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THE LOCALLY TRANSVERSAL LINEARIZATION (LTL) METHOD REVISITED: A SIMPLE ERROR ANALYSIS

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1. INTRODUCTION

Efforts to obtain useful information on the response of non-linear systems of engineering interest revolve around many existing approximate tools, analytical or numerical as most of them do not admit exact solutions. Amongst the available analytical techniques to solve initial value problems (IVPs), mention may be made of the classical Krylov–Bogoliubov averaging technique [1], the secular perturbation theory [2, 3], the method of multiple scales [4], an incremental harmonic balancing method [5], the homotopy method [6], a process analysis method [7], etc. A major deficiency of almost all the analytical techniques is in their inability to accurately predict even a reasonable spectrum of the complex response scenarios such as bifurcations to quasi-periodicity and chaos. Even though numerical techniques often provide a more versatile (and less elegant) alternative, their susceptibility to the choice of a wrong time step often reduces the reliability of the results obtained [8, 9]. A third route based on numeric-analytical approaches is also available. These techniques includes, among others, the phase space linearization (PSL) method proposed by Iyenger and Roy [10–12] and a frequency-domain-based multiharmonic balancing technique for predicting periodic orbits by Narayanan and Shekar [13]. There is nevertheless no precise, unified and a generally applicable method available till date to solve different kinds of non-linear IVPs with continuous or piecewise continuous vector fields.

It is well known that tangent spaces of non-linear ODEs, unlike linear ones, are ever changing function of the independent variable (space or time). Thus, it is generally not possible to replace the non-linear equation pathwise in terms of a linearized equation even over a small step size. The recently proposed LTL procedure [14–17] avoids this difficulty by constructing the linearized equations such that the linearized solution manifolds transversally intersect the (unknown) solution manifolds of the original non-linear equations at a countable set of points in the state space where the solution vector is sought. The most important feature of this method is that it attempts to precisely satisfy the governing non-linear ODEs of the physical problem at a given set of points along the axis of the independent variable. This is achieved by constructing the LTL-based ODEs such that the given non-linear ODEs are implicitly satisfied at a given point along the time axis. The procedure eventually leads to a set of non-linear algebraic equations in terms of the desired solution vector at the given point along the time axis. For a dynamical system posed in the form of an IVP, the non-linear algebraic equations are uncoupled for different chosen points along the time axis. Finally, the method is illustrated for a non-linear two-degree-of-freedom system.

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2. THE BASIC LTL METHOD

The brief LTL approach is first explained with the help of a non-linear oscillator described in the state space by the following system of ODEs for the dependent vector $x = \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}^{\mathrm{T}} \in R^n$:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = v\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right). \tag{1}$$

Introducing the augmented response vector $X = \{\{x_1 = x\}, \{x_2 = \mathrm{d}x/\mathrm{d}t\}\}^{\mathrm{T}} \in R^{2n}$, and denoting the scalar elements of the vector field $v = [v^{(1)}, v^{(2)}, \dots, v^{(n)}]^{\mathrm{T}}$, equation (1) may be written in the state space as n first order ODEs:

$$\frac{\mathrm{d}x_1^{(l)}(t)}{\mathrm{d}t} = x_2^{(l)}(t),\ l = 1, 2, \dots, n,$$

$$\frac{\mathrm{d}x_2^{(l)}(t)}{\mathrm{d}t} = v^{(l)}(x_1, x_2, t). \tag{2}$$

The superscript $^{\mathrm{T}}$ in the definition of the vectors indicates transposition. Using a vector notation, the above system may simply be written as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V(X, t), \tag{3}$$

where $V(X, t) = \{x_2^{(l)}, v^{(l)} \mid l = 1, 2, \dots, n\}^{\mathrm{T}}$ is a $2n$-dimensional augmented vector field, which is non-linear in X. The only independent variable, $t \in R$ is defined over the closed and compact domain $D = [0\ \ T] \subset R$. Let the subset of the time axis over $[0, T]$ be strictly ordered such that $0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = T$ and $h_i = t_i - t_{i-1}$, where $i \in Z^+$. It is now intended to construct a set of n conditionally linear system of ODEs, wherein the ith linear system should, in a sense, be a representative of the instantaneous non-linear flow at $t = t_i$. The LTL system has to be so constructed as to implicitly satisfy the governing non-linear ODEs at $t = t_i$. In the LTL scheme that follows, the ith locally linearized system is of the same dimension as the given non-linear system such that it is also n-dimensional and is preferably obtainable from the given non-linear system with the simplest and least alterations. Here a convenient and easily adaptable methodology of linearization using LTL, henceforth labelled as the basic LTL (BLTL) scheme, is first described. Towards this, the ith linearized ODEs are constructed by recasting equation (3) over $D_i = (t_{i-1}, t_i]$ as

