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WAVE PROPAGATION THROUGH CYLINDERS AND SPHERES AS COMPUTED BY ORDINARY EXPONENTIALS

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1. INTRODUCTION

Propagation of acoustic waves through layered cylinders and spheres can be computed from basic wave solutions in terms of Bessel and Hankel functions, see references [1–3] and references therein. The cylinder or sphere is then considered to be built up by homogeneous shells.

The basic wave solutions for a horizontally layered medium, on the other hand, are obtained in terms of exponentials [4]. It is much more convenient to handle exponentials than Bessel/Hankel functions. As shown in references [5, 6], there exist certain radial dependencies of medium velocities and density for a solid spherical shell for which the basic wave solutions are actually exponential. The spherical shell will of course not be homogeneous. The medium velocities, for example, will be proportional to the radius. Furthermore, a transformation of the radius variable is needed.

In the present paper, it is shown that basic wave solutions in terms of exponentials also exist for a layered cylinder, for certain radial dependencies of medium velocities and density. The shells may be fluid or solid, and combinations of fluid and solid regions are allowed ("fluid–solid media"). Compound-matrix theory is introduced for the cylindrical as well as spherical cases to handle the well-known numerical problems at high frequency. For reasons of computational efficiency, the compound matrices are factorized into sparse matrices which can be applied in sequence, cf. reference [7]. Striking similarities appear between the horizontally (section 3), cylindrically (section 4), and spherically (section 5) layered cases.

A cylinder or sphere with an arbitrary variation with radius of medium velocities and density can be approximated by a sequence of shells with the type of medium parameter variation considered here, that allows basic wave solutions in terms of exponentials rather than Bessel/Hankel functions. An interior homogeneous shell, however, is handled without propagation by including boundary conditions at its interface. Such a a small shell can be useful to avoid velocities unnaturally tending to zero at the centre. For an arbitrary parameter variation with radius, extrapolation techniques can be used to increase the order of convergence and to obtain error estimates. In this way, the approximation process can be automated, and the solution can be obtained within a prescribed error tolerance by adaptively increasing the number of shells. Focusing by Luneberg lenses is studied as a computational example (section 6).

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2. PRELIMINARIES

Monofrequency waves are considered with angular frequency ω. As is common practice, the time (t) factor $e^{-i\omega t}$ is suppressed in the formulas. Basic equations for the displacement vector u_i and the stress tensor τ_{ij}, as given in Cartesian three-dimensional co-ordinates (x_1, x_2, x_3), are [4]

$$-\rho\omega^2 u_i = \tau_{ij,j} + f_i, \tag{1}$$

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}. \tag{2}$$

Here, standard Cartesian-tensor notation is used and δ_{ij} is the Kronecker delta. The source function is given by f_i. The Lamé parameters are denoted λ and μ, and the density is denoted ρ. The medium P- and S-velocities are obtained as $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$ and $\beta = [\mu/\rho]^{1/2}$ respectively. The strain tensor e_{ij} that appears in equation (2) is defined as $e_{ij} = (u_{i,j} + u_{j,i})/2$.

For horizontally, cylindrically and spherically layered fluid-solid media, separation of variables is appropriate and the partial differential equation (1) can be simplified. With the variable z related to depth or radius, things will typically boil down to an ordinary differential equation (ODE) system of the type

$$\mathbf{y}'(z) = \mathbf{A}(z)\cdot\mathbf{y}(z) + \mathbf{f}(z) \tag{3}$$

with boundary conditions at a number of points ζ (different $\mathbf{G}$ for different ζ)

$$\mathbf{G}\cdot\mathbf{y}(\zeta) = \mathbf{0}. \tag{4}$$

Here, $\mathbf{y}(z)$ is an n-dimensional column vector, related to the displacement–stress vector, with $n = 4$ for a solid region and $n = 2$ for a fluid region. $\mathbf{A}(z)$ is an $(n \times n)$ matrix and the source function is incorporated by the n-dimensional column vector $\mathbf{f}(z)$. The integer m is used to denote the number of boundary conditions in a particular instance of equation (4), such that $\mathbf{G}$ is an $(m \times n)$ matrix. The boundary conditions appear at fluid-solid interfaces (where the tangential component of the traction vector must vanish) and at the interior and exterior regions (with radiation conditions, for example). It can be noted that $m = 2$ in the solid case and $m = 1$ in the fluid case.

