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FINITE ELEMENT MODELLING OF LAMINATED PIEZO-ELASTIC STRUCTURES: LYAPUNOV STABILITY ANALYSIS

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In this study, the dynamic stability of finite element modelling of laminated piezo-elastic structures is investigated. The criteria for stability are established based on the second method of Lyapunov, which considers the energy of the system. The results show that the equations of motion are asymptotically stable. However, energy dissipation through the piezoelectric effect continued at zero feedback gain. This implied that the structure was controlled unconditionally. Subsequently, a control strategy that satisfies the condition of no piezoelectric effects when the gain set to zero is developed.

1. INTRODUCTION

Due to their adaptable properties, piezoelectric structural materials can function as distributed sensors and actuators for monitoring and controlling the response of a structure. In sensor applications, the strains in the piezoelectric material can be determined by measuring the induced electric potential. In actuator applications, distributed forces can be effected by subjecting the piezoelectric material to an appropriate electric potential.

By integrating piezoelectric elements into advanced composite materials, the potential for forming a high-strength, high-stiffness, lightweight structure capable of self-monitoring, analyzing and adapting to changing operating conditions can be explored. Models simulating the control of various structural elements have been developed and studied for this purpose. Two approaches in active control using piezoelectric sensors have been adopted. One approach assumes the closed-circuit mode [1–3] and the other assumes the open-circuit mode [4–9].

The active control model developed through assuming a closed-circuit mode has been studied extensively by Miller *et al.* [3]. Results of the study show that applying suitable potentials to the piezoelectric elements via an amplifier can effect control of a structure. At zero amplifier gain, the structure is free from any piezoelectric control. Thus, the active control model resembles a passive one in this state.

Extensive research on an active control model of an open-circuit mode has also shown that control can be effected through the piezoelectric elements. However, these studies have not considered the bearing of the active control model at zero amplifier gain. Also, the potential generated by the sensor elements in this condition has been assumed to be insignificant implying that its influence in controlling the structure is not noticeable [4].

This study shows, using the model developed by Tzou and Tseng [4], that control forces persist at zero amplifier gain when the current active control model of the open-circuit mode is used. These control forces are significant to be ignored in controlling the structure. In light of this, an active control model of the open-circuit mode, which resembles the passive one when the amplifier gain is set to zero, is proposed and developed.

2. FORMULATION OF THE FINITE ELEMENT EQUATIONS OF MOTION

Figure 1 shows the schematic of an element of the piezo-elastic structure. The structure undergoing the action of the external forces is assumed to be perfectly bonded, elastic and orthotropic in behaviour. The deformation of the structure is assumed to take place under isothermal conditions. The frame of reference is taken to coincide with the principal axes of the material. Based on these assumptions, a linear constitutive model is adopted to predict the behaviour of the piezo-elastic structure.

2.1. CONSTITUTIVE EQUATION

The linear constitutive equations coupling the elastic field and the electric field can be expressed by the direct and inverse piezoelectric equations. These are written in the principal direction of the material, respectively, as follows:

$$\mathbf{D}' = \mathbf{e}'^{\mathrm{T}}\boldsymbol{\varepsilon} + \mathbf{g}'\mathbf{E}, \qquad \boldsymbol{\sigma} = \mathbf{C}'\boldsymbol{\varepsilon} - \mathbf{e}'\mathbf{E}, \tag{1, 2}$$

where $\mathbf{D}'$ represents the electric displacement vector, $\mathbf{e}'$ the piezoelectric constant matrix, $\boldsymbol{\varepsilon}$ the strain vector, $\mathbf{g}'$ the dielectric constant matrix, $\mathbf{E}$ the electric field vector, $\boldsymbol{\sigma}'$ the stress vector, $\mathbf{C}'$ the elasticity matrix, $\mathbf{e}'^{\mathrm{T}}$ the transpose of $\mathbf{e}'$ and the prime represent the local co-ordinate system. It is noted that brackets [ ] and braces { } representing matrices and vectors have been omitted. To transform these equations from the principal direction of the material to co-ordinates that are geometrically natural to the body, first order and second order transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are introduced. The resulting constitutive

