bmatrix} = \boldsymbol{\Omega}_R^2. \tag{76}$$

It is immediate to realize by direct comparison that the components on the diagonal matrix $\boldsymbol{\Omega}_R^2$ are the squares of the frequencies of vibration of the restrained system.

The same results may be obtained by applying formulae (47) and (48). In fact, in this case, from formula (50)

$$\boldsymbol{\alpha}_R = \boldsymbol{\Phi}_R^T \boldsymbol{M}_F \boldsymbol{\Phi}_F, \qquad \boldsymbol{M}_R = \boldsymbol{\alpha}_R \boldsymbol{\alpha}_R^T = \boldsymbol{I}, \qquad \boldsymbol{\Omega}_R^2 = \boldsymbol{\alpha}_R \boldsymbol{\Omega}^2 \boldsymbol{\alpha}_R^T. \tag{77–79}$$

where $\boldsymbol{\Phi}_F$ and $\boldsymbol{\Omega}_F$ are the matrices of the eigenvectors and natural frequencies of the free system.

9.2.3. *Damping characteristics*

Two basic assumptions will be made regarding damping. The first one is that the first modal damping ratio in the restrained model is 2%. The second one is that the damping ratio of the coupled system, with the superstructure considered as rigid, is $\xi_b = 10\%$. In this

TABLE 4

Damping ratios for the free and restrained systems in the case of mass-proportional damping. Approximate damping ratios for the coupled system

	Restrained system		Free system		Coupled system	
n	ξ_n	$\xi_n\omega_n$	ξ_n	$\xi_n\omega_n$	ξ_n	$\xi_n\omega_n$
1	0·020	0·2515	∞	0·2515	0·173	0·545
2	0·007	0·2515	0·012	0·2515	0·037	0·813
3	0·005	0·2515	0·006	0·2515	0·015	0·587
4	—	—	0·005	0·2515	0·007	0·350

way, the damping coefficient of the soil–foundation or foundation–isolation system may be calculated, using reference [3]

$$c_b = 2(m_r + m_b)\omega_b\xi_b. \tag{80}$$

where $m_r = 3m$ is the total mass of the restrained system and $\omega_b = \sqrt{k_b/(m_r + m_b)}$.

Further assumptions will be made on the damping of the restrained system but these will be considered separately.

9.2.3.1. *Mass-proportional damping.* In this case, the damping matrix of the restrained, free and coupled systems may be written as

$$\boldsymbol{D}_{R_m} = c_m\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \boldsymbol{D}_{F_m} = c_m\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad \boldsymbol{D}_{C_m} = c_m\begin{bmatrix}1+\delta & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \tag{81}$$

where $\delta = c_b/c_m$.

The modal damping ratios corresponding to the first two systems may be easily calculated providing the values reported in Table 4. Since the damping in the coupled model is of the non-classical type, the undamped normal modes do not uncouple the equations of motion. By applying the classical procedure, the damping matrix in principal co-ordinates is not diagonal. In the case at hand, it provides the result

$$\boldsymbol{D}^*_{C_m} = \boldsymbol{\Phi}_C^T\boldsymbol{D}_{C_m}\boldsymbol{\Phi}_C = c_m\begin{bmatrix}2{\cdot}168 & -1{\cdot}614 & 1{\cdot}247 & 0{\cdot}676\\ -1{\cdot}614 & 3{\cdot}231 & -1{\cdot}723 & -0{\cdot}934\\ 1{\cdot}247 & -1{\cdot}723 & 2{\cdot}331 & -0{\cdot}721\\ 0{\cdot}676 & -0{\cdot}934 & 0{\cdot}721 & 1{\cdot}392\end{bmatrix}, \tag{82}$$

where $\boldsymbol{\Phi}_C$ is the matrix of the eigenvectors of the coupled system.