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = A(X_i \triangleq X(t_i), t_i)\,\bar{X} + B(t), \tag{4}$$

where the $n \times n$ matrix $A(X_i, t_i)$ and the n-dimensional vector $B(t)$ are to be so chosen as to satisfy the identity:

$$A(X_i, t_i) X_i + B(t_i) = V(X_i, t_i). \tag{5}$$

Elements of the matrix $A(X_i, t_i)$ are functions of the still unknown (discrete) solution vector X_i and thus equation (4) is clearly linearized with conditionally constant coefficients. Since equation (4) is required to satisfy equation (3) at the left end of the domain segment D_i, the initial condition vector to equation (4) is $\bar{X}(t_{i-1}) \triangleq \bar{X}_{i-1} = X_{i-1}$. Now, suitable constraint equations need to be framed. Towards this, the solution of equation (4) is explicitly written as

$$\bar{X}(t) = \Psi(X_i, t, t_{i-1})\left\{X_{i-1} + \int_{t_{i-1}}^{t \leqslant t_i} \Psi^{-1}(X_{i-1}, t, t_{i-1})B(t)\,\mathrm{d}t\right\}, \tag{6}$$

where $\Psi(X_i, t,\ t_{i-1})$ is the fundamental solution matrix. While the first term on the right-hand side of equation (6) represents the complementary solution, the second one, involving integration, stands for the particular integral due to the vector function, $B(t)$. The constraint condition, i.e., $X_i \in M \cap \bar{M}$ (M is the solution manifold of the original non-linear equation (3) and $\bar{M}$ is the solution manifold of the LTL equation (4)), is now obtained by requiring that

$$X_i = \bar{X}_i \tag{7}$$

in equation (6). This leads to the following n algebraic non-linear equations in terms of the unknown set of vectors, X_i:

$$\Phi_i(X_i, X_{i-1}, t_i, t_{i-1}) = 0, \tag{8}$$

where the vector non-linear function Φ_i is given by

$$\Phi_i = X_i - \Psi(X_i, t_i, t_{i-1})\left\{X_{i-1} + \int_{t_{i-1}}^{t_i} \Psi^{-1}(X_i, t, t_{i-1})B(t)\,\mathrm{d}t\right\}. \tag{9}$$

Equation (8) consists of n algebraic and non-linear (transcendental) equations for the unknown vector x_i and may be solved using an approach similar to Newton–Raphson.

3. ERROR ESTIMATES

An estimate of error involved in the proposed LTL procedure may be obtained by expanding both the solution vector of the non-linear differential equation and the solution vector of the corresponding transversal LTL equation in implicit Taylor's series and comparing the terms in both the expansions that are similar. The steps involved in the error estimate are outlined in the following. First, the solution vector X corresponding to original system of non-linear differential equations (3) is expanded in a Taylor's series at $X_{i-1} = X(t_{i-1})$ using a step size $h_i = t_i - t_{i-1}$ as

$$\begin{aligned} X_i &= X_{i-1} + V(X_{i-1}, t_{i-1})h + O(h^2) \\ &= X_{i-1} + \left[V(X_i, t_i) - \frac{\mathrm{d}V(X_{i-1}, t_{i-1})}{\mathrm{d}t}h\right]h + O(h^2) \\ &= X_{i-1} + V(X_i, t_i)h + O(h^2). \end{aligned} \tag{10}$$

Similarly, the solution vector $\bar{X}$ corresponding to the LTL equation (4) is implicitly expanded in Taylor's series at $\bar{X}_{i-1} = X_{i-1}$ as

$$\begin{aligned} \bar{X}_i &= X_{i-1} + [A(X_i, t_i)X_{i-1} + B(t_{i-1})]h + O(h^2) \\ &= X_{i-1} + \left[A(X_i, t_i)(\bar{X}_i - \{A(X_i, t_i)X_{i-1} + B(t_{i-1})\}h) + B(t_i) - \frac{\mathrm{d}B(t_{i-1})}{\mathrm{d}t}h\right]h + O(h^2) \\ &= X_{i-1} + [A(X_i, t_i)\bar{X}_i + B(t_i)]h + O(h^2). \end{aligned} \tag{11}$$

For the above expansions to be valid, the vector functions V and B need to have C^1 continuity. Using the constraint equations (5) and (7), the above expression reduces to the right-hand side of equation (10) except for the remainder term of order $O(h^2)$. Hence, the vector $\bar{X}_i$ has a local error of $O(h^2)$. Thus, it is clear that the linearized velocity vector $\bar{x}_2 = \mathrm{d}\bar{x}/\mathrm{d}t$, has a local error of $O(h^2)$. The linearized displacement vector $\bar{x}_1$ will therefore, have an error of $O(h^3)$. Moreover, the global error orders are one integral order less than the corresponding local error orders.