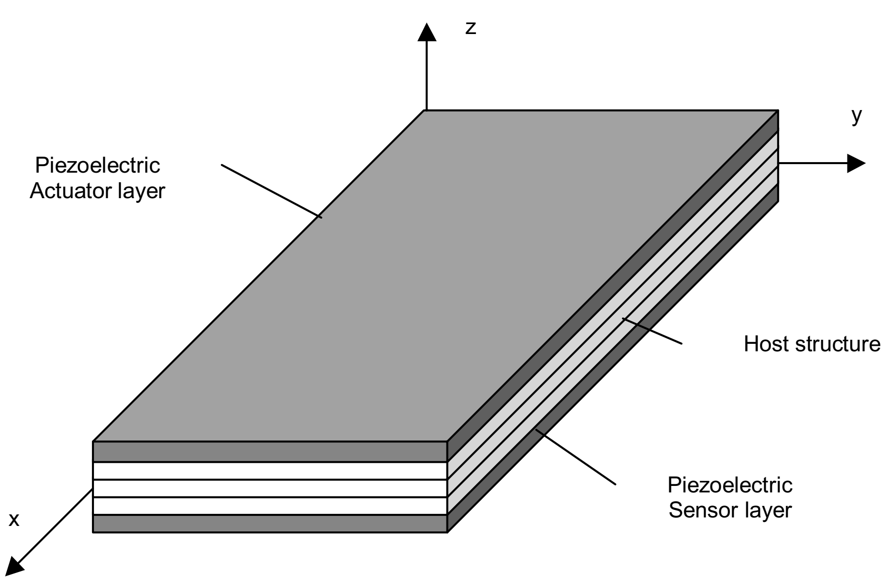

Figure 1. Schematic diagram of the piezoelectric structure.

equation of the piezo-elastic laminates oriented arbitrarily becomes

$$\mathbf{D} = \mathbf{T}_1^{-1}\mathbf{e}^{\mathrm{T}}\,\mathbf{T}_2^{-\mathrm{T}}\boldsymbol{\varepsilon} + \mathbf{T}_1^{-1}\mathbf{g}\mathbf{T}_1\mathbf{E}, \qquad \boldsymbol{\sigma} = \mathbf{T}_2^{-1}\mathbf{C}\mathbf{T}_2^{-\mathrm{T}}\boldsymbol{\varepsilon} - \mathbf{T}_2^{-1}\mathbf{e}\mathbf{T}_1\mathbf{E} \tag{3, 4}$$

written compactly, respectively, as

$$\mathbf{D} = \bar{\mathbf{e}}^{\mathrm{T}}\varepsilon + \bar{\mathbf{g}}\mathbf{E} \qquad \boldsymbol{\sigma} = \bar{\mathbf{C}}\varepsilon - \bar{\mathbf{e}}\mathbf{E}. \tag{5, 6}$$

Equation (5) describes the effect of elastic strain on the dielectric displacement. While, equation (6) describes the additional stress induced by the electric field.

2.2. PIEZO-ELASTIC DYNAMICS: A VARIATIONAL FORMULATION

The dynamic equations representing the behaviour of the piezo-elastic structure are derived using Hamilton's variational principle

$$\int_t \delta(T + W)\,\mathrm{d}t = 0 \tag{7}$$

where T represents the kinetic energy and W the work function. The kinetic energy for the body Ω is given as

$$T = \int_\Omega \frac{1}{2}\dot{\mathbf{u}}^{\mathrm{T}}\rho\dot{\mathbf{u}}\,\mathrm{d}\Omega, \tag{8}$$

where $\dot{\mathbf{u}}$ is the velocity vector and ρ the mass density. The work function is given as

$$W = \int_\Omega \frac{1}{2}(-\varepsilon^{\mathrm{T}}\bar{\mathbf{C}}\varepsilon + 2\varepsilon^{\mathrm{T}}\bar{\mathbf{e}}\mathbf{E} + \mathbf{E}^{\mathrm{T}}\bar{\mathbf{g}}\mathbf{E})\,\mathrm{d}\Omega + W_{ext}. \tag{9}$$

The work done by external mechanical and electrical forces is given as

$$W_{ext} = \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{P}_b\,\mathrm{d}\Omega + \int_{\Gamma_u} \mathbf{u}^T\mathbf{P}_s\,\mathrm{d}\Gamma_u + \mathbf{u}^{\mathrm{T}}\mathbf{P}_c - \int_{\Gamma_\phi} \phi Q\,\mathrm{d}\Gamma_\phi, \tag{10}$$

where ϕ represents the electric potential, Q the surface charge, $\mathbf{P}_b$, $\mathbf{P}_s$ and $\mathbf{P}_c$ the body, surface and concentrated load vectors, respectively. By substituting equations (8)–(10) into equation (7), Hamilton's variational principle requires that

$$\int_t \Bigg(\int_\Omega (\delta u^{\mathrm{T}}\rho\ddot{\mathbf{u}} + \{\mathbf{L}\delta\mathbf{u}\}^{\mathrm{T}}\bar{\mathbf{C}}\mathbf{L}\mathbf{u} + \{\mathbf{L}\delta\mathbf{u}\}^{\mathrm{T}}\bar{\mathbf{e}}\nabla\phi + \nabla\delta\phi^{\mathrm{T}}\bar{\mathbf{e}}\mathbf{L}\mathbf{u}$$