By neglecting the off-diagonal terms, approximate damping ratios of the coupled model may be obtained. These are reported in Table 4. Although in this case it has been easy to construct the damping matrices for the free and for the coupled models, it may be interesting to see how the theoretical results that have been derived in the paper apply in this case also. In particular, the modal damping ratios of the restrained model may be obtained from those of the free and coupled ones from the expression

$$\boldsymbol{D}_{R_m} = \boldsymbol{\alpha}_R \boldsymbol{D}_{F_m} \boldsymbol{\alpha}_R^{\mathrm{T}}. \tag{83}$$

This is a slightly extended form of equations (49) and (52) that accounts for non-classical damping. In numerical terms, it follows that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0{\cdot}828 & 0{\cdot}553 & -0{\cdot}091 & -0{\cdot}028 \\ 0{\cdot}237 & -0{\cdot}497 & -0{\cdot}828 & -0{\cdot}107 \\ 0{\cdot}091 & -0{\cdot}145 & 0{\cdot}237 & -0{\cdot}956 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0{\cdot}828 & 0{\cdot}237 & 0{\cdot}091 \\ 0{\cdot}553 & -0{\cdot}497 & -0{\cdot}145 \\ -0{\cdot}091 & -0{\cdot}828 & 0{\cdot}237 \\ -0{\cdot}028 & -0{\cdot}107 & -0{\cdot}956 \end{bmatrix}, \tag{84}$$

where c_m has been omitted because it appears on both sides of the equality.

It may be interesting to note that the modal damping ratios for the restrained and the free systems obey the same law and may be related to reach other only by the knowledge of the vibration frequencies. In fact, it is immediate to verify that the product between the damping ratios and the natural frequencies remains constant and it is this constant that defines the mass-proportional law. For the coupled system, instead, the modal damping ratios do not obey the same law. This is clearly illustrated in Table 4.

9.2.3.2. *Stiffness-proportional damping.* In this case, the damping matrices of the restrained and free systems may be written as

$$\boldsymbol{D}_{R_s} = c_s \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{D}_{F_s} = c_s \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{85}$$

where the damping coefficient c_s is evaluated as previously stated, namely 2% damping ratio in the first mode of the restrained system. The damping matrix for the coupled system takes, therefore, the form

$$\boldsymbol{D}_{C_s} = c_S \begin{bmatrix} 1+\tau & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{86}$$

where $\tau = c_b/c_S$.

By operating in the same way as with the mass-proportional case, the modal damping coefficients for the three considered cases are obtained as reported in Table 5. Once again the modal damping ratios of the restrained and free systems could have been obtained from

TABLE 5

Damping ratios for the free and restrained systems in the case of stiffness-proportional damping. Approximate damping ratios for the coupled system

	Restrained system		Free system		Coupled system	
n	ξ_n	ξ_n/ω_n	ξ_n	ξ_n/ω_n	ξ_n	ξ_n/ω_n
1	0·020	0·002	0	0·002	0·094	0·030
2	0·056	0·002	0·034	0·002	0·059	0·003
3	0·081	0·002	0·063	0·002	0·072	0·002
4	—	—	0·083	0·002	0·085	0·002

the stiffness-proportional damping law. In fact, as it may be noticed from Table 5, a constant ratio exists between the modal damping ratios and the corresponding frequencies as suggested by equation (37).

9.2.3.3. *Rayleigh damping.* In this case, the damping matrices may be written in the form

$$\boldsymbol{D}_{R_R} = c_m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_S \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{87}$$

$$\boldsymbol{D}_{F_R} = c_m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + c_s \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{D}_{C_R} = \boldsymbol{D}_{F_R} + \boldsymbol{D}_b, \tag{88, 89}$$

where $\boldsymbol{D}_b$ is the damping matrix of the soil system or the isolation system.

In the present case, the damping coefficients c_m and c_s have been evaluated in such a way that the distance of the modal damping ratios of the restrained system from the reference value of 2% is at a minimum in the sense of the least-squares method. The damping ratios calculated in this way for the three considered systems are reported in Table 6. Once more it may be seen that the damping ratios in the restrained and the free systems obey the Rayleigh damping law and therefore, could have been easily obtained from the knowledge of this law and the natural frequencies. If the damping matrix of the coupled system had obeyed the same law, this would have applied to the damping ratios of the coupled systems also, but unfortunately, as it may be seen from the computed values in Table 6, this is not the case.