Linear boundary-value problems of the type (3)–(4) can be solved with the compound-matrix method [8]. In particular, consider an $(m \times n)$ matrix $\mathbf{G} = (\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n)$ from (4). It is appropriate to introduce

$$Y_{[k_1,\ldots,k_m]}(\zeta) = \det(\mathbf{g}_{k_1}, \mathbf{g}_{k_2}, \ldots, \mathbf{g}_{k_m}), \tag{5}$$

where the k's take on values from 1 to n, and collect the $Y_{[k_1,\ldots,k_m]}(\zeta)$ with $1 \leqslant k_1 < k_2 < \cdots < k_m \leqslant n$ in a $(1 \times \binom{n}{m})$ row vector $\mathbf{Y}(\zeta) = (Y_{[1,\ldots,m]}(\zeta), Y_{[1,\ldots,m-1,m+1]}(\zeta), \ldots, Y_{[1,\ldots,m-1,n]}(\zeta), Y_{[1,\ldots,m-2,m,m+1]}(\zeta), \ldots, Y_{[n-m+1,\ldots,n]}(\zeta))$. $\mathbf{Y}(\zeta) = \mathbf{G}^{\Delta_m}$, the mth order compound matrix of $\mathbf{G}$. The m boundary conditions (4) are now characterized by $\mathbf{Y}(\zeta)$ and they can be conveniently transported to other points z by

$$\mathbf{Y}(z) = \mathbf{Y}(\zeta)\cdot[\mathbf{P}(\zeta, z)]^{\Delta_m}, \tag{6}$$

where $\mathbf{P}(z, \zeta)$ is the propagator matrix for system (3) defined by

$$\frac{\partial\mathbf{P}(z,\zeta)}{\partial z} = \mathbf{A}(z)\cdot\mathbf{P}(z,\zeta) \quad \text{and} \quad \mathbf{P}(\zeta,\zeta) = \mathbf{I}. \tag{7}$$

The compound matrix $[\mathbf{P}(\zeta, z)]^{\Delta_m}$, of dimension $\binom{n}{m} \times \binom{n}{m}$, comprises all $(m \times m)$ subdeterminants of $\mathbf{P}(\zeta, z)$. It is defined, for rows as well as columns, in analogy to the columns of $\mathbf{G}^{\Delta_m}$.

If the source function $\mathbf{f}$ does not vanish between ζ and z, however, the boundary conditions as transported to z will be non-homogeneous and inclusion of another compound vector $\mathbf{Z}$ will be needed [8].

3. HORIZONTAL STRATIFICATION

It is convenient to introduce the additional notation (x, y, z) for the Cartesian co-ordinates (x_1, x_2, x_3). The third co-ordinate z is the depth co-ordinate, and the horizontal stratification means per definition that the medium parameters are independent of x and y ("range-independence"). For P–SV waves, independent of y and without displacements in the y direction, Fourier transformation of the x co-ordinate is appropriate and the wave field can be synthesized from components

$$(u_x, u_z, \tau_{zx}, \tau_{zz}) = \mathrm{e}^{\mathrm{i}kx}(r_1, \mathrm{i}\, r_2, r_3, \mathrm{i}\, r_4), \tag{8}$$

where k is the horizontal wavenumber. The horizontal slowness is denoted by p, and $k = \omega p$.

For each k, the vector $\mathbf{y}(z) = (\omega r_1(z), \omega r_2(z), r_3(z), r_4(z))^{\mathrm{T}}$ for a solid region fulfills an ODE system of type (3) with system matrix $\mathbf{A}(z)$ given by [4, (7.28)]

$$\mathbf{A}(z) = \omega \begin{pmatrix} 0 & p & \mu^{-1}(z) & 0 \\ -\dfrac{\lambda(z)}{\lambda(z) + 2\mu(z)} p & 0 & 0 & \dfrac{1}{\lambda(z) + 2\mu(z)} \\ \dfrac{4\mu(z)[\lambda(z) + \mu(z)]}{\lambda(z) + 2\mu(z)} p^2 - \rho(z) & 0 & 0 & \dfrac{\lambda(z)}{\lambda(z) + 2\mu(z)} p \\ 0 & -\rho(z) & -p & 0 \end{pmatrix}. \tag{9}$$

For a homogeneous layer, the corresponding propagator matrix $\mathbf{P} = \mathbf{P}(z, \zeta)$ can be written [7, (11)–(14)]:

$$\mathbf{P} = \mathbf{C}^{-1} \cdot \mathbf{P}_T \cdot \mathbf{C}, \tag{10}$$

where

$$\mathbf{P}_T = \begin{pmatrix} CQ & 0 & STQ & 0 \\ 0 & CP & 0 & STP \\ -SDQ & 0 & CQ & 0 \\ 0 & -SDP & 0 & CP \end{pmatrix}, \tag{11}$$

$$CP = \cosh(\omega d\chi), \quad CQ = \cosh(\omega d\nu), \tag{12}$$

$$SDP = \chi^{-1} \sinh(\omega d\chi), \quad SDQ = \nu^{-1} \sinh(\omega d\nu), \tag{13}$$

$$STP = -\chi \sinh(\omega d\chi), \quad STQ = -\nu \sinh(\omega d\nu) \tag{14}$$

with $d = (z - \zeta)$,

$$\chi^2 = (p^2 - \alpha^{-2}), \quad \nu^2 = (p^2 - \beta^{-2}) \tag{15}$$

and

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & -p \\ 0 & 1 & -p & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\beta^2 p & 1 & 0 \\ 2\beta^2 p & 0 & 0 & 1 \end{pmatrix} \cdot \mathrm{diag}(1, 1, \rho^{-1}, \rho^{-1}), \tag{16}$$

$$\mathbf{C}^{-1} = \mathrm{diag}(1, 1, \rho, \rho) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2\beta^2 p & 1 & 0 \\ -2\beta^2 p & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & p & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