$$- \nabla\delta\phi^{\mathrm{T}}\bar{\mathbf{g}}\nabla\phi - \delta\mathbf{u}^{\mathrm{T}}\mathbf{P}_{\mathrm{b}})\,\mathrm{d}\Omega - \int_{\Gamma_u} \delta\mathbf{u}^{\mathrm{T}}\mathbf{P}_s\,\mathrm{d}\Gamma_u - \delta\mathbf{u}^{\mathrm{T}}\mathbf{P}_c + \int_{\Gamma_\phi} \delta(\phi Q)\,\mathrm{d}\Gamma_\phi \Bigg)\,\mathrm{d}t = 0. \tag{11}$$

In equation (11), the strains are represented as $\varepsilon = \mathbf{L}\mathbf{u}$ and the electric field as $\mathbf{E} = -\nabla\phi$, where $\mathbf{L}$ represents a matrix of differential operators with respect to space.

2.3. PIEZO-ELASTIC DYNAMICS: FINITE ELEMENT FORMULATION

The continuous displacement field is approximated by a discrete one using the finite element approximation functions such as

$$\mathbf{u} \approx \mathbf{u}^h = \mathbf{N}_u \tilde{\mathbf{u}}, \tag{12}$$

where $\mathbf{u}$ represents the continuous displacement field, $\mathbf{u}^h$ the approximate displacement field, $\tilde{\mathbf{u}}$ the discrete displacements, $\mathbf{N}$ the interpolation functions and h the association of the approximate displacements with the mesh. The continuous potential field is approximated by a discrete one using the finite element approximation functions such as

$$\phi \approx \phi^h = N_\phi \tilde{\phi}, \tag{13}$$

where ϕ represents the continuous potential field, ϕ^h the approximate potential field and $\tilde{\phi}$ the discrete potentials. Substituting equation (12) and (13) into equation (11), the finite element statement can be written as

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{\mathbf{u}}} \\ \ddot{\tilde{\phi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{\phi u} & -\mathbf{K}_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}} \\ \tilde{\phi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ -\mathbf{G} \end{Bmatrix}, \tag{14}$$

where

$$\mathbf{K}_{uu} = \int_\Omega \mathbf{B}_u^{\mathrm{T}} \bar{\mathbf{C}} \mathbf{B}_u \,\mathrm{d}\Omega, \qquad \mathbf{K}_{u\phi} = \int_\Omega \mathbf{B}_u^{\mathrm{T}} \bar{\mathbf{e}} \mathbf{B}_\phi \,\mathrm{d}\Omega,$$

$$\mathbf{K}_{\phi\phi} = \int_\Omega \mathbf{B}_\phi^{\mathrm{T}} \bar{C} \mathbf{B}_\phi \,\mathrm{d}\Omega, \qquad \mathbf{M}_{uu} = \int_\Omega \mathbf{N}_u^{\mathrm{T}} \rho \mathbf{N}_u \mathrm{d}\Omega,$$

$$\mathbf{F} = \int_\Omega \mathbf{N}_u^{\mathrm{T}} \mathbf{P}_B \,\mathrm{d}\Omega + \int_{\Gamma_u} \mathbf{N}_u^{\mathrm{T}} \mathbf{P}_s \,\mathrm{d}\Gamma_u + \mathbf{N}_u^{\mathrm{T}} \mathbf{P}_c, \qquad G = \int_{\Gamma_\phi} \mathbf{N}_\phi^{\mathrm{T}} \mathbf{Q} \,\mathrm{d}\Gamma_\phi. \tag{15}$$

2.4. ACTIVE VIBRATION MEASUREMENT AND CONTROL

The dynamic equations, which are given in terms of the displacements and potentials, are reduced to one given in terms of displacements only. That is,

$$\mathbf{M}_{uu} \ddot{\tilde{\mathbf{u}}} + \mathbf{K}_{uu} \tilde{\mathbf{u}} + \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \tilde{\mathbf{u}} = \mathbf{F} - \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} G. \tag{16}$$

The potential vector can be recovered using

$$\tilde{\phi} = \mathbf{K}_{\phi\phi}^{-1} (\mathbf{G} + \mathbf{K}_{\phi u} \tilde{\mathbf{u}}). \tag{17}$$

The system dynamics is governed by equation (16) and the distributed dynamic measurement can be calculated using equation (17). Note that $\mathbf{G}$ is usually zero in the distributed sensor layer [4]. Thus, the distributed sensor output is estimated by

$$\tilde{\phi} = \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \tilde{\mathbf{u}}. \tag{18}$$

This potential is regarded as the output signal from the piezoelectric sensor layer and it can be processed further to provide feedback to the piezoelectric actuators for active vibration control. Also, the sensor output in equation (18) is obtained from the mechanical displacements only.