One important aspect that should be noticed is that, by considering only the structural modes, the nearest constant damping ratio is equal to 1·9%.

9.2.3.4. *Arbitrary damping law.* In structural dynamics and earthquake engineering, damping is usually assigned in terms of modal damping ratios which on the basis of tests, measurements and professional consensus are assumed to be frequency independent. In the treatment that has been presented, it is obviously possible to derive the modal damping ratios of the restrained system from those of the free system, but the contrary is definitely not possible unless the damping obeys some damping law that is, at maximum level of

TABLE 6

Damping ratios for the free and restrained systems in the case of Rayleigh damping. Approximate damping ratios for the coupled system

	Restrained system		Free system		Coupled system	
n	$\hat{\xi}_n$†	ξ_n^*‡	$\hat{\xi}_n$†	ξ_n^*‡	$\hat{\xi}_n$†	ξ_n^*‡
1	0·020	0·020	∞	∞	0·158	0·065
2	0·018	0·018	0·017	0·017	0·042	0·017
3	0·021	0·021	0·019	0·019	0·027	0·019
4	—	—	0·022	0·022	0·024	0·022

†Damping ratio computed from the generalized damping matrix.
‡Damping ratio computed from the Rayleigh damping matrix.

complexity, of the Rayleigh type. In this section, a more general damping law than the Rayleigh one is considered. In particular, constant modal damping ratios will be assigned to the free system. The damping matrix for this system will be constructed by modal synthesis [3]. Two ways are possible to evaluate the modal damping ratios for the restrained system. The first one is to apply equation (52). The second one would be to construct the damping matrix for the restrained system from the damping matrix of the free system by deleting the rows and the columns pertaining to the restrained degrees of freedom. The modal damping ratios could then be derived by standard modal decomposition. In the present case, a 2% damping ratio has been assigned to the four modal contributions. In this way, the damping matrix for the free system is of the type

$$\boldsymbol{D}_{F_A} = \begin{bmatrix} 0{\cdot}921 & -0{\cdot}615 & -0{\cdot}183 & -0{\cdot}122 \\ -0{\cdot}615 & 1{\cdot}353 & -0{\cdot}555 & -0{\cdot}183 \\ -0{\cdot}183 & -0{\cdot}555 & 1{\cdot}354 & -0{\cdot}615 \\ -0{\cdot}122 & -0{\cdot}183 & -0{\cdot}615 & 0{\cdot}921 \end{bmatrix}. \tag{90}$$

It follows that the damping matrix for the restrained model is

$$\boldsymbol{D}_{R_A} = \begin{bmatrix} 1{\cdot}353 & -0{\cdot}555 & -0{\cdot}183 \\ -0{\cdot}555 & 1{\cdot}354 & -0{\cdot}615 \\ -0{\cdot}183 & -0{\cdot}615 & 0{\cdot}921 \end{bmatrix}. \tag{91}$$

By applying modal decomposition to the damping matrix of the restrained system, it may be seen that the undamped normal modes for the restrained system do not uncouple the damping terms, i.e., the transformed matrix is not diagonal:

$$\boldsymbol{\Phi}_R^T \boldsymbol{D}_{R_A} \boldsymbol{\Phi}_R = \begin{bmatrix} 0{\cdot}279 & -0{\cdot}111 & -0{\cdot}049 \\ -0{\cdot}111 & 1{\cdot}332 & -0{\cdot}037 \\ -0{\cdot}049 & -0{\cdot}037 & 2{\cdot}017 \end{bmatrix}. \tag{92}$$

