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# SOLUTION OF THE DUFFING EQUATION BY USING TARGET FUNCTION METHOD

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## 1. INTRODUCTION

Solution of the Duffing equation in non-linear vibration problem is studied in the paper. The governing equation for the problem was formulated in references [1, 2]. In the case of $\varepsilon$ being a small parameter, the equation is solved by using the Lindstedt–Poincare technique, the method of multiple scales, and the method of averaging [1, 2]. Almost all perturbation methods are based on small parameter $\varepsilon$ so that the approximate solutions can be expressed in a series of small parameters.

There are some shortcomings in the perturbation method. Clearly, in the case of $\varepsilon$ being a larger value, the perturbation method is no longer valid. Generally, in the perturbation method one can only get, for example, a two-term solution for the small parameter. Obviously, the rigidly increasing labor of derivation makes it practically impossible to obtain more terms by hand. Also, in this method it is not easy to judge how the approximation is achieved. The limitation of the perturbation method was also pointed out in references [3, 4].

Recently, the target function method for evaluating the buckling loading and the vibration frequency was suggested [5]. It was found that the idea of target function method is a general one, which can also be used for the present analysis.

The idea of target function method can be described as follows. Assume that the Duffing equation with an initial condition ($u = A$, $\mathrm{d}u/\mathrm{d}t = 0$ at the time $t = 0$) is integrated on the interval $(0, t_p)$. It is found that the function $v(t_p)$ is the mentioned target function, where the dependent variable $v(= \mathrm{d}u/\mathrm{d}t)$ is the velocity of motion. The second zero of the target function, denoted by $t_p = T_{p2}$, will be the period of motion. In fact, from (a) the mentioned initial condition, and (b) the conservation of energy in the vibration defined by the Duffing function, we have $u = A$ and $v = \mathrm{d}u/\mathrm{d}t = 0$ at the time $t = T_{p2}$. Thus, after comparing the conditions at the time $t = 0$ and at the time $t = T_{p2}$ we see that $T_{p2}$ is the period of the motion. Clearly, the suggested technique depends on the computer computation intensively. Particular advantages of the suggested method will be described in the section "remarks".

## 2. ANALYSIS

In the following analysis, the Duffing equation is defined by [1, 2]

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \omega_o^2 u(1 + \varepsilon u^2) = 0, \tag{1}$$

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where $\omega_o$ is the circular frequency, $\varepsilon$ is a constant which may not be a small value. The imposed boundary condition takes the form

$$u|_{t=0} = A, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = 0, \tag{2}$$

where $A$ is a positive value. After letting

$$v = \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{3}$$

$$F(u) = \omega_o^2 \int_0^u u(1 + \varepsilon u^2)\,\mathrm{d}u = \omega_o^2 \left(\frac{u^2}{2} + \frac{\varepsilon u^4}{4}\right) \tag{4}$$

and carrying out integration of equation (1), the following equation on the phase plane is obtainable [1, 2]:

$$\frac{v^2}{2} + F(u) = h, \tag{5}$$

where $h$ is obtained from equations (2), (4) and (5), and takes the value

$$h = F(u)|_{u=A} = \omega_o^2 \left(\frac{A^2}{2} + \frac{\varepsilon A^4}{4}\right). \tag{6}$$

Clearly, from equations (4) and (5) we see that the motion is stable at any vicinity of the position $u = 0$ and $v = 0$, and the motion is periodical.

The target function method is introduced as follows. For a given value $t_p$, we perform the integration for equation (1) with the initial boundary value condition (2) on the interval $(0, t_p)$, and obtain the functions $u(t)$ and $v(t)$ $(0 \leqslant t \leqslant t_p)$ and $v(t_p)$, where $v(t) = \mathrm{d}u/\mathrm{d}t$. The obtained $v(t_p)$ is called the target function in this paper. Obviously, the $v(t_p)$ value is a function of the given time $t_p$, and it is not equal to zero in general. In this case, we can define the target function by $H(t_p) = v(t_p)$. The governing equation of target function method takes the form

$$H(t_p) = v(t_p) = 0. \tag{7}$$

Assume that $t_p = T_{p1}$ and $t_p = T_{p2}$ are two successful zeros of the target function $v(t_p)$. It is easy to prove that the second zero $T_{p2}$ will be the period of motion for the Duffing equation (1) with the initial condition (2). In fact, from the definition of $T_{p1}$ and $T_{p2}$ and (4–6), we have