4. HIGHER ORDER LTL PROCEDURES

The present aim is to derive other forms of LTL schemes with capabilities to remain close to the original path as followed by the non-linear system with the given boundary conditions, provided that the chosen step size is sufficiently small. The basic form of LTL, presented in the previous section, only ensures transversal intersections and not a consistent closeness of the paths. In what follows, a method for deriving consistently higher orders and path-sensitive LTL-based ODEs is outlined. To derive, for instance, an LTL scheme, one order higher than the basic scheme of section 2, equation (1) is differentiated once to obtain

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = v^{(1)}\,(x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}, t), \tag{12}$$

where the right superscript over v denotes the order of differentiation with respect to t. Introducing the augmented response vector $\hat{X} = \{x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, x^{(n)}\}^{\mathrm{T}}$, equation (12) may be written in a vector form as

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = \hat{V}\,(\hat{W}, t), \tag{13}$$

where $\hat{X} \in R^n$, $\hat{V}(\hat{X}, t) = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}, v^{(1)}\,(\hat{X}, t)\}^{\mathrm{T}}$ is an $(n+1)$-dimensional vector field, which is non-linear in $\hat{X}$. The corresponding higher order LTL (HLTL) equation can be written as

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} = \tilde{A}(\hat{X}_i, t_i)\,\tilde{X} + \tilde{B}(t), \tag{14}$$

where $\tilde{X}(x) = \{\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots, \bar{x}^{(n-1)}, \bar{x}^{(n)}\}^{\mathrm{T}} = \{\bar{X}, \bar{t}^{(n)}\}^{\mathrm{T}}$ is the augmented HLTL solution vector. The $(n+1)\times(n+1)$ matrix $\tilde{A}(\hat{X}_i, t_i)$ (conditionally constant) and the $(n+1)$-dimensional vector $\tilde{B}(t)$ are to be so chosen as to satisfy the identity:

$$\tilde{A}(\hat{X}_i, t_i)\,\hat{X}_i + \tilde{B}(t_i) = \hat{V}(\hat{X}_i, t_i). \tag{15}$$

The augmented HLTL equation is solved, over the subdomain D_i, subject to the initial conditions $\hat{X}_{i-1} = \{X_{i-1}, v(X_{i-1}, t_{i-1})\}^{\mathrm{T}}$ and the constraint condition $\hat{X}_i = \tilde{X}_i$. The rest of the steps to obtain the required solution vector are the same as that outlined for BLTL case in section 2. The error estimates for HLTL essentially follow the same steps as in the section 3. Thus, the local and the global error orders for $\tilde{X}_i$ are, respectively, $O(h^2)$ and $O(h)$. In other words, the linearized acceleration vector $\mathrm{d}\tilde{x}_2/\mathrm{d}t$, has a local error of $O(h^2)$. The linearized velocity vector $\tilde{x}_2$, has a local error of order $O(h^3)$ and linearized displacement vector has a local error of order $O(h^4)$. Here it may be noted that, the acceleration vector, which has the highest derivative in t, is however artificially constructed for the HLTL system and does not constitute an element of the required solution vector. The elements belonging to $\bar{X} \subset \tilde{X}$, therefore, do not exceed a local error order of $O(h^3)$. Of course, it needs to be noted that this improved accuracy is obtainable only with a higher computational cost.

5. ERRORS IN THE PHASE-INDEPENDENT REGIME: A SPECIAL CASE

In general, solutions of ODEs with different initial conditions (while holding the other system parameters to be the same) are different. However, there is a special case in positively damped linear or non-linear engineering dynamical systems, wherein the system trajectories, $x^{(j)}(t)$, starting with distinct initial conditions $x^{(j)}(t_0)$, $t > t_0$ and $j = 1, 2, \ldots$. converge to a unique function, say $y(t)$, as $(t - t_0) \to \infty$. Should such a response regime, where the memory of initial conditions is asymptotically lost, exist, it is inferred to as phase-independent [18]. Due to the asymptotic uniqueness of the evolving trajectory, the intersection point with any approximate and transversal trajectory is unique in the phase-independent case. One thus concludes that in such a case the LTL method should pick up this intersection point nearly exactly irrespective of the chosen time step size, subject only to the errors inevitable in any root searching procedure (see Figure 9(a) and 9(b)).

