Journal of Sound and Vibration (2002) **256**(4), 665–680
doi:10.1006/jsvi.5015, available online at http://www.idealibrary.com on IDEAL®

A PARAMETRIC STUDY ON THE AXISYMMETRIC MODES OF VIBRATION OF MULTI-LAYERED SPHERICAL SHELLS WITH LIQUID CORES OF RELEVANCE TO HEAD IMPACT MODELLING

P. G. Young

School of Engineering and Computer Science, University of Exeter, Harrison Building, North Park Road, Exeter, Devon, EX8 5LW, England. E-mail: philippe.g.young@exeter.ac.uk.

(*Received* 3 *August* 2001, *and in final form* 18 *January* 2002)

The free vibration of spheres composed of inviscid compressible liquid cores surrounded by spherical layers of linear elastic, homogeneous and isotropic materials are studied using three-dimensional elasticity equations. The exact three-dimensional equations are first derived for an N-layered sphere with a liquid core and an extensive parametric study is then presented for the first few natural frequencies of the spheroidal modes of vibration. Non-dimensional frequency parameters are compared with values obtained using lower order membrane and shell theories. It is shown that for a remarkably wide range of geometric and material parameters, which encompasses values typical for the human head, the first ovalling mode of a fluid-filled shell behaves like a membrane filled with incompressible fluid and a simple closed-form expression is derived which closely approximates the natural frequencies obtained using the exact three-dimensional equations.

1. INTRODUCTION

The free oscillations of spheres has been of interest for a long time with work on the subject, in many cases, motivated by an interest in modelling oscillations of the Earth. Lamb [1] first obtained the equations governing the free vibration of the solid sphere and Chree [2] subsequently obtained these equations in the more convenient spherical co-ordinates (rather than in Cartesian co-ordinates). More recently, Sato and Usami [3, 4] have studied the vibrations of solid spheres and provided extensive numerical results. The vibration of hollow spheres was studied by Shah *et al.* [5, 6], using two-dimensional and exact three-dimensional theory, and natural frequency parameters for several shells with different inner-to-outer radius ratios were presented in graphical form. Jiang *et al.* [7] studied the free vibration behaviour of multi-layered hollow spheres and provided tabular results for a number of cases. Solid and hollow spheres are also treated in the book by Lapwood and Usami [8] on oscillations of the Earth.

The free vibration of fluid-filled spherical shells has been treated by a number of authors. Engin [9] developed a model of the human head consisting of a spherical shell filled with inviscid fluid using a thin-shell theory. Advani and Lee [10] investigated the vibration of a fluid-filled shell using higher order shell theory including transverse shear and rotational inertia. Results are presented both graphically, in the form of frequency spectra, and in tabular form comparing the higher order theories with the elementary theory for two

0022-460X/02/$35.00

different shell shear wave propagation speeds. More recently, Guarino and Elger [11] have looked at the frequency spectra of a fluid-filled sphere, both with and without a central solid sphere, in order to explore the use of auscultatory percussion as a clinical diagnostic tool. A higher order shell theory was again used to model the outer shell.

In the previous work on the free vibration of fluid-filled shells, frequency spectra or tabulated results were only presented for at most a few select cases. In the present study, numerical results are presented for a very wide range of material parameters (e.g., Young's modulus of shell, bulk modulus of fluid) and geometric parameters (thickness of shell, radius of fluid). However, rather than present results for a large range of modes for each case the emphasis has been placed on the behaviour of the first ovalling ($n = 2$) mode of spheroidal oscillation as material and geometric parameters are varied. The motivation for focusing our attention on this mode is a recent parametric study [12] on the response of a fluid-filled shell to a radially applied force, which has shown that both the onset of dynamic pressure effects in the brain and the magnitude of the observed pressures can be predicted very accurately by the ratio of the impact duration to the period of oscillation of the first ovalling mode. Numerical results obtained using the full three-dimensional elasticity equations are compared with results obtained using simpler membrane and shell theories to explore the range of applicability of these theories.

2. THEORY

2.1. THREE DIMENSIONAL FORMULATION OF FREQUENCY EQUATIONS FOR A FLUID-FILLED MULTI-LAYERED SHELL

The solution to the equations of motion, in spherical co-ordinates, for an isotropic, homogeneous, elastic medium are given by Sâto and Usami [3] and may be written

$$u_1 = -A_{mn}U_1(n, q, r, z)\,Sph_1, \qquad v_1 = -A_{mn}V_1(n, q, r, z)\,Sph_2,$$

$$w_1 = mA_{mn}W_1(n, q, r, z)\,Sph_3, \tag{1}$$

$$u_2 = 0, \qquad v_2 = mB_{mn}V_2(n, k, r, z)\,Sph_4, \qquad w_2 = -B_{mn}W_2(n, k, r, z)\,Sph_5, \tag{2}$$

$$u_3 = -C_{mn}U_3(n, k, r, z)\,Sph_1, \qquad v_3 = -C_{mn}V_3(n, k, r, z)\,Sph_2,$$

$$w_3 = mC_{mn}W_3(n, k, r, z)\,Sph_3, \tag{3}$$

where

$$U_1 = \frac{1}{q^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Z_{n+1/2}(qr)}{r^{1/2}}\right), \qquad V_1 = \frac{1}{q^2}\frac{Z_{n+1/2}(qr)}{r^{3/2}},$$

$$W_1 = V_1, \qquad V_2 = \frac{1}{n(n+1)}\frac{Z_{n+1/2}(kr)}{r^{1/2}},$$

$$W_2 = V_2, \qquad U_3 = \frac{n(n+1)}{k^2}\frac{Z_{n+1/2}(kr)}{r^{3/2}},$$

$$V_3 = \frac{1}{k^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{1/2}Z_{n+1/2}(kr)\right], \qquad W_3 = V_3,$$

$$Sph_1 = \mathrm{P}_n^m(\cos\theta)\left[\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right] m\phi \exp(\mathrm{i}\omega t), \qquad Sph_2 = \frac{\mathrm{d}}{\mathrm{d}\theta}\,\mathrm{P}_n^m(\cos\theta)\left[\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right] m\phi \exp(\mathrm{i}\omega t),$$

$$Sph_3 = \frac{\mathrm{P}_n^m(\cos\theta)}{\sin\theta}\left[\begin{smallmatrix}\sin\\ -\cos\end{smallmatrix}\right] m\phi \exp(\mathrm{i}\omega t), \qquad Sph_4 = \frac{\mathrm{P}_n^m(\cos\theta)}{\sin\theta}\left[\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right] m\phi \exp(\mathrm{i}\omega t),$$