Therefore, not only the resulting damping in the restrained system does not obey the same damping law as in the free system, but it also turns out to be of the non-classical type. It is interesting, however, to observe that the results contained in matrix (92) may be also obtained by the general formulation reported in this paper. In particular, from equation (49), it follows that:

$$\boldsymbol{\Phi}_R^T \mathbf{D}_{R_A} \boldsymbol{\Phi}_R = \alpha \boldsymbol{\Phi}_F^T \mathbf{D}_{F_A} \boldsymbol{\Phi}_F \boldsymbol{\alpha}^{\mathrm{T}}. \tag{93}$$

Furthermore from the application of equation (52) the approximate damping ratios may be obtained: $\xi_{1R_A} = 0{\cdot}011$, $\xi_{2R_A} = 0{\cdot}019$, $\xi_{3R_A} = 0{\cdot}020$. These are consistent with the diagonal terms in the matrix (92).

9.2.4. *Final remarks*

The applications reported in this section show how the results developed in the theoretical part apply to discrete systems. Similar applications can be worked out for discretized systems. The applications show also that it is possible to carry over the damping specifications from one system to another in the case of the Rayleigh damping law or in the simpler component laws (i.e., mass- and stiffness-proportional damping laws). In more general cases, this is not possible. Therefore, it seems reasonable at the present state of knowledge on structural damping to describe the classical damping in the structural part of coupled systems by means of the most appropriate Rayleigh law compatible with the values derived from tests, identification procedures from earthquake records or professional consensus.

10. CONCLUSIONS

In dynamic structural systems, damping is usually specified in terms of modal damping ratios evaluated experimentally from similar structures. In some problems, however, damping needs to be specified for different boundary conditions from those available from experiments. This is often the case in soil–structure interaction problems where a damping matrix may be required for an unrestrained system. Therefore, the problem of specifying damping for the same basic structural system under different support conditions is one of strong practical relevance.

This paper has addressed this problem and has shown that damping specified for one boundary condition can be easily carried onto any other boundary condition provided that the damping law is no more general than Rayleigh-type damping. For instance, any of the Caughey damping laws that do not coalesce in a Rayleigh damping law does not transform in the same law (that is with the same Caughey coefficients) by a change of the boundary conditions.

This condition applies independently to continuous, discrete and discretized systems.

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APPENDIX A: A PROPERTY OF BASIC STRUCTURAL OPERATORS

The basic structural operators are the mass, damping and stiffness tensors **M**, **D**, **K**. A property of these operators that has been used in the text will be proved here. If the operators are defined with reference to an unrestrained structural system, the corresponding operators for a system obtained from the previous one by suppressing some of its degrees of freedom are restrictions of the former operators. The components of the operators of the restrained system may be obtained from those of the unrestrained one by simply removing from their matrix representation, the rows and the columns corresponding to the suppressed degrees of freedom.

This operation may be easily implemented by means of the contraction tensor **C**. This tensor is defined as

$$\mathbf{C}\cdot\mathbf{e}_i = \mathbf{e}_i^r = \mathbf{e}_i\cdot\mathbf{C}^{\mathrm{T}}$$

and its function is to transform any N-dimensional vector in an M-dimensional vector with $M < N$. The components of the tensor **C** may be found from

$$C_{ij} = \mathbf{e}_i^r\cdot\mathbf{C}\cdot\mathbf{e}_j = \mathbf{e}_i^r\cdot\mathbf{e}_j^r = \delta_{ij} \quad (i = 1, \ldots, M;\, j = 1, \ldots, N).$$

For the case when $M = 2$ and $N = 3$, the matrix of the components of **C** takes the form

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore, the tensor **C** transforms the basis vectors of the unrestrained system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ into the basis vectors of the restrained system $\mathbf{e}_1^r$, $\mathbf{e}_2^r$.

Given that the basic operator **A** of the unrestrained system

$$\mathbf{A} = A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j,$$

the operator $\mathbf{A}^r$ of the corresponding restrained system may be obtained as

$$\mathbf{A}^r = \mathbf{C}\cdot\mathbf{A}\cdot\mathbf{C}^{\mathrm{T}} = A_{ij}\mathbf{C}\cdot\mathbf{e}_i\otimes\mathbf{e}_j\cdot\mathbf{C}^{\mathrm{T}} = A_{ij}\mathbf{e}_i^r\otimes\mathbf{e}_j^r = A_{ij}^r\mathbf{e}^r\otimes\mathbf{e}_j^r,$$

which confirms that $A_{ij}^r = A_{ij}$.