6. CERTAIN COMPUTATIONAL ISSUES

Computations of the fundamental solution matrix and its inverse, crucial for the construction of a linearized solution, require exponentiation of certain (possibly quite large) system matrices. Let it be required to numerically obtain $[\Phi(t, t_{i-1}) = \exp\{[M](t - t_{i-1})\}$. A computationally expedient way to do so is to divide the interval $(t_{i-1}, t]$ into 2^k equal sub-intervals and make use of the following identity:

$$\Phi(t, t_{i-1}) = \left\{\Phi\left(\left(t_{i-1} + \frac{t - t_{i-1}}{2^k}\right), t_{i-1}\right)\right\}^k. \tag{16}$$

Denoting the argument of the kth exponent RHS of the above equation as $\Phi_k(t_{i-1})$ and $h_k = 2^{-k}(t - t_{i-1})$, one may use a truncated Taylor expansion (retaining four or even fewer terms) to compute $\Phi_k(t_{i-1})$ as

$$\Phi_k(t_{i-1}) = I + [M]h_k + [M]^2 h_k^2/2 + \cdots, \tag{17}$$

where $[I]$ is an $n \times n$ identity matrix. A similar scheme may also be utilized to obtain the inverse $\Phi^{-1}(t, t_{i-1}) = \exp\{-[M](t - t_{i-1})\}$. In this case,

$$\Phi_k^{-1}(t_{i-1}) = I - [M]h_k + [M]^2 h_k^2/2 + \cdots. \tag{18}$$

At the time when this article was written, the attention of the authors was attracted to a recent paper by Leung [19], where a similar matrix exponentiation as outlined in this section has been utilized for the response calculation of linear engineering systems under deterministic and random loading conditions.

7. NUMERICAL ILLUSTRATIONS

Even though the focus of the present work is on the propositions and theoretical error estimates of transversal linearization scheme, a rather limited numerical illustration on the two-degree-of-freedom system is presented here. Presently, a two-degree-of-freedom system is defined by the following system of second order non-linear ODEs:

$$\ddot{x}_1 + 0{\cdot}4\dot{x}_1 + x_1 + k_1x_1^3 + k_2x_1x_2^2 = A_1\sin(\lambda_1 t),$$
$$\ddot{x}_2 + \dot{x}_2 + 2x_2 + k_3x_1^3 + k_4x_2^3 = A_2\sin(\lambda_2 t + \phi). \tag{19}$$

The corresponding LTL equations over the domain D_i in four-dimensional space may be written as

$$y_1 = \bar{x}_1,\ y_2 = \frac{\mathrm{d}\bar{x}_1}{\mathrm{d}t},\ y_3 = \bar{x}_2 \quad \text{and} \quad y_4 = \frac{\mathrm{d}\bar{x}_2}{\mathrm{d}t},$$

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ -(1 + k_1y_{1,i+1}^2) & -0{\cdot}4 & -k_2y_{1,i+1}y_{3,i+1} & 0 \\ 0 & 0 & 0 & 1 \\ -k_3y_{1,i+1}^2 & 0 & -(2 + k_4y_{3,i+1}^2) & 1 \end{Bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ A_1\sin(\lambda_1 t) \\ 0 \\ A_2\sin(\lambda_2 t + \phi) \end{Bmatrix}. \tag{20}$$

The solution may be obtained as explained in section 2 and is omitted for the sake of brevity.

8. NUMERICAL RESULTS

In this section a few numerical results are presented for the case of a non-linear two-degree-of-freedom system, using the BLTL method. In all the presentations to follow, the time step size, h_i, for integration has been made constant for all i. The difference between