$$Sph_5 = \frac{\mathrm{d}}{\mathrm{d}\theta}\,\mathrm{P}_n^m(\cos\theta)\left[\begin{smallmatrix}\sin\\ -\cos\end{smallmatrix}\right] m\phi \exp(\mathrm{i}\omega t),$$

where u_i, v_i, $w_i (i = 1, 2, 3)$ are the radial, colatitudinal and azimuthal components of displacement, respectively, and $Z_{n+1/2}$ are linear combinations of spherical Bessel functions of the first and second kind respectively. $\mathrm{P}_n^m(\cos\theta)$ are associated Legendre functions, n is the harmonic number, $q = \omega/c_p$, $k = \omega/c_s$, in which ω is the radian natural frequency, $c_p = [(\lambda + 2\mu)/\rho]^{1/2}$ is the speed of a longitudinal wave, $c_s = (\mu/\rho)^{1/2}$ is the speed of a shear wave and ρ is the material density. The Lamé parameters λ and μ are given by $\lambda = vE/[(1-2v)(1+v)]$ and $\mu = E/[2(1+v)]$, where v is the Poisson ratio and E is Young's modulus of elasticity, and the quantities A_{mn}, B_{mn} and C_{mn} are constant coefficients.

The stress components of interest in this work may be written as

$$rr_1 = A_{mn} Rr_1(n, q, r, z, \mu) Sph_1, \qquad r\theta_1 = -A_{mn} R\theta_1(n, q, r, z, \mu) Sph_2,$$

$$r\phi_1 = m A_{mn} R\phi_1(n, q, r, z, \mu) Sph_3, \tag{4}$$

$$rr_2 = 0, \qquad r\theta_2 = m B_{mn} R\theta_2(n, k, r, z, \mu) Sph_4, \qquad r\phi_2 = -B_{mn} R\phi_2(n, k, r, z, \mu) Sph_5, \tag{5}$$

$$rr_3 = -C_{mn} Rr_3(n, k, r, z, \mu) Sph_1, \qquad r\theta_3 = -C_{mn} R\theta_3(n, k, r, z, \mu) Sph_2,$$

$$r\phi_3 = m C_{mn} R\phi_3(n, k, r, z, \mu) Sph_3, \tag{6}$$

where

$$Rr_1 = (\lambda + 2\mu)\frac{Z_{n+1/2}(qr)}{r^{1/2}} + \frac{2\mu}{q^2}\left(\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Z_{n+1/2}(qr)}{r^{1/2}}\right) - n(n+1)\frac{Z_{n+1/2}(qr)}{r^{5/2}}\right),$$

$$R\theta_1 = \frac{2\mu}{q^2}\frac{\mathrm{d}}{\mathrm{d}r}\frac{Z_{n+1/2}(qr)}{r^{3/2}},$$

$$R\phi_1 = R\theta_1, \qquad R\theta_2 = \frac{\mu}{n(n+1)}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Z_{n+1/2}(kr)}{r^{1/2}}\right) - \frac{Z_{n+1/2}(kr)}{r^{3/2}}\right),$$

$$R\phi_2 = R\theta_2, \qquad Rr_3 = \frac{2\mu n(n+1)}{k^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Z_{n+1/2}(kr)}{r^{3/2}}\right),$$

$$R\theta_3 = \frac{\mu}{k^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{Z_{n+1/2}(kr)}{r^{1/2}}\right) + (n(n+1) - 2)\frac{Z_{n+1/2}(kr)}{r^{5/2}}\right), \qquad R\phi_3 = R\theta_3.$$

For the free vibration of a fluid-filled shell, the stress components will vanish on the outer surface of the shell at r_s giving

$$rr^{(s)}(r_s) = 0, \qquad r\theta^{(s)}(r_s) = 0, \qquad r\phi^{(s)}(r_s) = 0.$$

At the interface between two solid layers, $l, l+1$, continuity of displacement and of the three stress components given in equations (1)–(6) must exist, giving

$$rr^{(l)}(r_l) = rr^{(l+1)}(r_l), \qquad u^{(l)}(r_l) = u^{(l+1)}(r_l), \qquad r\theta^{(l)}(r_l) = r\theta^{l+1)}(r_l),$$

$$v^{(l)}(r_l) = v^{(l+1)}(r_l), \qquad r\phi^{(l)}(r_1) = r\phi^{(l+1)}(r_l), \qquad w^{(l)}(r_l) = w^{(l+1)}(r_l),$$

where r_l is the outside radius of the lth layer.

At the interface between the innermost solid layer and the liquid core there is continuity of the radial displacement and stress components, and the azimuthal and colatitudinal components of stress on the shell vanish, giving

$$rr^f(r_f) = rr^i(r_f), \qquad u^f(r_f) = u^i(r_f), \qquad r\theta^i(r_f) = 0, \qquad r\phi^i(r_f) = 0,$$

where r_f is the outside radius of the fluid (inside radius of the shell), and the superscript i refers to the innermost solid layer of shell and the superscript f to the fluid.

Upon substitution of equations (1)–(6) into the above boundary and continuity conditions, two uncoupled sets of equations are obtained, one governing the toroidal or first-class vibrations, which results from the use of equations (2) and (5), and the other governing spheroidal or second-class vibrations, which results from the use of equations (1), (3), (4) and (6). The toroidal modes of a shell filled with an inviscid fluid will be the same as those for the equivalent shell *in vacuo* and therefore only the spheroidal modes will be considered in the present study.

2.2. FREQUENCY EQUATIONS FOR A FLUID-FILLED MEMBRANE

The frequency equation for the $n > 0$ modes of vibration of a membrane filled with a compressible fluid can be obtained from the work by Engin [9] and is given by

$$\beta^4 - \beta^2(1 + 3v + \lambda_n - \gamma_n) - (1 - v^2)(2 - \lambda_n) + (1 - v - \lambda_n)\gamma_n = 0, \tag{7}$$

where $c_f = \sqrt{B/\rho_f}$ is the pressure wave speed in the fluid, B is the bulk modulus of the fluid and ρ_f is the density of the fluid, $c_s^* = \sqrt{E_s/\rho_s(1 - v^2)}$ is the plate velocity in the shell where E is Young's modulus of the shell, v the Poisson ratio and ρ_s the density, and

$$\beta = \frac{\omega R}{c_s^*}, \qquad \lambda_n = n(n+1), \qquad \gamma_n = \frac{R}{h}\frac{\rho_f}{\rho_s}\frac{c_f^2}{c_s^{*2}}\,\alpha\,\frac{j(n,\alpha)}{j'(n,\alpha)}, \qquad \alpha = \frac{\omega R}{c_f}$$

where R is the radius of the mid-surface of the shell.