The right-hand side of equation (16) contains two force terms, i.e., the mechanical forces and electrical forces. In active vibration control, the electrical forces are regarded as the feedback control forces $\mathbf{F}_f$. They are given as

$$\mathbf{F}_f = \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{G}. \tag{19}$$

$\mathbf{G}$ is taken as a function of the feedback potential in terms of the output signal from the distributed sensing layer. This output signal is obtained from equation (18). Hence the feedback force, associated with the velocity, can be written as

$$\mathbf{F}_f = \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\dot{\tilde{\mathbf{u}}}, \tag{20}$$

where $\mathbf{A}_v$ represents the velocity-driven amplifier gain matrix. Substituting equation (20) into the dynamic equation (16) yields

$$\mathbf{M}_{uu}\ddot{\tilde{\mathbf{u}}} + \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}K_{\phi u}\dot{\tilde{\mathbf{u}}} + \mathbf{K}_{uu}\tilde{\mathbf{u}} + \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\tilde{\mathbf{u}} = \mathbf{F}. \tag{21}$$

Equation (21) may be written compactly as

$$\mathbf{M}_{uu}\ddot{\tilde{\mathbf{u}}} + \mathbf{C}_p\dot{\tilde{\mathbf{u}}} + \mathbf{K}\tilde{\mathbf{u}} = \mathbf{F}, \tag{22}$$

where $\mathbf{K} = \mathbf{K}_{uu} + \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}$ and $\mathbf{C}_p = \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}$ represent damping due to the piezoelectric effect.

All that remains is to see whether the feedback control forces induced by the feedback voltage can effectively enhance the system damping and therefore suppress the vibration of the structure. This will be investigated using stability analysis based on Lyapunov's second method. From equation (22), it is known that equivalent damping is only relative to the feedback potential, which is determined by the amount of feedback gain $\mathbf{A}_v$. Thus, by adjusting the feedback gain, changes to the damping of the structure are made so that the goal of controlling and suppressing the vibration of the structure can be achieved.

3. STABILITY ANALYSIS

The second method of Lyapunov provides a means for determining the stability of a system without explicitly solving for the eigenvalues. Stability criteria are established based on a positive-definite scalar functional, namely the Lyapunov functional, which is chosen to be representative of the energy of the system. The criteria that emerge ensure that the total time derivative of the functional will be negative definite and consequently guarantee that the system represented by it will be asymptotically stable.

3.1. LYAPUNOV ENERGY FUNCTIONAL

The energy flux is obtained from the equation of motion by pre-multiplying equation (21) by $\dot{\mathbf{u}}^{\mathrm{T}}$ and integrating with respect to time. That is,

$$\int_t \dot{\tilde{\mathbf{u}}}^{\mathrm{T}}(\mathbf{M}_{uu}\ddot{\tilde{\mathbf{u}}} + \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-}\mathbf{K}_{\phi u}\dot{\tilde{\mathbf{u}}}\mathbf{K}_{uu}\tilde{\mathbf{u}} + \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\tilde{\mathbf{u}})\,\mathrm{d}t = \int_t \dot{\tilde{\mathbf{u}}}^{\mathrm{T}}\mathbf{F}\,\mathrm{d}t. \tag{23}$$

The external forces will be omitted since they are considered to be bounded. Thus $\int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{F}\, \mathrm{d}t < \infty$. In light of $\mathrm{d}\tilde{\mathbf{u}} = \dot{\tilde{u}}\, \mathrm{d}t$ and $\mathrm{d}\dot{\tilde{\mathbf{u}}} = \ddot{\tilde{\mathbf{u}}}\, \mathrm{d}t$, equation (23) suggests that the Lyapunov's functional V may be taken as

$$\int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{M}_{uu}\, \mathrm{d}\dot{\tilde{\mathbf{u}}} + \int \tilde{\mathbf{u}}^{\mathrm{T}} \mathbf{K}_{uu}\, \mathrm{d}\tilde{\mathbf{u}} = - \int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} \mathbf{A}_v \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \dot{\tilde{\mathbf{u}}}\, \mathrm{d}t - \int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \tilde{\mathbf{u}}\, \mathrm{d}t. \quad (24)$$