The reciprocal of the contraction operator is the extension operator which transforms the basis vectors of the restrained system into the basis vectors of the unrestrained one

$$\mathbf{E}\cdot\mathbf{e}_j^r = \mathbf{e}_j = \mathbf{e}_j^r\cdot\mathbf{E}^{\mathrm{T}}.$$

The components of the extension operator **E** may be calculated as

$$E_{ij} = \mathbf{e}_i\cdot\mathbf{E}\cdot\mathbf{e}_j^r = \delta_{ij} \quad (i = 1, \ldots, N;\; j = 1, \ldots, M).$$

It is obvious that $\mathbf{E} = \mathbf{C}^{\mathrm{T}}$. The extension operator $\mathbf{E}$ enables the representation of a basic operator of a restrained system in the larger vector space of the unrestrained system.

$$\mathbf{A}^e = \mathbf{E}\cdot\mathbf{A}^r\cdot\mathbf{E}^{\mathrm{T}} = \mathbf{A}^r_{ij}\mathbf{E}\cdot\mathbf{e}^r_i\otimes\mathbf{e}^r_j\cdot\mathbf{E}^{\mathrm{T}} = \mathbf{A}^r_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = \mathbf{A}^e_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \quad (i = 1,\dots,M;\ j = 1,\dots,N).$$

Having defined the operator of the restrained system in the larger vector space of the unrestrained one, it is evident as to how it is possible to calculate its components from the operator of the unrestrained system.

$$A^r_{pq} = \mathbf{e}_p\cdot\mathbf{A}^r\cdot\mathbf{e}_q = A_{pq} = \mathbf{e}_p\cdot\mathbf{A}\cdot\mathbf{e}_q \quad (p,q = 1,\dots,M).$$

The property that has shown to be valid for the basic structural operators is not valid for functions of operators such as powers and roots which often occur in the applications.

Take for instance the operator

$$(\mathbf{A}^r)^2 = \mathbf{A}^r\cdot\mathbf{A}^r = \mathbf{C}\cdot\mathbf{A}\cdot\mathbf{C}^{\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{A}\cdot\mathbf{C}^{\mathrm{T}}$$

which is different from

$$\mathbf{C}\cdot\mathbf{A}^2\cdot\mathbf{C}^{\mathrm{T}}.$$

This shows that the components of $(\mathbf{A}^r)^2$ cannot be obtained from those of $\mathbf{A}^2$ by removing rows and columns corresponding to the suppressed degrees of freedom.

APPENDIX B: ON THE COMPONENTS OF SECOND ORDER CARTESIAN TENSORS

Let $\mathbf{V}$ be an N-dimensional linear vector space and $\mathbf{V}_S \subset \mathbf{V}$ an M-dimensional linear subspace n $\mathbf{V}$. Let $(\mathbf{f}_i, i = 1,\dots,M)$ be an orthonormal basis in $\mathbf{V}_S$ and $(\mathbf{r}_i, i = M+1,\dots,N)$ an orthonormal basis in $\mathbf{V} - \mathbf{V}_S$. Then $[(\mathbf{f}_i,\ i = 1,\dots,M), (\mathbf{r}_i,\ i = M+1,\dots,N)]$ is an orthonormal basis in $\mathbf{V}$. A second order Cartesian tensor $\mathbf{T}$ in $\mathbf{V}$ is defined as

$$\mathbf{T} = T_{ij}\mathbf{f}_i\otimes\mathbf{f}_j + T_{ij}\mathbf{f}_i\otimes\mathbf{r}_j + T_{ij}\mathbf{r}_i\otimes\mathbf{f}_j + T_{ij}\mathbf{r}_i\otimes\mathbf{r}_j$$