The matrices $\mathbf{C}$ and $\mathbf{C}^{-1}$ have here been written in factorized form. The compound matrix $\mathbf{P}^{\Delta_2}$ can then also be obtained in factorized form, which is most useful for computational efficiency [7, 9]. A further factorization of $\mathbf{P}_T$ as $\mathbf{P}_T = \mathbf{D}^{-1} \cdot \mathbf{P}_E \cdot \mathbf{D}$, where $\mathbf{P}_E = \mathrm{diag}(\mathrm{e}^{-\omega\nu d}, \mathrm{e}^{-\omega\chi d}, \mathrm{e}^{\omega\nu d}, \mathrm{e}^{\omega\chi d})$ is also appropriate in this context [7, (22)–(25)].

3.1. FLUID REGIONS

For a fluid region, for which $\tau_{zx} = 0$, $\mathbf{y} = (\omega r_2, r_4)^{\mathrm{T}}$ is considered and an ODE system of type (3) is obtained with system matrix $\mathbf{A}(z)$ given by

$$\mathbf{A}(z) = \omega \begin{pmatrix} 0 & [\alpha(z)^{-2} - p^2]/\rho(z) \\ -\rho(z) & 0 \end{pmatrix}. \tag{18}$$

For a homogeneous layer, the corresponding propagator matrix $\mathbf{P} = \mathbf{P}(z, \zeta)$ can be written

$$\mathbf{P} = \mathrm{diag}(1, \rho) \cdot \begin{pmatrix} CP & STP \\ -SDP & CP \end{pmatrix} \cdot \mathrm{diag}(1, \rho^{-1}) . \tag{19}$$

4. CYLINDRICAL STRATIFICATION

Cylindrical co-ordinates (r, θ, z) are introduced such that $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, $x_3 = z$. The medium parameters are assumed to be independent of z and θ. Restriction is made to waves that are independent of z without displacements in the axis direction z. A Fourier series for the θ co-ordinate is appropriate and the wave field can be synthesized from components

$$(u_\theta, u_r, \tau_{r\theta}, \tau_{rr}) = \mathrm{e}^{\mathrm{i}m\theta}(V, \mathrm{i}U, S, \mathrm{i}R). \tag{20}$$

For each integer m, the vector $(V(r), U(r), S(r), R(r))^{\mathrm{T}}$ for a solid region fulfills an ODE system of type (3) with z replaced by r and system matrix $\mathbf{A}(r)$ given by

$$\mathbf{A}(r) = \begin{pmatrix} 1/r & m/r & \mu^{-1}(r) & 0 \\ -\dfrac{\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{m}{r} & -\dfrac{\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{1}{r} & 0 & \dfrac{1}{\lambda(r)+2\mu(r)} \\ \xi(r)\left(\dfrac{m}{r}\right)^2 - \rho(r)\omega^2 & \xi(r)\dfrac{m}{r^2} & -2/r & \dfrac{\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{m}{r} \\ \xi(r)\dfrac{m}{r^2} & \xi(r)\dfrac{1}{r^2} - \rho(r)\omega^2 & -m/r & -\dfrac{2\mu(r)}{\lambda(r)+2\mu(r)}\dfrac{1}{r} \end{pmatrix} \tag{21}$$

where $\xi(r) = 4\mu(r)[\lambda(r) + \mu(r)]/(\lambda(r) + 2\mu(r))$, which follows from (1) to (2) together with the co-ordinate transformation relations in reference [4, section 2.6].

Instead of $(V(r), U(r), S(r), R(r))^{\mathrm{T}}$, however, the vector

$$\mathbf{y} = ((\omega a/r)\, V, (\omega a/r)\, U, (r/a)^2\, S, (r/a)^2\, R)^{\mathrm{T}} \tag{22}$$

will be considered, where a is a chosen reference radius. To obtain a propagator matrix which can be expressed in terms of exponentials, the appropriate type of cylindrical shell turns out to be the one for which

$$\lambda(r) = \lambda_0\, a^2/r^2, \quad \mu(r) = \mu_0\, a^2/r^2, \quad \rho(r) = \rho_0\, a^4/r^4, \tag{23}$$

where λ_0, μ_0, ρ_0 are constants. In particular, the P- and S-velocities depend on r according to

$$\alpha(r) = \alpha_0\, r/a, \quad \beta(r) = \beta_0\, r/a, \tag{24}$$

where $\alpha_0 = [(\lambda_0 + 2\mu_0)/\rho_0]^{1/2}$ and $\beta_0 = [\mu_0/\rho_0]^{1/2}$. The variable r also needs to be changed:

$$r/a = \exp(z/a), \tag{25}$$

where z is the new variable, not to be confused with the $z = x_3$ co-ordinate.