From equation (7), limiting cases of a membrane *in vacuo* and a membrane filled with incompressible fluid can be obtained:

(1) *Membrane in vacuo*. Letting $\gamma_n = 0$ the frequency equation for a membrane *in vacuo* is

$$\beta^4 - \beta^2(1 + 3v + \lambda_n) - (1 - v^2)(2 - \lambda_n) = 0. \tag{8}$$

(2) *Membrane filled with incompressible fluid.* For a membrane filled with incompressible fluid $\alpha\, j(n,\alpha)/j'(n,\alpha) \approx \alpha^2/n$ and equation (7) can be re-written as

$$\beta^4(1+\tau) - \beta^2(1+3v+\lambda_n-(1-v-\lambda_n)\,\tau) - (1-v^2)(2-\lambda_n) = 0, \tag{9}$$

where $\tau = (\rho_f/\rho_s)\,(R/h)\,1/n$. By inspection it is clear that for $\tau = 0$ equation (9) is identical to the frequency equation for a membrane *in vacuo* given by equation (8).

It is convenient at this stage to introduce the following non-dimensional parameter:

$$\Omega' = \omega \mathrm{R} \sqrt{(4\pi/3)\ \frac{R}{h} \times \frac{\rho_f + 3\,(h/R)\,\rho_s}{E_s}}$$

which for thin shells is approximately equal to $\Omega = \omega\sqrt{mass/hE_s}$ where *mass* is the mass of the fluid-filled shell (shell and fluid combined).

Re-writing equation (9) in terms of Ω' gives

$$\Omega'^4\,\frac{(1-v^2)^2}{(n\tau+3)^2}\,(1+\tau)(3/4\pi)^2 - \Omega'^2\,\frac{1-v^2}{n\tau+3}\,(1+3v+\lambda_n-(1-v-\lambda_n)\,\tau)(3/4\pi)$$

$$-(1-v^2)(2-\lambda_n) = 0. \tag{10}$$

As $\tau = (\rho_f/\rho_s)\ (R/h)\ 1/n \to 0$

$$\Omega'^2 = (4\pi/3)\,\frac{3(1+3v+\lambda_n) +/- 3\sqrt{(1+3v+\lambda_n)^2 + 4(2-\lambda_n)(1-v^2)}}{2(1-v^2)} \tag{11}$$

which, as previously discussed, is identical to the solution for a shell *in vacuo*.

$$\text{As } \tau = \frac{\rho_f}{\rho_s}\,\frac{R}{h}\,\frac{1}{n} \to \infty\,, \qquad \Omega'^2(3/4\pi)\,\frac{(v-1+\lambda_n)}{n} - (2-\lambda_n) = 0$$

from which

$$\Omega' = \sqrt{(4\pi/3)\ \frac{n(\lambda_n-2)}{\lambda_n+v-1}}\,. \tag{12}$$

Interestingly, varying τ between 0 and infinity for the $n = 2$ mode results in only a relatively small change in the non-dimensional natural frequency Ω'as shown in Figure 1. For the $n = 2$ mode equation (12) becomes

$$\Omega' = \sqrt{(32\Pi/(3(5+v))} \tag{13}$$

which is a good approximation for the whole range of fluid-to-shell density ratios ρ_f/ρ_s as well as for a membrane *in vacuo*. The rationale for the unusual form used for the non-dimensional parameter (Ω') is now apparent as it highlights the surprising insensitivity

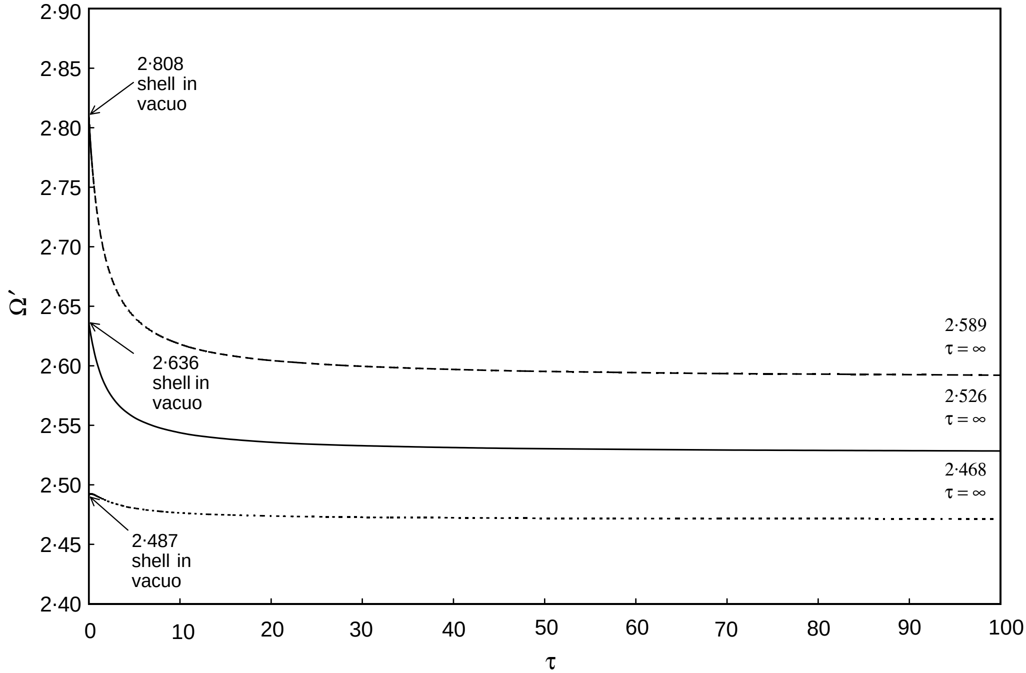

Figure 1. Frequency parameter Ω' against $\tau = (\rho_f/\rho_s)\ (R/h)\ 1/n$ using equation (10) for a membrane filled with compressible fluid. As τ tends to zero the results are identical to those for a membrane *in vacuo* (given by equation (11)). – – – $v = 0{\cdot}0$; —— $v = 0{\cdot}25$; - - - - - $v = 0{\cdot}49$.

of the natural frequencies for the first ovalling mode to the distribution of density between the shell and the fluid. In addition, in practice, it is far simpler to establish the mass of the whole system rather than the relative densities of the skull and brain.