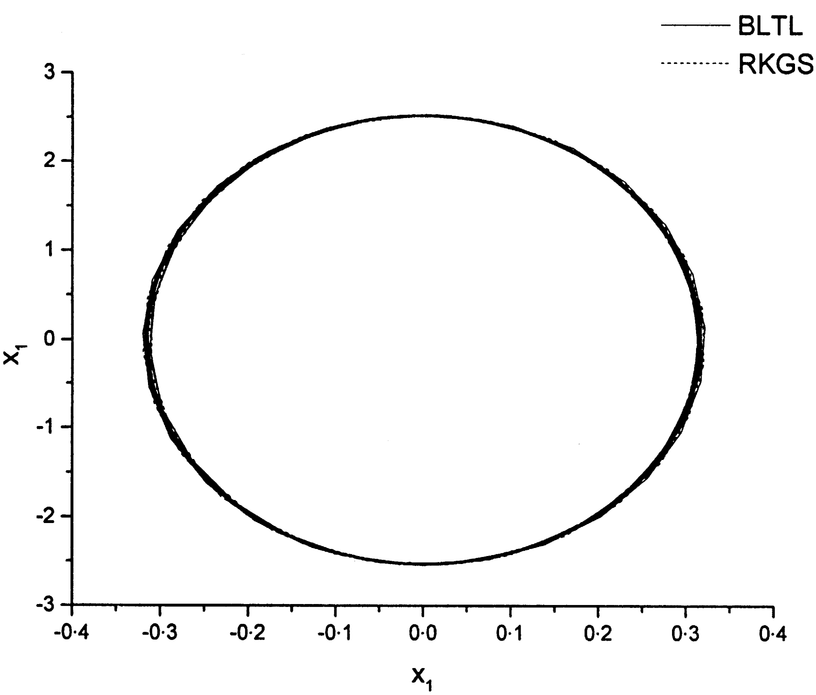

Figure 1. A typical elliptic one-periodic orbit, $k_1 = 0{\cdot}5$, $k_2 = 0{\cdot}6$, $k_3 = 0{\cdot}6$, $k_4 = 0{\cdot}3$, $A_1 = 20$, $A_2 = 8{\cdot}0$, $\lambda_1 = 8$, $\lambda_2 = 6{\cdot}0$, $\phi = 0{\cdot}0$, $h = 0{\cdot}01$.

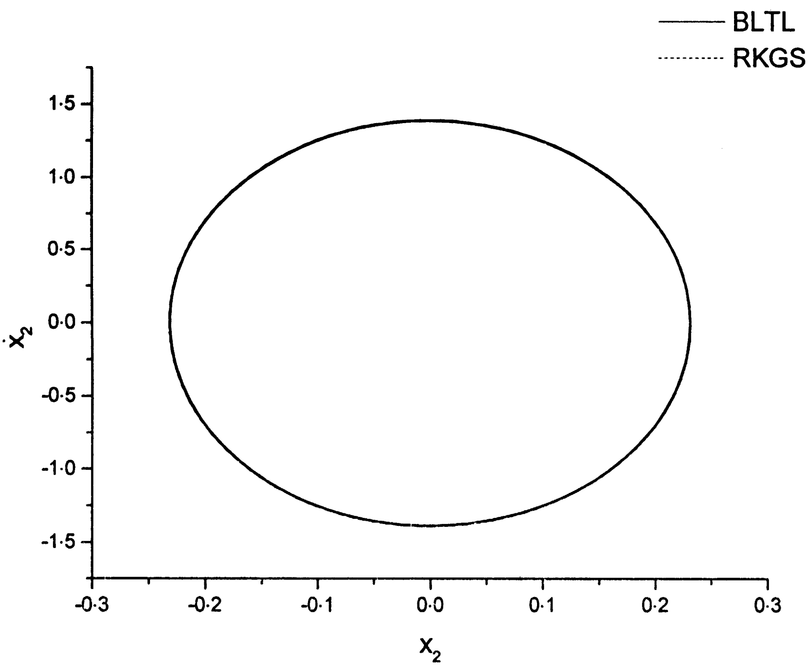

Figure 2. A typical one-periodic orbit, $k_1 = 0{\cdot}5$, $k_2 = 0{\cdot}6$, $k_3 = 0{\cdot}6$, $k_4 = 0{\cdot}3$, $A_1 = 20$, $A_2 = 8{\cdot}0$, $\lambda_1 = 8$, $\lambda_2 = 6{\cdot}0$, $\phi = 0{\cdot}0$, $h = 0{\cdot}01$.

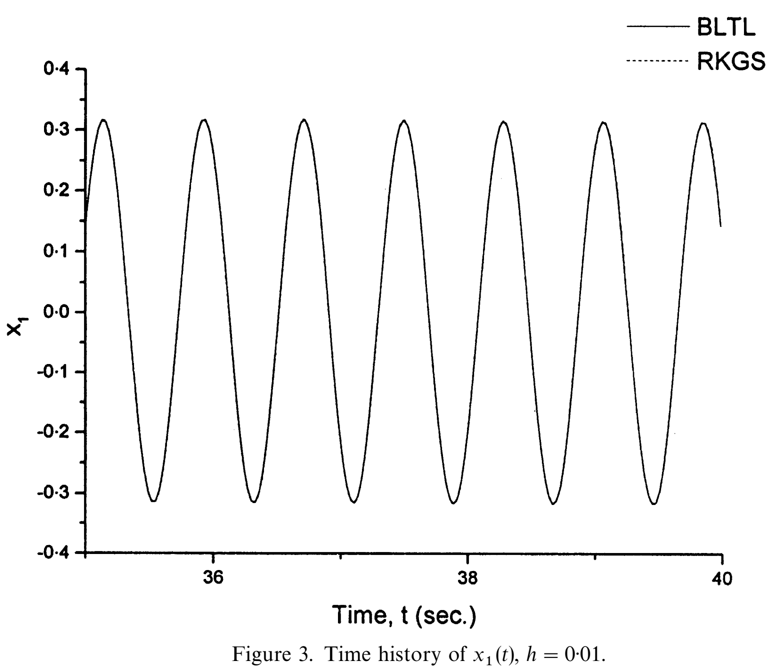

Figure 3. Time history of $x_1(t)$, $h = 0{\cdot}01$.