It can now be verified that $\mathbf{y}(z)$ fulfills an ODE system of type (3) with constant system matrix $\mathbf{A}(z)$ given by

$$\mathbf{A}(z) = \omega \begin{pmatrix} 0 & m/a\omega & \mu_0^{-1} & 0 \\ -\dfrac{\lambda_0}{\lambda_0+2\mu_0}\dfrac{m}{a\omega} & -\dfrac{2(\lambda_0+\mu_0)}{\lambda_0+2\mu_0}\dfrac{1}{a\omega} & 0 & \dfrac{1}{\lambda_0+2\mu_0} \\ \xi_0\left(\dfrac{m}{a\omega}\right)^2 - \rho_0 & \xi_0\dfrac{m}{(a\omega)^2} & 0 & \dfrac{\lambda_0}{\lambda_0+2\mu_0}\dfrac{m}{a\omega} \\ \xi_0\dfrac{m}{(a\omega)^2} & \xi_0\dfrac{1}{(a\omega)^2} - \rho_0 & -m/a\omega & \dfrac{2(\lambda_0+\mu_0)}{\lambda_0+2\mu_0}\dfrac{1}{a\omega} \end{pmatrix} \tag{26}$$

where $\xi_0 = 4\mu_0(\lambda_0 + \mu_0)/(\lambda_0 + 2\mu_0)$, and that the corresponding propagator matrix $\mathbf{P} = \mathbf{P}(z, \zeta)$ can be written

$$\mathbf{P} = \mathbf{C}^{-1} \cdot \mathbf{P}_T \cdot \mathbf{C} \tag{27}$$

where $\mathbf{P}_T$ is still given by equations (11)–(14) but with

$$\chi^2 = \left(\frac{m}{a\omega}\right)^2 - \alpha_0^{-2} + 4(1 - \beta_0^2/\alpha_0^2)\,\frac{1}{(a\omega)^2}, \tag{28}$$

$$v^2 = \left(\frac{m}{a\omega}\right)^2 - \beta_0^{-2} \tag{29}$$

and

$$\mathbf{C} = \begin{pmatrix} \psi_0 & -m/a\omega & -\mu_0^{-1} & m/2\mu_0 \\ -m/a\omega & -\psi_0 & -m/2\mu_0 & -\mu_0^{-1} \\ 1 & -m & -a\omega/2\mu_0 & 0 \\ m & 1 & 0 & a\omega/2\mu_0 \end{pmatrix} \tag{30}$$

$$\mathbf{C}^{-1} = \left(2 + \frac{a^2\omega^2}{2\beta_0^2}\right)^{-1} \cdot \begin{pmatrix} -a\omega & 0 & 2 & m \\ 0 & a\omega & -m & 2 \\ -2\mu_0 & -2\mu_0 m & 2\mu_0\psi_0 & -2\mu_0 m/a\omega \\ 2\mu_0 m & -2\mu_0 & -2\mu_0 m/a\omega & -2\mu_0\psi_0 \end{pmatrix}, \tag{31}$$

where $\psi_0 = m^2/a\omega - a\omega/2\beta_0^2$. It is useful, however, to write $\mathbf{C}$ and $\mathbf{C}^{-1}$ in terms of sparse matrix factors. By some Gaussian elimination algebra, it follows that

$$\mathbf{C} = \operatorname{diag}\left(\frac{1}{a\omega}, \frac{1}{a\omega}, 1, 1\right) \begin{pmatrix} -1 & 0 & 0 & m \\ 0 & 1 & -m & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \operatorname{diag}\left(2 + \frac{a^2\omega^2}{2\beta_0^2}, 2 + \frac{a^2\omega^2}{2\beta_0^2}, 1, 1\right) \tag{32}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ m & 0 & 0 & 1 \end{pmatrix} \operatorname{diag}\left(1, 1, \frac{a\omega}{2\mu_0}, \frac{a\omega}{2\mu_0}\right),$$

$$\mathbf{C}^{-1} = \operatorname{diag}\left(1, 1, \frac{2\mu_0}{a\omega}, \frac{2\mu_0}{a\omega}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m & 1 & 0 \\ -m & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

$$\times \operatorname{diag}\left(\left(2 + \frac{a^2\omega^2}{2\beta_0^2}\right)^{-1}, \left(2 + \frac{a^2\omega^2}{2\beta_0^2}\right)^{-1}, 1, 1\right) \tag{33}$$

$$\times\begin{pmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0 & m\\ 0 & 1 & -m & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\operatorname{diag}(a\omega, a\omega, 1, 1)\,.$$