3. NUMERICAL RESULTS AND DISCUSSION

3.1. SINGLE-LAYER SPHERICAL SHELL FILLED WITH FLUID

Numerical results were computed for fluid-filled spheres consisting of a single homogeneous isotropic outer shell with an inviscid perfect liquid core. Results are presented in terms of the non-dimensional frequency parameter $\Omega = \omega\sqrt{(mass/hE)}$ and are, *a priori*, a function of the following four non-dimensional parameters: the ratio of Young's modulus of the shell to the bulk modulus of the fluid (E/B); the ratio of the density of the fluid to the density of the shell (ρ_f/ρ_s), the Poisson ratio of the shell material v; and the ratio of the thickness of the shell to outer radius of the fluid (h/r_f). Because of the primary interest in head injury modelling, the non-dimensional geometric and material ratios are also given in the tables in terms of the baseline values used by Engin [9], that is $(\rho_f/\rho_s)_o = 1000/2140$, $(E/B)_o = 13{\cdot}79/2{\cdot}18$, and $(h/r_f)_o = (0{\cdot}00381/0{\cdot}0762) = 1/20$. Non-dimensional frequency parameters Ω for the first $n = 2$ spheroidal mode of vibration are given in Tables 1–3 for the Poisson ratio equal to 0·25 as obtained using the full three-dimensional elasticity solution. Results were computed for a very wide range of compressibility ratios E/B and thickness ratios h/r_f. The percentage error in the non-dimensional frequency parameter obtained

TABLE 1

Frequency parameter Ω for the first n = 2 spheroidal mode for density ratio $(\rho_f/\rho_s) = 1000/2140 = (\rho_f/\rho_s)_o \times 1$ *(approximately equal to the typical density ratio of brain to skull bone) and for* $v = 0{\cdot}25$

$(h/r_f) = (h/r_f)_0\times$	$h/r_f =$	$(E/B) = (E/B)_0\times$	1/100	1/4	1/2	1	2	4	8	16	32	64
		$E/B =$	0·063	1·58	3·16	6·33	12·7	25·3	50·6	101	202	405
1/100	0·0005	3-D	2·527	2·527	2·527	2·526	2·526	2·525	2·523	2·520	2·513	2·499
		Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Bending	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Membrane incomp.	0%	0%	0%	0%	0%	0%	0%	0%	1%	1%
1/2	0·025	3-D	2·541	2·537	2·532	2·524	2·508	2·476	2·414	2·297	2·094	1·791
		Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Bending	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Membrane incomp.	0%	0%	0%	1%	1%	3%	5%	11%	22%	42%
1	0·05	3-D	2·556	2·550	2·544	2·531	2·505	2·453	2·355	2·177	1·890	1·524
		Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Bending	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Membrane incomp.	0%	0%	1%	1%	2%	4%	9%	18%	35%	68%
		Acoustic	—	—	—	—	—	—	—	– 60%	– 31%	– 15%
2	0·1	3-D	2·588	2·579	2·570	2·554	2·520	2·451	2·314	2·066	1·699	1·303
		Membrane	0%	0%	0%	– 1%	– 1%	– 1%	– 1%	– 1%	– 1%	– 1%
		Bending	0%	0%	0%	0%	0%	0%	0%	0%	– 1%	– 1%
		Membrane incomp.	0%	0%	0%	1%	2%	5%	11%	25%	52%	98%
		Acoustic	—	—	—	—	—	—	– 71%	– 35%	– 16%	– 7%
4	0·2	3-D	2·662	2·654	2·645	2·628	2·590	2·513	2·345	2·020	1·582	1·169
		Membrane	– 3%	– 3%	– 3%	– 3%	– 3%	– 3%	– 4%	– 4%	– 4%	– 4%
		Bending	1%	1%	1%	0%	0%	– 1%	– 2%	– 3%	– 3%	– 3%
		Membrane incomp.	– 3%	– 2%	– 2%	– 1%	0%	3%	11%	28%	64%	122%
		Acoustic	—	—	—	—	—	– 92%	– 46%	– 19%	– 8%	– 3%
8	0·4	3-D	2·836	2·830	2·823	2·810	2·781	2·711	2·526	2·107	1·528	1·147
		Membrane	– 8%	– 8%	– 8%	– 8%	– 9%	– 10%	– 11%	– 12%	– 8%	– 11%
		Bending	2%	2%	1%	1%	0%	– 2%	– 6%	– 9%	– 7%	– 10%
		Membrane incomp.	– 8%	– 8%	– 7%	– 7%	– 6%	– 3%	4%	24%	71%	128%
		Acoustic	—	—	—	—	—	– 71%	– 30%	– 10%	– 7%	– 1%
16	0·8[†]	3-D	3·150	3·147	3·143	3·137	3·122	3·080	2·930	2·415	1·772	1·267
		Membrane	– 16%	– 16%	– 16%	– 17%	– 17%	– 19%	– 23%	– 25%	– 24%	– 24%
		Bending	6%	6%	6%	5%	3%	– 2%	– 14%	– 22%	– 23%	– 23%
		Membrane incomp.	– 16%	– 16%	– 16%	– 16%	– 15%	– 14%	– 10%	10%	49%	109%
		Acoustic	—	—	—	—	—	– 65%	– 23%	– 5%	– 1%	0%
32	1·6	3-D	3·506	3·505	3·503	3·500	3·498	3·489	3·451	3·095	2·266	1·614
		FE	3·508	3·508	3·506	3·504	3·500	3·490	3·447	2·982	2·161	1·537
		Acoustic	—	—	—	—	—	– 85%	– 32%	– 4%	– 1%	0%

† Approximate ratio of the radius of the Earth's core to the thickness of the mantle.