In equation (24), the mass matrix $\mathbf{M}_{uu}$ is positive definite and the stiffness matrix $\mathbf{K}_{uu}$ is positive definite. When $\tilde{\mathbf{u}} \neq \mathbf{0}$ and $\dot{\tilde{\mathbf{u}}} \neq \mathbf{0}$, the left-hand side (LHS) of V is positive definite. Since the LHS is equal to the right-hand side (RHS), the RHS is also positive-definite. However, asymptotic stability is guaranteed when the time derivative of equation (24) is negative definite: i.e. when

$$\dot{V} < 0, \quad (25)$$

where

$$\dot{V} = - \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} \mathbf{A}_v \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \dot{\tilde{\mathbf{u}}} - \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \tilde{\mathbf{u}}. \quad (26)$$

The validity of expression (25) as a sufficient condition for asymptotic stability is supported through physical insight. Since the Lyapunov functional is based on the energy of the system, equation (26) represents the energy flux of the system. Thus, when the gain is set to zero, it is required that there should be no energy loss through the piezoelectric influence. The dissipative forces such as structural damping, inter-laminate friction and non-ideal boundary constraints ensure that the system energy flux of the passive dynamic structure is still negative definite. As long as the active piezoelectric elements do not add energy to the system, then asymptotic stability is ensured. The measure of active vibration suppression achieved in any given design will depend solely on choosing the appropriate potential control functions that are linked to the gain. These cause $\dot{V}$ to be large in magnitude and negative in sign so as to extract energy rapidly.

3.2. STABILITY CRITERIA

Sufficient conditions for the actuator control inputs to ensure asymptotic stability follow from expressions (25) and (26). Introducing the relation $\tilde{\mathbf{u}} \approx \dot{\tilde{\mathbf{u}}}\, \Delta t$ from $\mathrm{d}\tilde{\mathbf{u}} = \dot{\tilde{\mathbf{u}}}\, \mathrm{d}t$, equation (26) can be written as

$$\dot{V} = - \dot{\tilde{\mathbf{u}}}^{\mathrm{T}} \mathbf{K}_{u\phi} \mathbf{K}_{\phi\phi}^{-1} (\mathbf{A}_v \mathbf{K}_{\phi\phi}^{-1} + \Delta t \mathbf{I}) \mathbf{K}_{\phi u} \dot{\tilde{\mathbf{u}}}. \quad (27)$$

Since the stiffness matrices are positive definite, then equation (27) can be negative definite if

$$\mathbf{A}_v \mathbf{K}_{\phi\phi}^{-1} + \Delta t \mathbf{I} > \mathbf{0}. \quad (28)$$

A general and sufficient condition for stability implied by this statement is that the set of potential control functions must ensure the negative definiteness of $\dot{V}$. However, it will be practical to express a stronger condition on each particular actuator layer such that if this condition is satisfied for each actuator layer then asymptotic stability is ensured. This implies that the condition $\mathbf{A}_v > - \mathbf{K}_{\phi\phi} \Delta t$ must be satisfied for every actuator layer. A more practical approach will be to require that $\mathbf{A}_v > \mathbf{0}$ instead.

For the amplifier gain $\mathbf{0} < \mathbf{A}_v < \mathbf{1}$, the potential fed to the actuator layer will be less than the potential observed from the sensor layer since

$$\tilde{\phi}_f = \mathbf{A}_v \tilde{\phi}_s, \tag{29}$$

where $\tilde{\phi}_f$ represent the feedback potential, $\tilde{\phi}_s$ the sensor potential and $\mathbf{A}_v$ the gain. For $\mathbf{A}_v = \mathbf{1}$, the sensor potential is the same as the actuator potential, whereas the condition $\mathbf{A}_v = \mathbf{0}$ describes a zero actuator potential. Equation (27) suggests that there will be energy dissipation occurring through the piezoelectric effect when $\mathbf{A}_v = \mathbf{0}$. This implies that the active structure is controlled unconditionally. Practically, it is required that there should be no energy dissipation through the piezoelectric effect when the gain is set to zero [3]. It is evident, however, that the condition $\mathbf{A}_v > \mathbf{0}$ must be satisfied for asymptotic stability to exist.

Based on equation (21), the second term on the LHS represents accessible feedback control elements whereas the fourth term represents inaccessible feedback control elements. In order to achieve the full benefit of the feedback system, all feedback control states must be accessible. Although this requirement may not be satisfied generally, the design procedure as presented by equation (21) may still be valid. That is, the required value of the feedback gain may be computed, based on the accessible control states, to achieve the desired response of the structure. In view of this, an alternative active control strategy will be developed to satisfy the condition of no piezoelectric effects when the gain is set to zero.