Since the compound matrix of a product of matrices equals the product of the compound matrices of the factor matrices, relation (27) immediately gives a factorization of $\mathbf{P}^{\Delta_2}$. The factor $\mathbf{P}_T^{\Delta_2}$ is handled as described in references [7, 9]. Concerning $\mathbf{C}^{\Delta_2}$, it follows from (32) that

$$\mathbf{C}^{\Delta_2} = (2\mu_0)^{-1}\left(2+\frac{a^2\omega^2}{2\beta_0^2}\right)\operatorname{diag}\left(\frac{1}{a\omega}, 1, 1, 1, 1, a\omega\right)\cdot$$

$$\times\begin{pmatrix}-1 & m & 0 & 0 & -m & m^2\\ 0 & 1 & 0 & 0 & 0 & m\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -m\\ 0 & 0 & 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & -2 & -2 & 0 & -4\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\cdot$$

$$\times\operatorname{diag}\left(2+\frac{a^2\omega^2}{2\beta_0^2}, 1, 1, 1, 1, \left(2+\frac{a^2\omega^2}{2\beta_0^2}\right)^{-1}\right)\cdot \tag{34}$$

$$\times\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ -1 & 0 & -1 & -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ m & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ -m & 0 & 0 & 0 & 1 & 0\\ -m^2 & -m & 0 & 0 & m & 1\end{pmatrix}$$

$$\times\operatorname{diag}\left(\frac{2\mu_0}{a\omega}, 1, 1, 1, 1, \frac{a\omega}{2\mu_0}\right).$$

A factorized expression for $(\mathbf{C}^{-1})^{\Delta_2}$ is obtained analogously from (33). The relation "$Y_5 = -Y_2$" [7, (9)] will be maintained during the implied compound-matrix propagation of a (1×6) row vector $\mathbf{Y}$ entering from the left.

4.1. FLUID REGIONS

For a fluid region, for which $\tau_{r\theta} = 0$, the m-dependent vector $(U(r), R(r))^{\mathrm{T}}$ fulfills an ODE system of type (3) with z replaced by r and system matrix $\mathbf{A}(r)$ given by

$$\mathbf{A}(r) = \begin{pmatrix}-1/r & [\alpha(r)^{-2} - (m/\omega r)^2]/\rho(r)\\ -\rho(r)\omega^2 & 0\end{pmatrix}. \tag{35}$$

Instead of $(U(r), R(r))^{\mathrm{T}}$, however, the vector

$$\mathbf{y} = ((\omega r/a)\, U\,, R\,)^{\mathrm{T}} \tag{36}$$

is considered, where a is a chosen reference radius. To obtain a propagator matrix which can be expressed in terms of exponentials, the appropriate type of fluid cylindrical shell turns out to be the one for which

$$\lambda(r) = \lambda_0\, r^2/a^2, \quad \rho(r) = \rho_0, \tag{37}$$

where λ_0, ρ_0 are constants. The P-velocity will depend on r according to $\alpha(r) = \alpha_0\, r/a$ where $\alpha_0 = [\lambda_0/\rho_0]^{1/2}$. The variable r needs to be changed to z as in (25).

It can now be verified that $\mathbf{y}(z)$ fulfills an ODE system of type (3) with constant system matrix $\mathbf{A}(z)$ given by

$$\mathbf{A}(z) = \omega\begin{pmatrix} 0 & [\alpha_0^{-2} - (m/a\omega)^2]/\rho_0 \\ -\rho_0 & 0 \end{pmatrix}. \tag{38}$$

This is of the same type as (18) for the horizontally stratified case, and the propagator-matrix expression (19) is applicable, with ρ replaced by ρ_0, α by α_0, and p by $m/a\omega$.

5. SPHERICAL STRATIFICATION

Spherical co-ordinates (r, θ, ϕ) are introduced such that $x_1 = r\sin(\theta)\cos(\phi)$, $x_2 = r\sin(\theta)\sin(\phi)$, $x_3 = r\cos(\theta)$. The medium parameters are assumed to be independent of θ and ϕ. Restriction is made to spheroidal waves, and the more simple toroidal waves are thus neglected. The wave field can be synthesized from components, $l = 0, 1, \ldots,$ and $m = -l, \ldots, l$,

$$(Lu_\theta, u_r, L\,\tau_{r\theta}\,, \tau_{rr}\,) = (V\partial Y_l^m/\partial\theta\,, U\, Y_l^m\,, S\,\partial Y_l^m/\partial\theta\,, R\, Y_l^m\,), \tag{39}$$

where $L = [l(l+1)]^{1/2}$ and Y_l^m are the spherical surface harmonics given by

$$Y_l^m(\theta, \phi) = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos(\theta))\mathrm{e}^{\mathrm{i}m\phi} \tag{40}$$

with associated Legendre functions P_l^m. Furthermore, $L\sin(\theta)\, u_\phi = V\,\partial Y_l^m/\partial\phi$ and $L\sin(\theta)\,\tau_{r\phi} = S\,\partial Y_l^m/\partial\phi$. It is understood that $u_\theta, u_\phi, \tau_{r\theta}, \tau_{r\phi}$ vanish when $l = 0$.