TABLE 2

Frequency parameter Ω for the first $n = 2$ spheroidal mode for density ratio $(\rho_f/\rho_s) = 1000/2140 \times 10\,000 = (\rho_f/\rho_s)_o \times 10\,000$ (tending towards a massless shell—i.e., density shell $\rightarrow 0$) and for $v = 0{\cdot}25$

$(h/r_f) = (h/r_f)_0 \times$	$h/r_f =$	$(E/B) = (E/B)_0 \times$ / $E/B =$	1/100 / 0·063	1/4 / 1·58	1/2 / 3·16	1 / 6·33	2 / 12·7	4 / 25·3	8 / 50·6	16 / 101	32 / 202	64 / 405
1/100	0·0005	3-D	2·526	2·526	2·526	2·526	2·525	2·525	2·523	2·519	2·512	2·499
		Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		Membrane incomp.	0%	0%	0%	0%	0%	0%	0%	0%	1%	1%
1/2	0·025	3-D	2·521	2·517	2·511	2·500	2·480	2·439	2·360	2·221	1·995	1·684
		Membrane incomp.	0%	0%	1%	1%	2%	4%	7%	14%	27%	50%
		Acoustic	—	—	—	—	—	—	—	− 92%	− 51%	− 27%
1	0·05	3-D	2·520	2·509	2·499	2·478	2·437	2·360	2·219	1·994	1·682	1·332
		Membrane	0%	0%	0%	0%	0%	0%	1%	1%	1%	1%
		Membrane incomp.	0%	1%	1%	2%	4%	7%	14%	27%	50%	90%
		Acoustic	—	—	—	—	—	—	− 92%	− 51%	− 27%	− 13%
2	0·1	3-D	2·523	2·502	2·481	2·440	2·363	2·223	1·995	1·684	1·332	1·004
		Membrane	0%	0%	0%	0%	0%	1%	1%	2%	2%	2%
		Membrane incomp.	0%	1%	2%	4%	7%	14%	27%	50%	90%	152%
		Acoustic	—	—	—	—	—	− 92%	− 51%	− 27%	− 13%	− 6%
4	0·2	3-D	2·549	2·507	2·465	2·385	2·241	2·010	1·692	1·335	1·004	0·734
		Membrane	− 1%	− 1%	− 1%	0%	0%	1%	2%	3%	4%	5%
		Membrane incomp.	− 1%	1%	2%	6%	13%	26%	49%	89%	152%	244%
		Acoustic	—	—	—	—	− 90%	− 50%	− 26%	− 13%	− 6%	− 3%
8	0·4	3-D	2·623	2·533	2·446	2·292	2·045	1·712	1·345	1·009	0·736	0·528
		Membrane	− 4%	− 3%	− 2%	− 1%	1%	4%	6%	8%	9%	9%
		Membrane incomp.	− 4%	0%	3%	10%	24%	48%	88%	150%	243%	379%
		Acoustic	—	—	—	− 86%	− 48%	− 25%	− 12%	− 6%	− 3%	− 1%
16	0·8	3-D	2·666	2·485	2·323	2·067	1·724	1·351	1·011	0·737	0·529	0·376
		Membrane	− 4%	− 3%	− 1%	3%	7%	11%	15%	16%	17%	18%
		Membrane incomp.	− 4%	2%	9%	22%	47%	87%	150%	243%	377%	572%
		Acoustic	—	—	− 84%	− 46%	− 24%	− 12%	− 6%	− 2%	− 1%	0%
32	1·6	3-D	2·425	2·165	1·956	1·661	1·322	1·001	0·733	0·528	0·376	0·267
		Acoustic	—	− 98%	− 55%	− 29%	− 14%	− 7%	− 3%	− 1%	0%	0%

TABLE 3

Frequency parameter Ω for the first $n = 2$ spheroidal mode for density ratio $(\rho_f/\rho_s) = 1000/2140 \times 10\,000 = (\rho_f/\rho_s)_o/10\,000$ (*tending towards massless fluid—i.e., density fluid* $\to 0$) *and for* $v = 0{\cdot}25$

$(h/r_f) = (h/r_f)_0 \times$	$h/r_f =$	$(E/B) = (E/B)_0 \times$ / $E/B =$	1/100 / 0·063	…	32768 / 207277
1/100	0·0005	3-D	2·634	…	2·607
		3-D *in vacuo*	2·634	…	2·634
		Membrane	2·634	…	2·607
		Membrane incomp.	2·634	…	2·634
		B. incomp	2·634	…	2·634
1/2	0·025	3-D	2·638	…	2·637
		3-D *in vacuo*	2·636	…	2·636
		Membrane	2·636	…	2·636
1	0·05	3-D	2·642	…	2·642
		3-D *in vacuo*	2·642	…	2·642
		Membrane	2·637	…	2·636
		Membrane incomp.	2·637	…	2·637
2	0·1	3-D	2·659	…	2·658
		3-D *in vacuo*	2·659	…	2·659
		Membrane	2·638	…	2·637
		Membrane incomp.	2·638	…	2·638
4	0·2	3-D	2·714	…	2·714
		3-D *in vacuo*	2·714	…	2·714
		Membrane	2·640	…	2·640
		Membrane incomp.	2·640	…	2·640
8	0·4	3-D	2·869	…	2·869
		3-D *in vacuo*	2·869	…	2·869
		Membrane	2·649	…	2·649
		Membrane incomp.	2·649	…	2·649
16	0·8	3-D	3·167	…	3·167
		3-D *in vacuo*	3·167	…	3·167
		Membrane	2·672	…	2·672
		Membrane incomp.	2·672	…	2·672
32	1·6	3-D	3·507	…	3·507
		3-D *in vacuo*	3·508	…	3·508

using membrane theory for a membrane filled with a compressible fluid and with an incompressible fluid, and for a compressible fluid in a rigid spherical cavity (acoustic modes, as obtained using the frequency equation derived from the work of Guarino and Elger [11]) are also presented (where the percentage discrepancy is very large, the errors for these approximate theories have been omitted in the tables and are shown as a dash). The non-dimensional frequency parameters Ω' obtained using equation (10) have been converted to Ω by correcting the approximation for mass (as h/r_f becomes larger, the error in the approximation for the mass increases). In Table 1, the results are given for a density ratio $\rho_f/\rho_s = 1000/2140$ ($(\rho_f/\rho_s)_o \times 1$) which is approximately that of brain and skull bone, in Table 2 $\rho_f/\rho_s = 1000/2140 \times 10\,000$ ($(\rho_f/\rho_s)_o \times 10\,000$) which approximates a massless shell filled with fluid and in Table 3 $\rho_f/\rho_s = (1000/2140)/10\,000$ ($(\rho_f/\rho_s)_o/10\,000$) which approximates a shell filled with massless fluid.