3.3. ACTIVE CONTROL FORMULATION

In order to arrive at certain easily implementable strategies which will be sufficient to ensure both stability and active dissipation of energy, one may exploit the duality which exists between piezoelectric spatially distributed sensors and actuators. That is, it is sufficient to ensure stability and active vibration control provided that each actuator layer is associated with an identically spatial varying sensor layer which is collocated on the opposite side of the mid-plane and that the actuator control inputs are always identical in sign to the potential induced by the corresponding sensors.

In equation (14), the dynamic forces will change the state of the forces associated with the inertia, stiffness and piezoelectricity respectively. In the passive state of a laminated composite structure with embedded piezoelectric layers, the potential is observed on the layer identified as a piezoelectric actuator, as well as on the layer identified as the piezoelectric sensor except that it is opposite in sign. During active control, the feedback voltage $\tilde{\phi}_f$ is fed onto the actuator layer at the same points as those used to measure the potential sensed by the actuator layer $\tilde{\phi}_S^A$ during the passive state. Thus, the effective voltage is given as

$$\tilde{\phi}_{eff} = \tilde{\phi}_f - \tilde{\phi}_S^A. \tag{30}$$

Equation (30) demonstrates that the feedback potential must first overcome the voltage generated by the piezoelectric effect on the actuator layer before any control can be effected. This feedback potential is derived from the sensor potential as given in equation (29). Substituting equation (29) into equation (30) yields the feedback voltage as

$$\tilde{\phi}_f = \mathbf{A}_v \tilde{\phi}_S = \tilde{\phi}_{eff} + \tilde{\phi}_S^A. \tag{31}$$

In view of equation (18), the feedback voltage may be given in terms of the velocity as

$$\tilde{\phi}_f = \mathbf{A}_v \mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \dot{\tilde{\mathbf{u}}}. \tag{32}$$

Substituting equation (32) into the first of equation (14) gives the dynamic equation in terms of the displacements as

$$\mathbf{M}_{uu}\ddot{\tilde{\mathbf{u}}} + \mathbf{K}_{u\phi}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\dot{\tilde{\mathbf{u}}} + \mathbf{K}_{uu}\tilde{\mathbf{u}} = \mathbf{F}. \tag{33}$$

Omitting the external forces, this suggests that Lyapunov's functional may be taken as

$$V = \int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}}\mathbf{M}_{uu}\,\mathrm{d}\dot{\tilde{\mathbf{u}}} + \int \tilde{\mathbf{u}}^{\mathrm{T}}\mathbf{K}_{uu}\,\mathrm{d}\tilde{\mathbf{u}} = -\int \dot{\tilde{\mathbf{u}}}^{\mathrm{T}}\mathbf{K}_{u\phi}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\dot{\tilde{\mathbf{u}}}\,\mathrm{d}t. \tag{34}$$

When $\dot{\tilde{\mathbf{u}}} \neq \mathbf{0}$ and $\tilde{\mathbf{u}} \neq \mathbf{0}$, the LHS is positive definite. Therefore,

$$\dot{V} = -\dot{\tilde{\mathbf{u}}}^{\mathrm{T}}\mathbf{K}_{u\phi}\mathbf{A}_v\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{\phi u}\dot{\tilde{\mathbf{u}}}. \tag{35}$$

With $\mathbf{A}_v > \mathbf{0}$, $\dot{V}$ is negative definite. This implies that equation (33) is asymptotically stable. This may be extended so that the system represented by it is asymptotically stable. It is evident that the control approach represented by equation (33) requires less computational effort as compared to that of equation (21).

4. NUMERICAL RESULTS

An in-house program was used to evaluate the dynamic response represented by equations (21) and (33). A composite beam, $0{\cdot}1 \times 0{\cdot}005 \times 0{\cdot}00005\ \mathrm{m}^3$, sandwiched between 0·00005 m thick piezoelectric sensor/actuator was considered to compare the two feedback approaches. A uniformly distributed load was applied to the entire beam. Properties for the PZT and the composite material are shown in Table 1.

Omitting the dynamic effects and setting the feedback gain to zero, the static response of the current control model given by equation (21) is shown in Figure 2 and that of the proposed control model given by equation (33) is shown in Figure 3. It is observed that the proposed model exhibits higher displacement amplitude than the current control model.