$$v(T_{p1}) = 0, \qquad u(T_{p1}) = -A, \tag{8a}$$

$$v(T_{p2}) = 0, \qquad u(T_{p2}) = A. \tag{8b}$$

After considering two points: (a) only the arguments $\mathrm{d}^2u/\mathrm{d}t^2$, $u$, $u^3$ are involved in equation (1), (b) the condition at the time $t = 0$ shown by equation (2) is the same as the condition at the time $t = T_{p2}$ shown by equation (8b), it follows the periodical property of the motion

$$u(t) = u(t + T_{p2}). \tag{9}$$

In conclusion, the period of the motion is investigated by the following steps. First, for a given $t_p$, we carry out an integration for equation (1) with condition (2) and obtain the functions $u(t)$ and $v(t)$ $(0 \leqslant t \leqslant t_p)$, and $v(t_p)$. The numerical solution of the ordinary differential equation can be performed by using Runge–Kutta integration rule [6].

In this case, the target function $v(t_p)$ may not be equal to zero in general. Our goal is to obtain the zeros of the target function $v(t_p)$, particularly, the second zero $T_{p2}$ which is the period of the motion. To this end, the half-division technique is suggested to evaluate the zeros of function.

Finally, the suggested method mainly depends on the numerical solution of the ordinary differential equation and numerical evaluation of the zeros of a given function. The mentioned computations are successful using a FORTRAN program on computer.

For the harmonic motion case ($\varepsilon = 0$ in equation (1)), we have the period of motion and the circular frequency of motion

$$T_o = \frac{2\pi}{\omega_o}, \qquad \omega_o = \frac{2\pi}{T_o}. \tag{10}$$

Similarly, in the present case, the period of motion $T_{p2}$ and the circular frequency of motion $\omega_p$ can be expressed by the relation

$$T_{p2} = \frac{2\pi}{\omega_p}, \qquad \omega_p = \frac{2\pi}{T_{p2}}. \tag{11}$$

Furthermore, the calculated circular frequency can be expressed as

$$\omega_p = \alpha\omega_o, \tag{12}$$

where $\alpha$ is a magnified (or reduced) factor for the circular frequency.

A particular advantage of the suggested method is that one can obtain the motion of the Duffing equation in addition to the circular frequency. The obtained displacement may be expressed in the form

$$u(t) = \frac{c_0}{2} + \sum_{k=1}^{M} c_k \cos(k\omega_p t), \quad (0 \leqslant t \leqslant T_{p2}). \tag{13}$$

Clearly, the involved Fourier coefficients can easily be evaluated from the obtained displacement $u(t)$ $(0 \leqslant t \leqslant T_{p2})$. It is found that since the function $F(u)$ is even one with respect to the argument $u$, thus, the Fourier coefficients $c_0, c_2, c_4, \ldots$ always equals zero, the magnified factor $\alpha$ and the calculated coefficients $c_1, c_3, c_5 \ldots$ depend on the $A$ and $\varepsilon$, and they are listed in Table 1. For the computation, we used the $M = 360$ divisions in the Runge–Kutta method for the numerical solution of ODE [6]. The circular frequency can also be evaluated by [1]

$$T_{p2} = \sqrt{2} \int_{-A}^{A} (h - F(u))^{-1/2} \, \mathrm{d}u. \tag{14}$$

We found that the calculated results using the target function method coincide with the ones from equation (14) very well within the range of four digital values.

In the meantime, if $\varepsilon$ is a small parameter, using the perturbation method, the approximate solution is [1, 2]

$$\alpha = 1 + \frac{3\varepsilon A^2}{8}, \qquad c_1 = A\left(1 - \frac{\varepsilon A^2}{32}\right), \qquad c_3 = \frac{\varepsilon A^3}{32}. \tag{15}$$

TABLE 1

*$\alpha$ value and the calculated Fourier coefficients for the solution of the Duffing equation* $d^2u/dt^2 + \omega_o^2 u(1 + \varepsilon u^2) = 0$ *with the condition* $u(0) = A$ *and* $u'(0) = 0$ (*see equations* (12, 13))