the results obtained from BLTL and HLTL methods compare very well and can hardly be represented graphically. Hence, all the results reported here are obtained using the BLTL method alone. An elliptic one-periodic orbit has been obtained via the BLTL method and a sixth order Runge–Kutta scheme (RKGS) in Figures 1 and 2. The comparison of results is

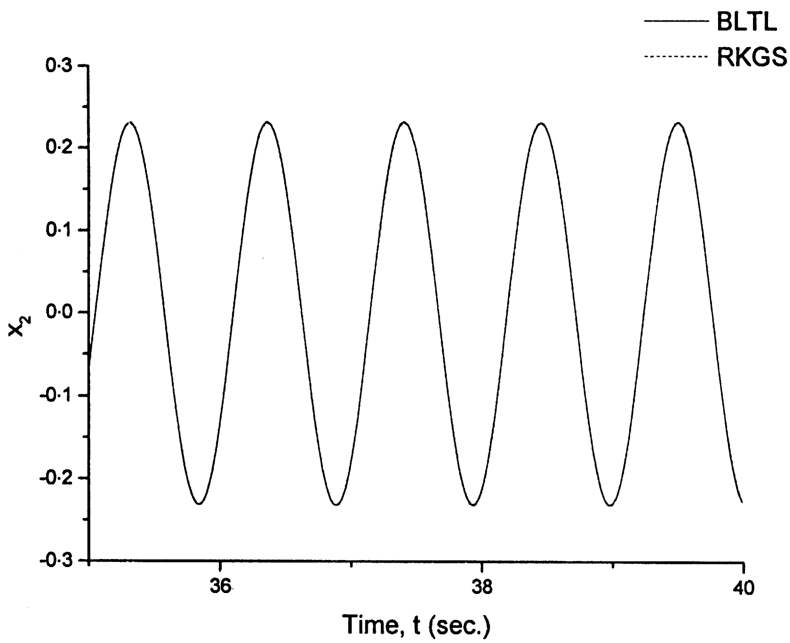

Figure 4. Time history of $x_2(t)$, $h = 0{\cdot}01$.

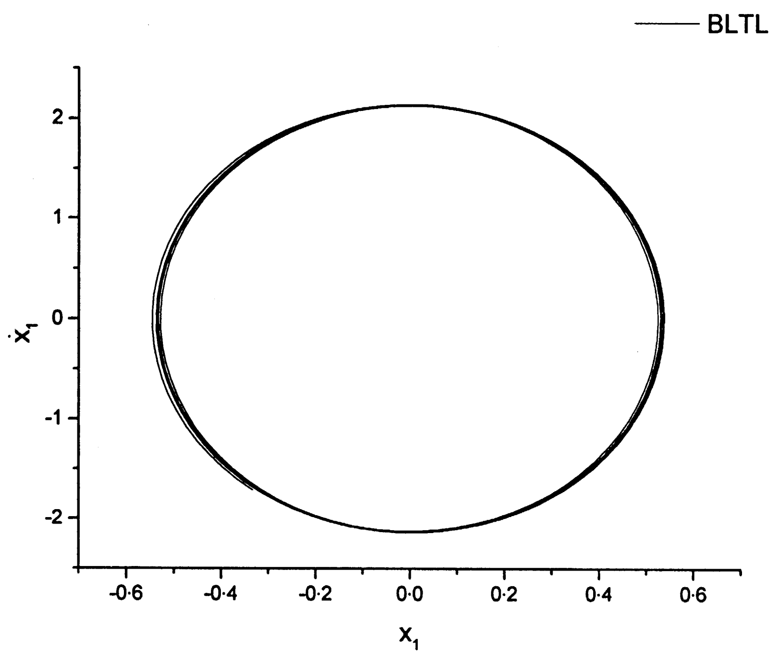

Figure 5. A quasi-periodic orbit of the two-degree-of-freedom system using BLTL method, $k_1 = 0{\cdot}5$, $k_2 = 0{\cdot}5$, $k_3 = 0{\cdot}6$, $k_4 = 0{\cdot}3$, $A_1 = 8$, $A_2 = 4{\cdot}0$, $\lambda_1 = 4$, $\lambda_2 = 4\pi$, $\phi = 0{\cdot}0$, $h = 0{\cdot}01$.

quite good as may be observed from the phase-plots in Figures 1 and 2. In Figures 3 and 4, the time history plots of the displacement components x_1 and x_2 are given. In Figures 5 and 6 the phase plots and the time history plots are reported for different sets of parameters (k_1,