$$= F_{ij}\mathbf{f}_i\otimes\mathbf{f}_j + C^{fr}_{ij}\mathbf{f}_i\otimes\mathbf{r}_j + C^{rf}_{ij}\mathbf{r}_i @ \mathbf{f}_j + R_{ij}\mathbf{r}_i\otimes\mathbf{r}_i = \mathbf{F} + \mathbf{C}^{fr} + \mathbf{C}^{rf} + \mathbf{R},$$

where $\mathbf{F}$ is a second order Cartesian tensor in $\mathbf{V}_S$, $\mathbf{R}$ is a second order Cartesian tensor in $\mathbf{V} - \mathbf{V}_{\mathrm{S}}$, $\mathbf{C}^{\mathrm{fr}}$ and $\mathbf{C}^{\mathrm{rf}}$ are coupling second order Cartesian tensors in $\mathbf{V}_{\mathrm{S}}\times(\mathbf{V} - \mathbf{V}_{\mathrm{S}})$ and in $(\mathbf{V} - \mathbf{V}_{\mathrm{S}})\times\mathbf{V}_{\mathrm{S}}$. If the tensor $\mathbf{T}$ is symmetric, then it may be seen that

$$\mathbf{C}^{\mathrm{fr}} = \mathbf{C}; \quad \mathbf{C}^{\mathrm{rf}} = \mathbf{C}^{\mathrm{T}}.$$

For the sake of simplicity, consider only symmetric second order tensors:

$$\mathbf{T} = \mathbf{F} + \mathbf{C} + \mathbf{C}^{\mathrm{T}} + \mathbf{R}.$$

It is easy to show that *the components of any of the tensors* $\mathbf{F}$, $\mathbf{C}$, $\mathbf{C}^{\mathrm{T}}$ *and* $\mathbf{R}$ *may be expressed in terms of the components of the tensor* $\mathbf{T}$. For instance, the components of the tensor $\mathbf{F}$, which is the one that matters in the present application, may be calculated as

$$F_{pq} = \mathbf{f}_p\cdot\mathbf{F}\cdot\mathbf{f}_q = \mathbf{f}_p\cdot(F_{ij}\mathbf{f}_i\otimes\mathbf{f}_j)\cdot\mathbf{f}_q = F_{ij}(\mathbf{f}_p\cdot\mathbf{f}_i)(\mathbf{f}_j\cdot\mathbf{f}_q) = F_{ij}\delta_{ip}\delta_{jq} = T_{ij}\delta_{ip}\delta_{jq} = T_{pq},$$

where advantage has been taken of the fact that

$$\mathbf{F} = \boldsymbol{F}_{ij}\mathbf{f}_i \otimes \mathbf{f}_i = \boldsymbol{T}_{ij}\mathbf{f}_i \otimes \mathbf{f}_j.$$

On the other hand, it may be shown that

$$\boldsymbol{T}_{pq} = \mathbf{f}_p \cdot \mathbf{T} \cdot \mathbf{f}_q = \mathbf{f}_p \cdot (\mathbf{F} + \mathbf{C} + \mathbf{C}^{\mathrm{T}} + \mathbf{R}) \cdot \mathbf{f}_q = \mathbf{f}_p \cdot \mathbf{F} \cdot \mathbf{f}_q + \mathbf{f}_p \cdot \mathbf{C} \cdot \mathbf{f}_q + \mathbf{f}_p \cdot \mathbf{C}^{\mathrm{T}} \cdot \mathbf{f}_q + \mathbf{f}_p \cdot \mathbf{R} \cdot \mathbf{f}_q = \boldsymbol{F}_{pq}$$

because

$$\mathbf{f}_p \cdot \mathbf{C} \cdot \mathbf{f}_q = \mathbf{f}_p \cdot \mathbf{C}^{\mathrm{T}} \cdot \mathbf{f}_q = \mathbf{f}_p \cdot \mathbf{R} \cdot \mathbf{f}_q = 0$$

as shown by direct calculation.