It can be seen from the tables that for $h/r_f \leqslant 0{\cdot}2$, the results obtained for a membrane filled with compressible fluid are within 5% of the three-dimensional results for the whole range of E/B and ρ_f/ρ_s values studied here. Results obtained using combined membrane and bending theory (using the Love–Kirchhoff approximation therefore neglecting rotatory inertia and shear deformation effects) are also included in Table 1 ($\rho_f/\rho_s = 1000/2140$) for comparison and are accurate over a larger range of thickness ratios than results obtained using the membrane only theory, as might be expected, but the range is not dramatically increased. Results obtained using the finite element method with eight-noded brick elements (as well as degenerate tetrahedral and wedge elements) and using full Gaussian integration are also given for the largest thickness ratio $h/r_f = 1{\cdot}6$ and can be seen to agree very well with results obtained using the exact solution. The non-dimensional natural frequency parameters as obtained using the frequency equation for a membrane filled with incompressible fluid (equation (10)) are remarkably good over a wide range of values of E/B. As was previously shown, this mode can be approximated very accurately by the closed-form expression (13) (which is exact for a membrane filled with incompressible fluid and with ρ_f/ρ_s tending to infinity).

In Table 3, in addition to three-dimensional results and comparison results obtained using membrane theories (compressible and incompressible theory), results for a shell *in vacuo* using three-dimensional elasticity theory are also given. As was shown for a membrane filled with incompressible fluid, as ρ_f/ρ_s tends to zero (massless fluid) the frequencies for a shell filled with incompressible fluid using three-dimensional elasticity equations are identical to those of a shell *in vacuo*.

In Table 4, the non-dimensional frequency parameters Ω for the first $n = 3$, 4, 5 and 6 spheroidal modes of vibration are given for a density ratio of $\rho_f/\rho_s = 1000/2140$ (approximately brain and skull) and for the Poisson ratio equal to 0·25 as obtained using the full three-dimensional elasticity solution. Again results are presented for a very wide range of compressibility ratios E/B and thickness ratios h/r_f. The percentage error obtained using simpler membrane theory for a membrane filled with a compressible fluid and incompressible fluid (expressions (7) and (10), respectively), and for a compressible fluid in a rigid spherical cavity (acoustic modes) are also presented for selected cases. It can be seen that for increasing mode number the range of applicability of the membrane theory becomes more restricted, as expected. None-the-less membrane results for the first $n = 3$ mode are within 3% of results obtained using the three-dimensional theory for h/r_f ratios up to 0·1. In Figure 2, the mode shapes for the first $n = 2$–6 modes are shown.

3.2. THREE-LAYER SPHERICAL SHELL FILLED WITH FLUID

The sandwich structure of the skull was approximated using a three-layer shell of total thickness h with the inner and outer layers h_t (table) of half the thickness of the middle layer h_d (diploe), that is $h_t = h/4$ and $h_d = h/2$. The ratio of Young's modulus of the middle layer E_d to Young's modulus of the inner and outer layers E_t was varied in such a way as to keep the effective membrane Young's modulus $E_{membrane}$ to bulk modulus ratio of the sandwich shell constant $E_{membrane}/B = (E_t + E_d)/(2B) = 13{\cdot}79/2{\cdot}18$. Three total thickness to radius of fluid ratios were studied $h/r_f = 0{\cdot}1$, 0·2 and 0·4. Up to four ratios of Young's modulus of the middle layer E_d to Young's modulus of the inner and outer sandwich layers E_t were

TABLE 4

Frequency parameter Ω for the first n = 3, 4, 5 and 6 spheroidal modes shown in Figure 2 for density ratio $(\rho_f/\rho_s) = 1000/2140 = (\rho_f/\rho_s) \times 1$ *and for* $v = 0{\cdot}25$

$(h/r_f) = (h/r_f)_0 \times$	$h/r_f =$	Mode number	$(E/B) = (E/B)_0 \times$ / $E/B =$ / shell theory	1/100	1/4	1/2	1	2	4	8	16	32	64
			$E/B =$	0·063	1·58	3·16	6·33	12·7	25·3	50·6	101	202	405
1/100	0·0005	3	3-D	3·341	3·341	3·341	3·341	3·340	3·339	3·337	3·333	3·325	3·308
			Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		4	3-D	3·955	3·955	3·955	3·955	3·954	3·953	3·951	3·947	3·938	3·922
			Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		5	3-D	4·472	4·472	4·472	4·472	4·471	4·470	4·468	4·464	4·456	4·439
			Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		6	3-D	4·928	4·928	4·928	4·928	4·927	4·926	4·924	4·920	4·912	4·896
			Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
1/2	0·025	3	3-D	3·310	3·306	3·301	3·292	3·274	3·238	3·167	3·032	2·789	2·408
			Membrane	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
		4	3-D	3·866	3·862	3·858	3·849	3·832	3·798	3·730	3·598	3·352	2·941
			Membrane	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	0%	0%
		5	3-D	4·340	4·336	4·332	4·324	4·308	4·276	4·213	4·089	3·850	3·430
			Membrane	− 2%	− 2%	− 2%	− 2%	− 2%	− 2%	− 2%	− 2%	− 1%	− 1%
		6	3-D	4·790	4·786	4·783	4·775	4·760	4·731	4·672	4·554	4·322	3·897
			Membrane	− 4%	− 4%	− 4%	− 4%	− 4%	− 4%	− 4%	− 3%	− 3%	− 2%
1	0·05	3	3-D	3·307	3·300	3·293	3·280	3·253	3·198	3·091	2·885	2·535	2·057
			Membrane	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	− 1%	0%	0%
			Acoustic	—	—	—	—	—	—	—	64%	32%	15%
		4	3-D	3·872	3·866	3·860	3·848	3·822	3·771	3·669	3·468	3·098	2·551
			Membrane	− 3%	− 3%	− 3%	− 3%	− 3%	− 3%	− 3%	− 2%	− 2%	− 1%
		5	3-D	4·423	4·417	4·412	4·400	4·376	4·327	4·229	4·030	3·647	3·036
			Membrane	− 7%	− 7%	− 7%	− 7%	− 7%	− 7%	− 6%	− 6%	− 4%	− 3%
		6	3-D	5·052	5·046	5·040	5·028	5·004	4·955	4·855	4·647	4·232	3·540
			Membrane	− 13%	− 13%	− 13%	− 13%	− 13%	− 12%	− 12%	− 11%	− 9%	− 5%
2	0·1	3	3-D	3·364	3·355	3·347	3·329	3·294	3·221	3·070	2·775	2·304	1·768
			Membrane	− 3%	− 3%	− 3%	− 3%	− 3%	− 3%	− 3%	− 3%	− 2%	− 1%
			Acoustic	—	—	—	—	—	—	74%	36%	16%	7%
		4	3-D	4·070	4·062	4·053	4·036	4·001	3·928	3·774	3·452	2·891	2·220
			Membrane	− 10%	− 10%	− 10%	− 10%	− 10%	− 10%	127%	86%	− 5%	− 3%