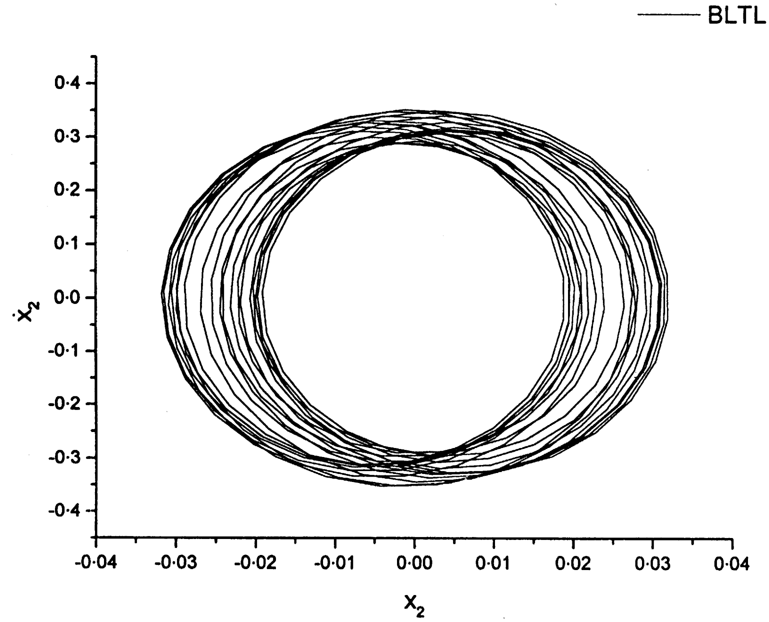

Figure 6. A quasi-periodic orbit of the two-degree-of-freedom system using BLTL method.

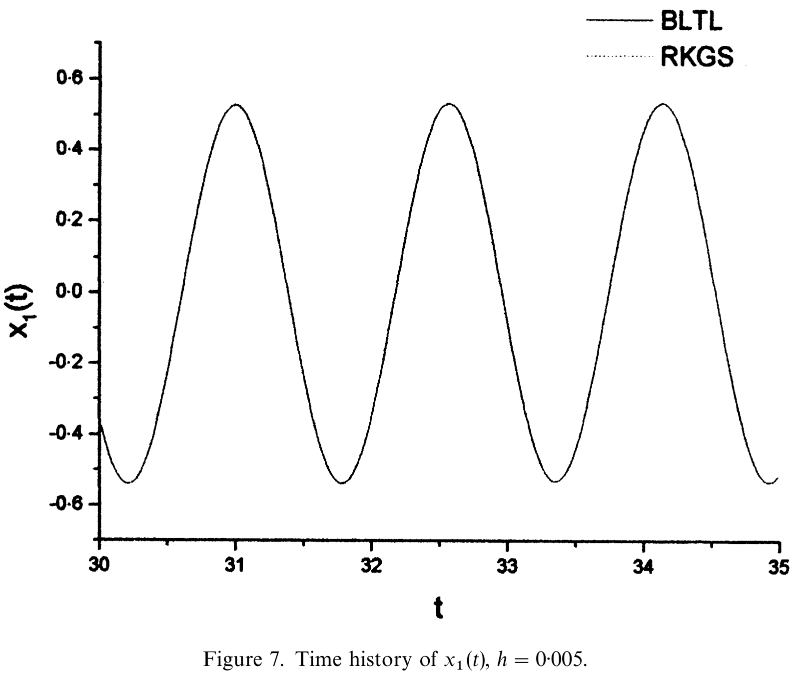

Figure 7. Time history of $x_1(t)$, $h = 0{\cdot}005$.

k_2, k_3 and k_4) and different frequencies. Figures 7 and 8 show the time history plots of $x_1(t)$ and $x_2(t)$ as obtained from RKGS and BLTL, respectively. In the phase-independent response regime, the LTL method picks up the intersection point nearly exactly irrespective of the chosen time step size, subject only to the errors inevitable in any root searching procedure. However, RKGS is sensitive to the chosen time step (vide Figures 9(a) and 9(b)).