TABLE 1

Material properties of the composite structure and piezoelectric material

Property	PZT	Graphite/epoxy
E_1	$0{\cdot}2\mathrm{E}+10\ \mathrm{N/m}^2$	$0{\cdot}98\mathrm{E}+11\ \mathrm{N/m}^2$
E_2	$0{\cdot}2\mathrm{E}+10\ \mathrm{N/m}^2$	$0{\cdot}79\mathrm{E}+10\ \mathrm{N/m}^2$
G_{12}	$0{\cdot}775\mathrm{E}+9\ \mathrm{N/m}^2$	$0{\cdot}56\mathrm{E}+10\ \mathrm{N/m}^2$
v_{12}	0·29	0·29
v_{21}	0·28	0·28
ρ	$1800\ \mathrm{kg\ m}^3$	$1520\ \mathrm{kg\ m}^3$
e_{31}	$0{\cdot}046\ \mathrm{C\ m}^2$	0·0
e_{32}	$0{\cdot}046\ \mathrm{C\ m}^2$	0·0
e_{33}	0·0	0·0
g_{11}	0·1062E–9 F/m	0·0
g_{22}	0·1062E–9 F/m	0·0
g_{33}	0·1062E–9 F/m	0·0

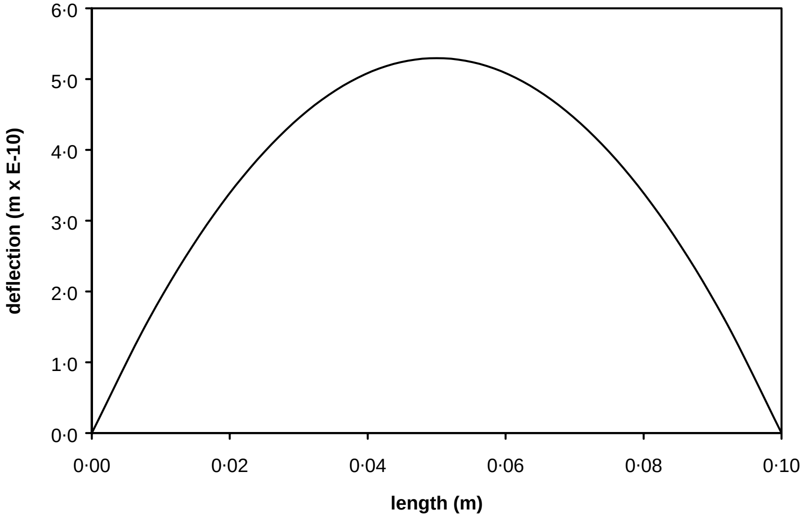

Figure 2. Beam deflection using the current active control model at zero gain.

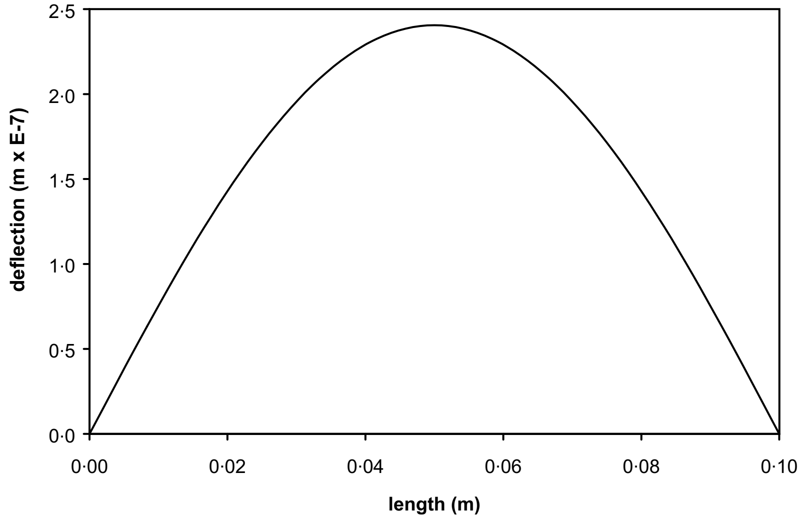

Figure 3. Beam deflection using the proposed control model at zero gain.

Since the external force is the same in both models, it implies that the current control model has a higher stiffness than the proposed control model. It is noted also that at this state, the proposed control model resembles the passive model, with a known accuracy.

Setting the gain to zero and excluding structural damping—for the dynamic response, Table 2 shows the first three natural frequencies obtained using the present control models. It is observed that the current control model yields higher natural frequencies than the passive model and the proposed control model yieldsthe same natural frequencies as the passive model.