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $A = 1$ case | | | | | | | | | | |
| $\varepsilon =$ | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1·0 |
| $\alpha$ | 1·0367 | 1·0720 | 1·1060 | 1·1389 | 1·1708 | 1·2017 | 1·2318 | 1·2612 | 1·2898 | 1·3178 |
| $c_1$ | 0·9971 | 0·9945 | 0·9923 | 0·9903 | 0·9885 | 0·9869 | 0·9854 | 0·9841 | 0·9828 | 0·9817 |
| $c_3$ | 0·0029 | 0·0054 | 0·0077 | 0·0096 | 0·0114 | 0·0130 | 0·0144 | 0·0157 | 0·0169 | 0·0180 |
| $c_5$ | 0·0000 | 0·0000 | 0·0001 | 0·0001 | 0·0001 | 0·0002 | 0·0002 | 0·0002 | 0·0003 | 0·0003 |
| $\alpha$* | 1·0375 | 1·0750 | 1·1125 | 1·1500 | 1·1875 | 1·2225 | 1·2625 | 1·3000 | 1·3375 | 1·3750 |
| $c_1$* | 0·9969 | 0·9937 | 0·9906 | 0·9875 | 0·9844 | 0·9812 | 0·9781 | 0·9750 | 0·9719 | 0·9687 |
| $c_3$* | 0·0031 | 0·0063 | 0·0094 | 0·0125 | 0·0156 | 0·0188 | 0·0219 | 0·0250 | 0·0281 | 0·0313 |
| $A = 1$ case | | | | | | | | | | |
| $\varepsilon =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\alpha$ | 1·3178 | 1·5691 | 1·7844 | 1·9760 | 2·1504 | 2·3116 | 2·4622 | 2·6040 | 2·7385 | 2·8666 |
| $c_1$ | 0·9817 | 0·9741 | 0·9698 | 0·9671 | 0·9653 | 0·9639 | 0·9629 | 0·9620 | 0·9614 | 0·9608 |
| $c_3$ | 0·0180 | 0·0253 | 0·0293 | 0·0318 | 0·0336 | 0·0348 | 0·0358 | 0·0366 | 0·0372 | 0·0377 |
| $c_5$ | 0·0003 | 0·0006 | 0·0009 | 0·0010 | 0·0011 | 0·0012 | 0·0013 | 0·0013 | 0·0014 | 0·0014 |
| $A = 2$ case | | | | | | | | | | |
| $\varepsilon =$ | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1·0 |
| $\alpha$ | 1·1389 | 1·2612 | 1·3719 | 1·4739 | 1·5691 | 1·6587 | 1·7435 | 1·8244 | 1·9017 | 1·9760 |
| $c_1$ | 1·9806 | 1·9681 | 1·9594 | 1·9530 | 1·9481 | 1·9442 | 1·9410 | 1·9384 | 1·9362 | 1·9343 |
| $c_3$ | 0·0193 | 0·0314 | 0·0398 | 0·0459 | 0·0506 | 0·0543 | 0·0573 | 0·0598 | 0·0619 | 0·0636 |
| $c_5$ | 0·0002 | 0·0005 | 0·0008 | 0·0011 | 0·0013 | 0·0015 | 0·0016 | 0·0018 | 0·0019 | 0·0020 |
| $\alpha$* | 1·1500 | 1·3000 | 1·4500 | 1·6000 | 1·7500 | 1·9000 | 2·0500 | 2·2000 | 2·3500 | 2·5000 |
| $c_1$* | 1·9750 | 1·9500 | 1·9250 | 1·9000 | 1·8750 | 1·8500 | 1·8250 | 1·8000 | 1·7750 | 1·7500 |
| $c_3$* | 0·0250 | 0·0500 | 0·0750 | 0·1000 | 0·1250 | 0·1500 | 0·1750 | 0·2000 | 0·2250 | 0·2500 |
| $A = 2$ case | | | | | | | | | | |
| $\varepsilon =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\alpha$ | 1·9760 | 2·6040 | 3·1071 | 3·5392 | 3·9240 | 4·2743 | 4·5979 | 4·9002 | 5·1849 | 5·4548 |
| $c_1$ | 1·9343 | 1·9241 | 1·9199 | 1·9177 | 1·9162 | 1·9153 | 1·9146 | 1·9140 | 1·9136 | 1·9132 |
| $c_3$ | 0·0636 | 0·0731 | 0·0770 | 0·0791 | 0·0804 | 0·0813 | 0·0819 | 0·0824 | 0·0828 | 0·0831 |
| $c_5$ | 0·0020 | 0·0027 | 0·0030 | 0·0031 | 0·0032 | 0·0033 | 0·0034 | 0·0034 | 0·0034 | 0·0035 |

*Results from the perturbation method [2].

For comparison, the relevant results are also listed in Table 1. From Table 1 we see that if 2% is a permissible error, the perturbation technique is acceptable when $\varepsilon < 0{\cdot}6$ ($\varepsilon < 0{\cdot}1$) for $A = 1$ ($A = 2$) respectively.

## 3. ANALYSIS IN GENERAL CASE

If the initial velocity in the vibration is not equal to zero, the initial conditions are as follows:

$$u|_{t=0} = A_1, \qquad \left.\frac{du}{dt}\right|_{t=0} = B_1. \tag{16}$$

Without loss of generality, it is assumed that two values $A_1$ and $B_1$ are positive. In fact, the problem can be solved in two ways.