TABLE 4

Continued

			$(E/B) = (E/B)_0 \times$	1/100	1/4	1/2	1	2	4	8	16	32	64
			$E/B =$	0·063	1·58	3·16	6·33	12·7	25·3	50·6	101	202	405
$(h/r_f) = (h/r_f)_0 \times$	$h/r_f =$	Mode number	shell theory										
		5	3-D	4·937	4·928	4·919	4·900	4·862	4·781	4·605	4·220	3·519	2·683
			Membrane	− 20%	− 20%	− 20%	− 20%	− 20%	− 19%	− 18%	− 15%	− 10%	− 5%
		6	3-D	6·061	6·051	6·040	6·017	5·970	5·869	5·643	5·129	4·201	3·157
			Membrane	− 31%	− 31%	− 31%	− 31%	− 31%	− 30%	− 29%	− 25%	− 16%	− 8%
4	0·2	3	3-D	3·618	3·609	3·599	3·579	3·538	3·446	3·234	2·788	2·168	1·593
			Membrane	− 11%	− 11%	− 11%	− 11%	− 11%	− 11%	− 10%	− 8%	− 6%	− 5%
			Acoustic	—	—	—	—	—	90%	43%	17%	7%	3%
		4	3-D	4·751	4·740	4·729	4·705	4·654	4·538	4·251	3·614	2·760	2·009
		5	3-D	6·242	6·228	6·212	6·180	6·110	5·941	5·488	4·507	3·359	2·421
		6	3-D	8·066	8·047	8·026	7·981	7·879	7·616	6·858	5·400	3·949	2·827
8	0·4	3	3-D	4·231	4·223	4·214	4·194	4·150	4·037	3·697	2·961	2·177	1·561
			Acoustic	—	—	—	—	—	56%	20%	6%	2%	1%
		4	3-D	5·996	5·970	5·957	5·928	5·859	5·660	4·989	3·806	2·752	1·962
		5	3-D	8·052	8·035	8·017	7·974	7·866	7·501	6·260	4·620	3·310	2·353
		6	3-D	10·320	10·298	10·273	10·214	10·052	9·397	7·448	5·407	3·857	2·738
16	0·8	3	3-D	5·087	5·082	5·076	5·064	5·033	4·935	4·450	3·366	2·418	1·720
			Acoustic	—	—	—	—	93%	39%	9%	2%	1%	0%
		4	3-D	7·259	7·253	7·245	7·226	7·177	6·973	5·836	4·255	3·036	2·155
		5	3-D	9·531	9·522	9·513	9·490	9·423	9·032	7·099	5·113	3·639	2·581
		6	3-D	11·793	11·785	11·775	11·751	11·673	10·996	8·306	5·953	4·232	3·000
32	1·6	3	3-D	5·734	5·733	5·731	5·716	5·723	5·705	5·563	4·299	3·078	2·185
			Acoustic	—	—	—	—	—	53%	11%	1%	0%	0%
		4	3-D	7·814	7·813	7·812	7·811	7·806	7·791	7·421	5·416	3·856	2·735
		5	3-D	9·673	9·672	9·672	9·671	9·670	9·662	9·032	6·493	4·617	3·273
		6	3-D	11·368	11·367	11·367	11·367	11·366	11·364	10·551	7·554	5·368	3·804