The undamped time response at zero gain to a step function is shown for the current and the proposed control models in Figures 4 and 5 respectively. Consistent with the above results, the aurrent control model exhibits lower amplitude and higher natural frequency

TABLE 2

First three natural frequencies using the present active control models

	Passive	Current	Proposed
ω_1^2	$1{\cdot}1220 \times 10^9$	$4{\cdot}9563 \times 10^{11}$	$1{\cdot}1220 \times 10^9$
ω_2^2	$1{\cdot}7909 \times 10^{10}$	$1{\cdot}9855 \times 10^{12}$	$1{\cdot}7909 \times 10^{10}$
ω_3^2	$9{\cdot}1247 \times 10^{10}$	$4{\cdot}4938 \times 10^{12}$	$9{\cdot}1247 \times 10^{10}$

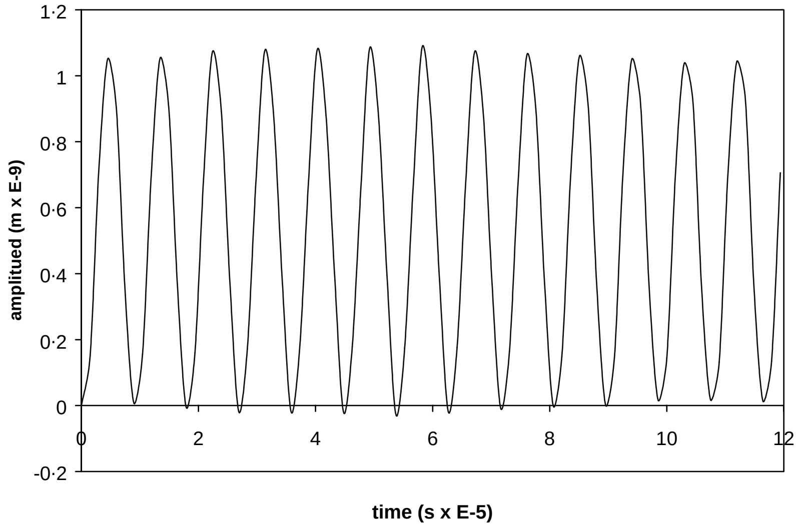

Figure 4. Time response of the current active control model at zero gain to a step force.

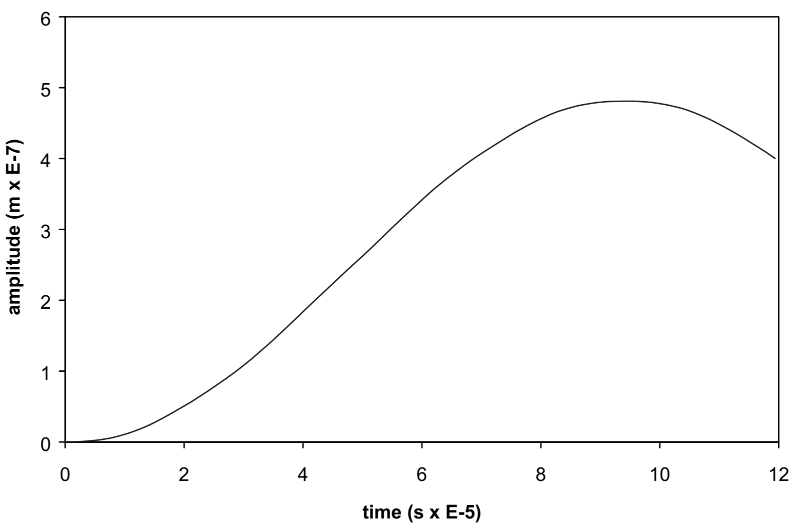

Figure 5. Time response of the proposed active control model at zero gain to a step force.

than the proposed model. For the damped case, it is readily seen by inspection that the damping force for the current control model will be higher than that of the proposed model by a factor $\mathbf{K}_{\phi\phi}^{-1}$.

The above results show that the current control model at zero gain under-predicts the deflection in a static and dynamic response and over-predicts the natural frequencies. This indicates that the model represents a structure which is stiffer than its design stiffness. The additional stiffness, above the design stiffness, indicates that piezoelectric control forces persist at zero gain. Contrary to this, it is required that the beam be free from piezoelectric control forces at this state, a phenomenon exhibited by the proposed control model. The additional stiffness introduced in the current model is not representative of the desired assumption and/or application. Hence, structures constructed with the aid of the current control model will respond differently from their predicted behaviour thus resulting in undesired responses and inaccuracies.

5. CONCLUSION

A current control strategy for laminated piezo-elastic structures using an open-circuit mode has been investigated. The equations of motion were found to be asymptotically stable. However, energy dissipation through the piezoelectric effect was observed at zero feedback gain. This implied that the active structure was controlled unconditionally—which is not representative of the desired application. As a result, a control strategy that satisfied the condition of no piezoelectric effects when the gain is set to zero was then developed. This strategy was based on the second method of Lyapunov, where a Lyapunov functional indicative of the total system mechanical energy was chosen. Asymptotic stability of the structure was shown to be ensured if the feedback control functions are greater than zero; and the feedback potential is greater than the induced piezoelectric potential of the actuator layer. With these criteria satisfied, active vibration control became uniquely dependent on the feedback gain.

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