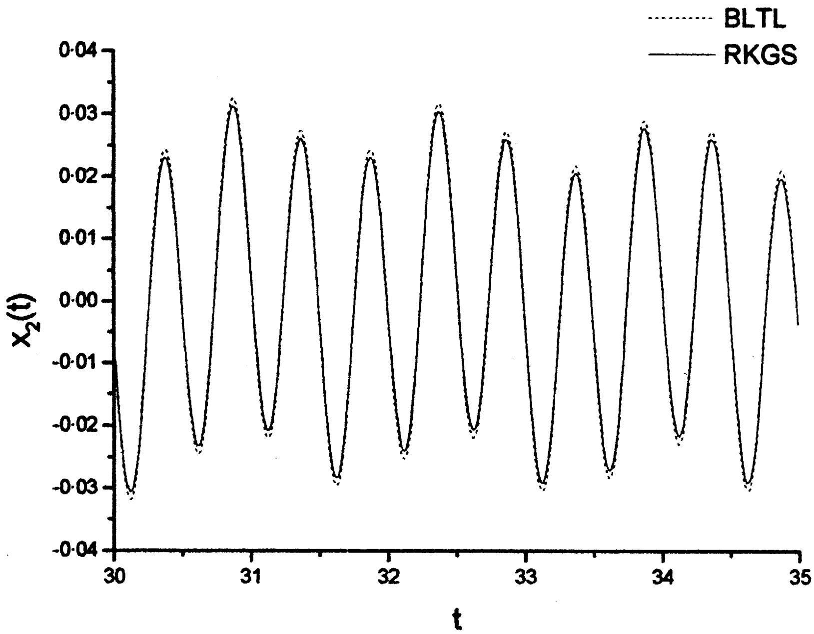

Figure 8. Time history of $x_2(t)$, $h = 0{\cdot}005$.

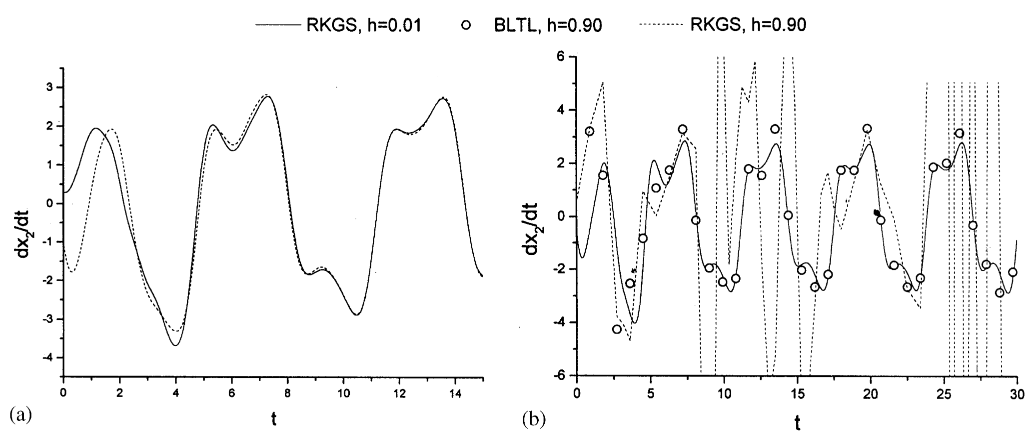

Figure 9 (a) Demonstration of phase-independence $k_1 = 0{\cdot}5$, $k_2 = 0{\cdot}6$, $k_3 = 0{\cdot}6$, $k_4 = 0{\cdot}3$, $A_1 = 4{\cdot}0$, $A_2 = 8{\cdot}0$, $\lambda_1 = 3{\cdot}14$, $\lambda_2 = 1{\cdot}0$. (b). Performance of BLTL and RKGS for large time step sizes. $k_1 = 0{\cdot}5$, $k_2 = 0{\cdot}6$, $k_3 = 0{\cdot}6$, $k_4 = 0{\cdot}3$, $A_1 = 4{\cdot}0$, $A_2 = 8{\cdot}0$, $\lambda_1 = 3{\cdot}14$, $\lambda_2 = 1{\cdot}0$.

9. DISCUSSION AND CONCLUSIONS

In this letter, a simple error analysis for non-linear initial value problems, as solved via the locally transversal linearization (LTL) method developed by the authors earlier [14–17], is proposed for the first time. In this method, corresponding to a given non-linear dynamical system, a set of conditionally linear dynamical systems are derived such that each linear system has its validity over a chosen step size and satisfies the non-linear vector field at the two boundary points of the time interval under consideration. In other words, given a time interval and the known initial state, the conditionally linear system is so constructed as to transversally intersect the non-linear trajectory at the end of the interval. The proposed error analysis in the LTL procedure is based on expanding both the solution vector of the non-linear differential equation and the solution vector of the corresponding transversal LTL equation in implicit Taylor's series and comparing the terms in both the expansions that are similar.

The conditional linearization achieved via the principles of LTL may be construed either as a flow or a map. Either way, the analytical nature of the flow or the diffeomorphism may be exploited to obtain certain useful results. For example, the basic concept of LTL may as well be made applicable to a large class of non-linear boundary value problems, governed by non-linear ODEs as shown in reference [17]. Efforts are presently on to apply the same principles for boundary-value problems governed by non-linear partial differential equations.

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