For each l, m pair, the vector $(V(r), U(r), S(r), R(r))^{\mathrm{T}}$ for a solid region fulfills an ODE system of type (3) with z replaced by r and system matrix $\mathbf{A}(r)$ given by [4, (8.32)]

$$\mathbf{A}(r) = \begin{pmatrix} 1/r & -L/r & \mu^{-1}(r) & 0 \\ \dfrac{\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{L}{r} & -\dfrac{2\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{1}{r} & 0 & \dfrac{1}{\lambda(r)+2\mu(r)} \\ \xi(r)\left(\dfrac{L}{r}\right)^2 - \rho(r)\omega^2 - \dfrac{2\mu(r)}{r^2} & -\eta(r)\,\dfrac{L}{r^2} & -3/r & -\dfrac{\lambda(r)}{\lambda(r)+2\mu(r)}\dfrac{L}{r} \\ -\eta(r)\,\dfrac{L}{r^2} & 2\eta(r)\,\dfrac{1}{r^2} - \rho(r)\omega^2 & L/r & -\dfrac{4\mu(r)}{\lambda(r)+2\mu(r)}\dfrac{1}{r} \end{pmatrix}, \tag{41}$$

where $\xi(r) = 4\mu(r)[\lambda(r)+\mu(r)]/(\lambda(r)+2\mu(r))$ and $\eta(r) = 2\mu(r)[3\lambda(r)+2\mu(r)]/(\lambda(r)+2\mu(r))$.

Instead of $(V(r), U(r), S(r), R(r))^{\mathrm{T}}$, however, the vector

$$\mathbf{y} = (\omega(a/r)^{1/2}\, V, \omega(a/r)^{1/2}\, U\,,\ (r/a)^{5/2}\, S,\ (r/a)^{5/2}\, R\,)^{\mathrm{T}} \tag{42}$$

will be considered, where a is a chosen reference radius. To obtain a propagator matrix which can be expressed in terms of exponentials, steps analogous to (23)–(25) turn out to be appropriate. Following references [5, 6], a spherical shell is considered for which

$$\lambda(r) = \lambda_0\, a^2/r^2, \quad \mu(r) = \mu_0\, a^2/r^2, \quad \rho(r) = \rho_0\, a^4/r^4 \tag{43}$$

where λ_0, μ_0, ρ_0 are constants. It follows that $\alpha(r) = \alpha_0\, r/a$ and $\beta(r) = \beta_0\, r/a$, where $\alpha_0 = [(\lambda_0 + 2\mu_0)/\rho_0]^{1/2}$ and $\beta_0 = [\mu_0/\rho_0]^{1/2}$. The variable r is changed to z according to $r/a = \exp(z/a)$.

It can now be verified that $\mathbf{y}(z)$ fulfills an ODE system of type (3) with constant system matrix $\mathbf{A}(z)$ given by

$$\mathbf{A}(z) = \omega \begin{pmatrix} 1/2a\omega & -L/a\omega & \mu_0^{-1} & 0 \\ \dfrac{\lambda_0}{\lambda_0 + 2\mu_0}\dfrac{L}{a\omega} & -\dfrac{5\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}\dfrac{1}{2a\omega} & 0 & \dfrac{1}{\lambda_0 + 2\mu_0} \\ \xi_0\left(\dfrac{L}{a\omega}\right)^2 - \rho_0 - \dfrac{2\mu_0}{(a\omega)^2} & -\eta_0\dfrac{L}{(a\omega)^2} & -1/2a\omega & -\dfrac{\lambda_0}{\lambda_0 + 2\mu_0}\dfrac{L}{a\omega} \\ -\eta_0\dfrac{L}{(a\omega)^2} & 2\eta_0\dfrac{1}{(a\omega)^2} - \rho_0 & L/a\omega & \dfrac{5\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}\dfrac{1}{2a\omega} \end{pmatrix}, \tag{44}$$

where $\xi_0 = 4\mu_0(\lambda_0 + \mu_0)/(\lambda_0 + 2\mu_0)$ and $\eta_0 = 2\mu_0(3\lambda_0 + 2\mu_0)/(\lambda_0 + 2\mu_0)$, and that the corresponding propagator matrix $\mathbf{P} = \mathbf{P}(z, \zeta)$ can be written

$$\mathbf{P} = \mathbf{C}^{-1} \cdot \mathbf{P}_T \cdot \mathbf{C} \tag{45}$$

where $\mathbf{P}_T$ is still given by (11)–(14) but with

$$\chi^2 = \left(\frac{L}{a\omega}\right)^2 - \alpha_0^{-2} + (25/4 - 8\beta_0^2/\alpha_0^2)\frac{1}{(a\omega)^2} \tag{46}$$

$$v^2 = \left(\frac{L}{a\omega}\right)^2 - \beta_0^{-2} - \frac{7}{4(a\omega)^2} \tag{47}$$

and

$$\mathbf{C} = \left[2a\omega\left(8 + \frac{a^2\omega^2}{\beta_0^2}\right)\right]^{-1} \cdot \begin{pmatrix} \psi_0 - 3 & L & -5a\omega/2\mu_0 & -a\omega L/\mu_0 \\ -3L & \psi_0 + 2 & -a\omega L/\mu_0 & 5a\omega/2\mu_0 \\ 2a\omega & 2a\omega L & -a^2\omega^2/\mu_0 & 0 \\ 2a\omega L & -4a\omega & 0 & -a^2\omega^2/\mu_0 \end{pmatrix}, \tag{48}$$