First is the equivalent method. Clearly, the motion that is determined by condition (16) could be equivalent to the one determined by condition (2). This method is reached simply

TABLE 2

*The calculated α values under different initial conditions*

| $A_1/A$ | 0·2 | 0·4 | 0·6 | 0·8 | 1·0 | |
|---|---|---|---|---|---|---|
| α | 2·15041512 | 2·15041512 | 2·15041613 | 2·15041613 | 2·15041610 | 2·15041620* |

*From equation (14).

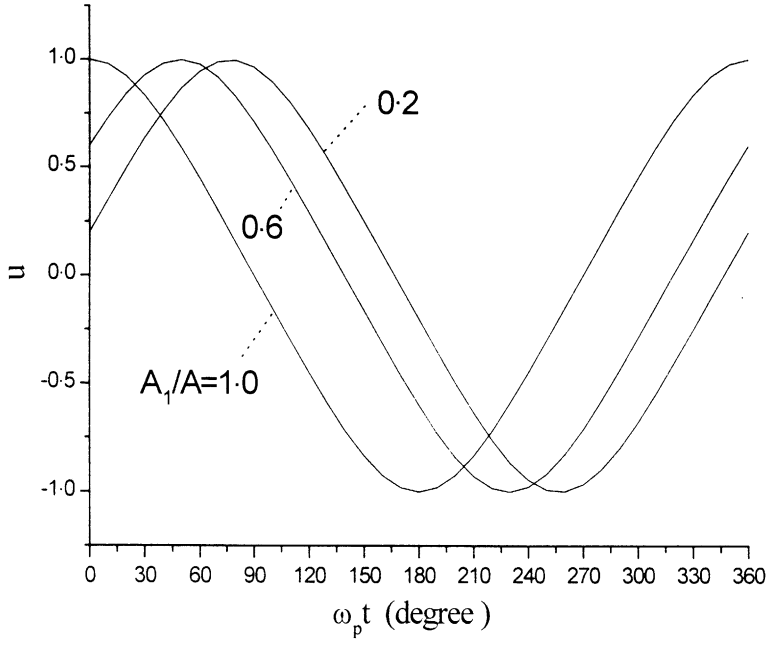


Figure 1. Motions of vibration under various initial conditions.

by substituting two sets of the initial condition: (a) $u = A$ and $v = 0$ (from equation (2)), (b) $u = A_1$ and $v = B_1$ (from equation (16)), in equation (5). Thus, we have

$$B_1^2 + \omega_o^2\left(A_1^2 + \frac{\varepsilon A_1^4}{2}\right) = \omega_o^2\left(A^2 + \frac{\varepsilon A^4}{2}\right). \tag{17}$$

This is to say, instead of solving the problem of the initial condition (17), we can solve the problem of condition (2) where the value $A$ is determined by using equation (17).

The second method is directly using the concept of target function. In this case, the target function may be defined by $H(t_p) = v(t_p) - B_1$, and the governing equation of target function method takes the form

$$H(t_p) = v(t_p) - B_1 = 0. \tag{18}$$

As before, the second zero of the function, $T_{p2}$, will be the period of motion.

In order to examine the second method, the following computation was performed under the conditions: (a) taking $\omega_o = 1$ without loss of generality, (b) $\varepsilon = 5$, (c) $A = 1$, (c) $A_1/A = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0$, (d) $B_1$ determined by equation (17) from the given values of $A$, $A_1$ and $\varepsilon$, (e) Three-hundred and sixty divisions is used in the solution of the differential equation. In this case, the calculated values of the factor α are listed in Table 2. The listed result proves that high accuracy has been achieved in the second method.

In addition, the motion of vibration for three cases $A_1/A = 0{\cdot}2$, $0{\cdot}6$ and $1{\cdot}0$ are plotted in Figure 1. As expected, from Figure 1 we see that the motions plotted are merely shifting in a horizontal direction to each other.

## 4. REMARKS

Previously, when the computer was not available as nowadays, investigators paid attention to the solution which can be performed by hand or very elementary computation. In contrast, the present study mainly depends on the successful numerical solutions and computer computation. Particular advantages for the method are as follows. Since there is no difference in the numerical solution between the linear and non-linear differential equations, the difficulty caused by the non-linearity disappears if target function method is used. Also, a highly accurate computation scheme is used, for example, 360 divisions are assumed in the numerical solution of ODE, the obtained result must be very near to the exact solution. Finally, all necessary information, including the motion of vibration and the period of motion, can be obtained from the solution.

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