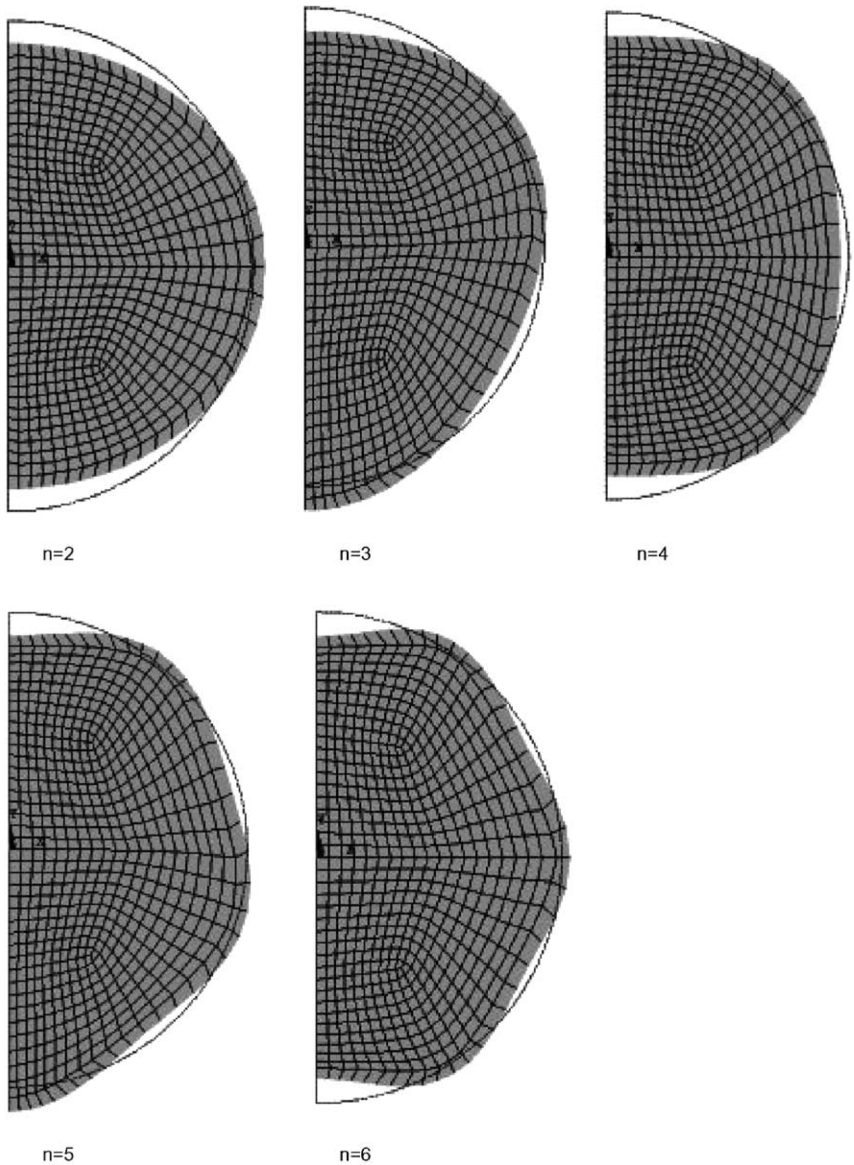

Figure 2. Mode shapes for the $n = 2$ to 6 spheroidal modes of vibration for a fluid-filled sphere. Deformed outline mode shape shown overlaid on un-deformed axi-symmetric finite element mesh.

considered $Ed/Et = 1$ (uniform homogeneous outer shell), $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. These Young's modulus ratios correspond, respectively, to the four flexural ratios 1, 1·25, 1·45 and 1·58; where the flexural ratio is defined as the ratio of the flexural rigidity of the three layer shell over the flexural rigidity of a uniform shell with equivalent membrane stiffness (it can easily be shown that the flexural ratio is approximately equal to $(7 + E_d/E_t)/(4(1 + E_d/E_t))$ neglecting the shift in the neutral axis due to curvature). In Table 5, the non-dimensional frequency parameters $\Omega = \sqrt{(mass/hE)}$ for the first $n = 2$ spheroidal mode of vibration as obtained using the three-dimensional elasticity equations are given for a density ratio $\rho_f/\rho_s = 1000/2140$ (same assumed density across the three shell layers). It can be seen that

TABLE 5

Frequency parameter Ω for the first $n = 2$ spheroidal mode for density ratio $(\rho_f/\rho_s) = 1000/2140 = (\rho_f/\rho_s)_o \times 1$ for a three layer shell and for $v = 0{\cdot}25$

		$(E_{membrane}/B) =$			1
		$(E_{membrane}/B)_0 \times$			6·33
$(h/r_f) = (h/r_f)_0 \times$	$h/r_f =$	$E_{membrane}/B =$	E_d/E_t	Flexural ratio	
2	0·1	One-layer	1	1	2·554
		Three-layer	1/2	1·25	2·559
		Three-layer	1/4	1·45	2·562
4	0·2	One-layer	1	1	2·628
		Three-layer	1/2	1·25	2·642
		Three-layer	1/4	1·45	2·648
		Three-layer	1/8	1·58	2·666
8	0·4	One-layer	1	1	2·810
		Three-layer	1/2	1·25	2·837
		Three-layer	1/4	1·45	3·103

Ω is relatively insensitive to increases in the flexural rigidity which is to be expected since, as was shown in Tables 1–3, membrane action predominates.

4. CONCLUSIONS

Results of a parametric study on the spheroidal modes of a fluid-filled shell have been presented and some interesting conclusions can be drawn from this study. In particular, the sensitivity of the natural frequency of the first $n = 2$ spheroidal mode of oscillation to changes in material and geometric parameters was explored using full three-dimensional elasticity theory and lower order membrane and shell theories and it was shown that:

- For a membrane filled with an incompressible fluid, the non-dimensional frequency parameter Ω for the $n = 2$ mode was shown to be remarkably insensitive to the ratio of membrane density to fluid density; in other words, regardless of whether the mass of a fluid-filled membrane is concentrated in the shell or distributed throughout the fluid it will have a very similar natural frequency for this mode. Furthermore, the frequency equation for the axi-symmetric modes of a membrane filled with an incompressible massless fluid was shown to be identical to that for a shell *in vacuo* (equi-voluminal mode). The fact that this mode is equi-voluminal is highly relevant when considering the influence of a hole in the shell (to model for example the foramen magnum). It explains why the onset of dynamic effects is not observed to be influenced ostensibly by the presence of a hole in numerical simulations carried out by the author. It also explains why, for a remarkably wide range of values of the compressibility ratio E/B, results obtained assuming an incompressible fluid core are very accurate (until the first $n = 2$ acoustic mode frequency becomes of the same order as the predominantly structural mode of interest in this study).
- Fluid-filled spherical shells can be modelled as membranes filled with compressible fluid for shell thickness ratios up to at least 0·2 with at most a 5% error in the predicted

natural frequencies. For an average head radius of approximately 8 cm, this is equivalent to a skull thickness of up to 1·6 cm. An interesting implication of the predominantly membrane behaviour observed is that the three-layer sandwich structure of the skull will not have a stiffening effect on this mode compared with a single homogeneous layer with equivalent membrane stiffness.

- For a wide range of compressibility ratios E/B and thickness ratios h/r_f, which encompass values typical for the human head, fluid-filled shells can be accurately modelled as membranes filled with incompressible fluid. Indeed, over the whole range of possible density ratios, for $h/r_f < 0{\cdot}4$ and for $E/B\ x\ h/r_f < 1$, the maximum error in using membrane filled with incompressible fluid expressions (equation (10)) less than 10% for the first spheroidal mode. Furthermore, the non-dimensional $n = 2$ frequency parameter of a membrane filled with incompressible fluid can be very closely approximated by a very simple closed-form expression (equation (13)). From this equation, the period of oscillation T_Ω of the $n = 2$ mode is given by

$$T_\Omega = \sqrt{((3\Pi(5 + v)mass)/(8hE))}, \tag{14}$$

where E is the effective membrane Young's modulus for a multi-layered shell. The period of oscillation as given by equation (14) is predominantly a function of the total mass of the fluid-filled shell and the membrane stiffness (hE). It is not a function of the relative density ratio of the shell and fluid (ρ_f/ρ_s), nor is it a function of the bulk modulus of the fluid and only a weak function of the Poisson ratio of the shell. Obtaining a closed-form expression for predicting the period of oscillation of the $n = 2$ mode of a fluid-filled shell over a very wide range of parameter values is very useful as it has previously been shown that both the onset of dynamic pressure effects in the fluid and the magnitude of the observed pressures in the fluid for a shell subjected to an applied force–time history can be predicted very accurately by the ratio of the impact duration to the period of oscillation of this mode [12].

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