$$\mathbf{C}^{-1} = \begin{pmatrix} -2a\omega & 0 & 5 & 2L \\ 0 & -2a\omega & 2L & -5 \\ -4\mu_0 & -4\mu_0 L & 2\mu_0(\psi_0 - 3)/a\omega & -6\mu_0 L/a\omega \\ -4\mu_0 L & 8\mu_0 & 2\mu_0 L/a\omega & 2\mu_0(\psi_0 + 2)/a\omega \end{pmatrix}, \tag{49}$$

where $\psi_0 = 2L^2 - a^2\omega^2/\beta_0^2$. It is useful, however, to write $\mathbf{C}$ and $\mathbf{C}^{-1}$ in terms of sparse matrix factors. By some Gaussian elimination algebra, it follows that

$$\mathbf{C} = \mathrm{diag}\left(\frac{1}{2a\omega}, \frac{1}{2a\omega}, 1, 1\right)\begin{pmatrix} -1 & 0 & 0 & -2L \\ 0 & -1 & -2L & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\times\,\mathrm{diag}\left(1, 1, \left(8 + \frac{a^2\omega^2}{\beta_0^2}\right)^{-1}, \left(8 + \frac{a^2\omega^2}{\beta_0^2}\right)^{-1}\right). \tag{50}$$

$$\times\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\mathrm{L} & 1 & 0 \\ -\mathrm{L} & 0 & 0 & 1 \end{pmatrix}\mathrm{diag}\left(1, 1, \frac{a\omega}{2\mu_0}, \frac{a\omega}{2\mu_0}\right)$$

$$\mathbf{C}^{-1} = \mathrm{diag}\left(1, 1, \frac{2\mu_0}{a\omega}, \frac{2\mu_0}{a\omega}\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L & 1 & 0 \\ L & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}.$$

$$\times\,\mathrm{diag}\left(1, 1, 8 + \frac{a^2\omega^2}{\beta_0^2}, 8 + \frac{a^2\omega^2}{\beta_0^2}\right) \tag{51}$$

$$\times\begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 2L \\ 0 & -1 & 2\mathrm{L} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\mathrm{diag}(2a\omega, 2\mathrm{a}\omega, 1, 1)\,.$$

Relation (45) immediately gives a factorization of $\mathbf{P}^{\Delta_2}$. The factor $\mathbf{P}_T^{\Delta_2}$ is again handled as described in references [7, 9]. Concerning $\mathbf{C}^{\Delta_2}$ and $(\mathbf{C}^{-1})^{\Delta_2}$, factorized expressions can be obtained directly from (50)–(51). The relation "$Y_5 = -Y_2$" [7, (9)] will again be maintained during the implied compound-matrix propagation of a (1×6) row vector $\mathbf{Y}$ entering from the left.

5.1. FLUID REGIONS

For a fluid region, for which $\tau_{r\theta} = 0$, the l, m-dependent vector $(U(r), R(r))^{\mathrm{T}}$ fulfills an ODE system of type (3) with z replaced by r and system matrix $\mathbf{A}(r)$ given by

$$\mathbf{A}(r) = \begin{pmatrix} -2/r & [\alpha(r)^{-2} - (L/\omega r)^2]/\rho(r) \\ -\rho(r)\omega^2 & 0 \end{pmatrix}. \tag{52}$$

Instead of $(U(r), R(r))^{\mathrm{T}}$, however, the vector

$$y = (\omega(r^2/a^2)\, U\,, R\,)^{\mathrm{T}} \tag{53}$$

is considered, where a is a chosen reference radius. To obtain a propagator matrix which can be expressed in terms of exponentials, the appropriate type of fluid spherical shell turns out to be the one for which

$$\lambda(r) = \lambda_0\, r^3/a^3, \quad \rho(r) = \rho_0\, r/a \tag{54}$$

where λ_0, ρ_0 are constants. The P-velocity will depend on r according to $\alpha(r) = \alpha_0\, r/a$ where $\alpha_0 = [\lambda_0/\rho_0]^{1/2}$. The variable r is changed to z in the familiar way: $r/a = \exp(z/a)$.

It can now be verified that $\mathbf{y}(z)$ fulfills an ODE system of type (3) with constant system matrix $\mathbf{A}(z)$ given by

$$\mathbf{A}(z) = \omega\begin{pmatrix} 0 & [\alpha_0^{-2} - (L/a\omega)^2]/\rho_0 \\ -\,\rho_0 & 0 \end{pmatrix}. \tag{55}$$

This is of the same type as (18) for the horizontally stratified case, and the propagator-matrix expression (19) is applicable, with ρ replaced by ρ_0, α by α_0, and p by $L/a\omega$.

6. DISCUSSION

An interior or exterior homogeneous region without sources is conveniently handled without propagation by including boundary conditions at its interface. The required compound matrix $\mathbf{G}^{\Delta}$ of a boundary-condition matrix $\mathbf{G}$ as in (4) is easily obtained from vectors that span the solution space. The cylindrical case can be treated using reference [10, sections 6.9 and 2.13]. For a homogeneous solid sphere, see reference [11, (2.2.28–30) and (2.3.6–8)].

For media with an arbitrary variation, with depth or radius, of medium velocities and density, a sequence of approximating media is considered. Each approximating medium is built up by layers or shells with the specific variation considered previously, for example in (23). It can be useful to initialize propagation from boundary conditions at the interface of a small interior homogeneous shell, however, since (23) is not appropriate at the centre ($r = 0$). Each medium in the sequence represents a refinement of its predecessor, and extrapolation techniques can be used to speed up convergence. Let the parameter h represent the fineness of a medium discretization, in depth or radius, and suppose that some field quantity S is to be computed. If the discretization is done in analogy with the mid-point rule, an asymptotic expansion is anticipated according to

$$S(h) = S_0 + \sum_{v=1}^{s} c_v h^{2v} + \mathcal{O}(h^{2s+2})\,. \tag{56}$$

Expansion (56) has been verified by numerical experiments. For the extrapolation, the Bulirsch–Stoer rational technique is suggested, which was also used in reference [12]. Error estimates are conveniently obtained from differences in the extrapolation table.

As a computational example, an acoustic Luneberg lens [13] is considered. This is a sphere, of radius a say, for which the P-velocity varies according to $\alpha(r) = [2 - (r/a)^2]^{-1/2}\alpha_0$, where α_0 is the P-velocity of the surrounding fluid. The Luneberg lens focuses an incoming plane wave at the opposite surface of the lens. This is illustrated in

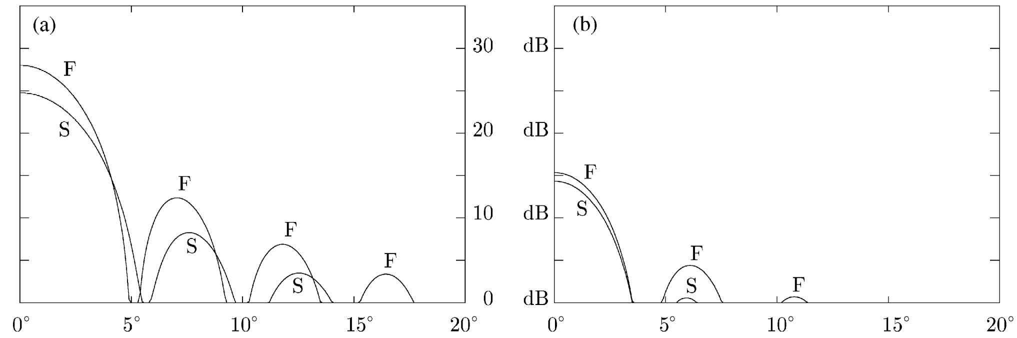

Figure 1. Focal polar patterns (lens gain at the opposite side of the lens for an incoming plane wave) for a fluid Luneberg lens (F) and a solid one with $\beta(r) = \alpha(r)/2$ (S). The frequency is such that $k_0 a = 40$, where k_0 is the wavenumber in the surrounding fluid. The density is constant throughout the medium. Panels (a) and (b) are for the spherical and cylindrical cases respectively.

Figure 1a, which shows the focal polar patterns for a fluid lens (F) and a solid lens with $\beta(r) = \alpha(r)/2$ (S). The fluid result was obtained in a different way, with a particular eigenfunction expansion, in reference [13, Figure 6]. Corresponding results for a cylindrical lens are shown in Figure 1b, with less focusing gain as expected.

7. CONCLUSIONS

It is well-known that wave propagation through a spherically symmetric object composed of concentric solid and fluid shells can be computed by spherical Bessel functions, whereby each shell is divided into homogeneous sub-shells. As recently shown in the seismological literature [5, 6], however, wave propagation in the earth can be computed by ordinary exponentials if the earth is divided into particular concentric shells, in each of which the P- and S-velocities vary proportionally to the radius. A number of advantages are obtained by using exponentials rather than spherical Bessel functions. Analytical scaling, differentiation, and integration, for example, become quite trivial.

In the present paper, the exponential-function technique has been developed for wave propagation through cylinders as well as spheres. Fluid and solid regions, and combinations thereof, have been considered. For fluid shells, it turned out that the formalism becomes *identical* to the one for handling fluid regions built up by homogeneous layers in horizontally stratified range-independent media. For solid shells, compound-matrix techniques were introduced to overcome the numerical cancellation problems for high-frequency computations in the evanescent regime. Each (6×6) compound matrix is factorized as a product of *sparse* matrices, which are applied in sequence to the pertinent row vector entering from the left. Computational efficiency is enhanced significantly in this way. The central compound-matrix factors, involving the exponential and/or trigonometric functions, become formally identical to corresponding matrices for the range-independent case. Although differences appear concerning the remaining compound-matrix factors, which are no longer independent of frequency, a compound-matrix code for wave propagation in range-independent media can easily be adapted to handle cylindrical- and spherical-shell applications.

Extrapolation techniques are useful to reduce the number of sub-shells needed for accurate computations when the elastic parameters vary in an arbitrary way with radius.

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