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DAMPED VIBRATION ANALYSIS OF A TWO-DEGREE-OF-FREEDOM DISCRETE SYSTEM

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1. INTRODUCTION

A two-degree-of-freedom (T-d.o.f.) system is the simplest model of a complex discrete system having multi-degrees of freedom (M-d.o.f.). As is well known, the title system constitutes an interesting introduction to the behavioural investigation of systems with an arbitrarily large number of degrees of freedom. Almost every vibration monograph [1–22] contains an important chapter dealing with a linear T-d.o.f. system. Vibrations of discrete systems having two degrees of freedom are the subject of many papers, e.g., references [23–41].

The motion of a T-d.o.f. system is normally expressed by two coupled non-homogeneous ordinary differential equations. Formulation of their exact analytical solutions in a general form for the system with damping is not easy. Solving, for example the free vibration problem, the characteristic quartic algebraic equation is obtained. Considering the case of light damping, one assumes that this equation has four complex roots in the form of two pairs of conjugate complex numbers. There is no simple direct relation between these anticipated roots and arbitrary values of the physical parameters characterizing the vibrating system. The solutions are presented in a basic form by corresponding combinations of products of decaying exponential and trigonometric time functions. Other possible solutions for different damping cases are not usually discussed in the literature. The aim of this work is to perform a full theoretical vibration analysis of a T-d.o.f. system with arbitrary damping. Such an analysis is possible, and exact analytical solutions can be determined for a certain simplified linear, viscously damped model whose physical parameters, namely, masses, viscous damping coefficients, and spring constants are assumed to be identical.

In this paper, free and forced vibrations caused by external exciting forces being arbitrary time functions are investigated and their exact analytical solutions are derived. While underdamped cases are usually only significant in the study of mechanical vibrations, it seems that all possible solutions presented in this work give a better understanding of the vibration phenomena occurring in arbitrarily damped T-d.o.f. system. Moreover, a T-d.o.f. forced vibration analysis is of great technical importance, because it makes possible to examine the dynamic vibration absorption phenomenon [1–22] and to design different types of discrete dynamic vibration absorbers (DDVAs). Problems of theory, behaviour and applications of DDVAs have been treated by numerous investigators and many references, e.g., [1, 14, 20, 30–41], are devoted to various concepts of them.

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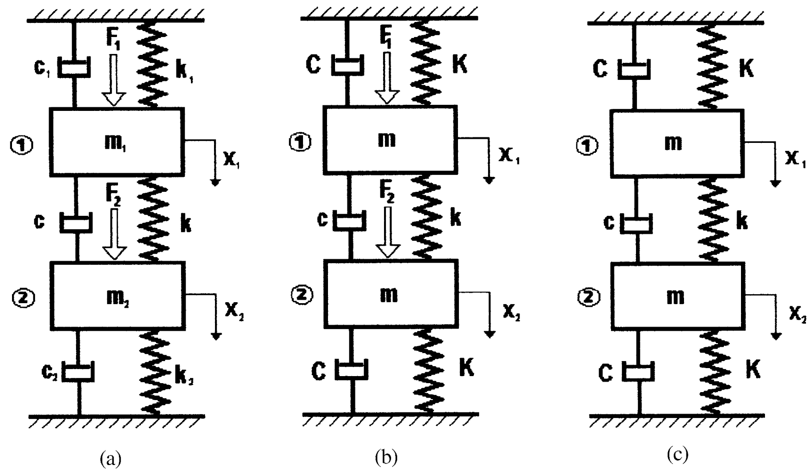

Figure 1. The physical model of a two-degree-of-freedom (T-d.o.f.) discrete system: (a) classical general model; (b) simplified model assumed; and (c) model analyzed for free vibrations.

2. FORMULATION OF THE PROBLEM

The classical general model of a two-degree-of-freedom discrete vibratory system represented by a spring–mass–damper system is depicted in Figure 1(a). The system is assumed to be linear and viscously damped. The external exciting forces acting on the masses are arbitrary time functions. The vibrations of the system under discussion are governed by the following non-homogeneous ordinary differential equations [5, 7, 11, 12, 21]:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k(x_1 - x_2) = F_1(t),$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_2x_2 + k(x_2 - x_1) = F_2(t), \tag{1}$$

where $x_i = x_i(t)$ is the mass displacement, $F_i = F_i(t)$ is the external exciting force, c_i, c are the viscous damping coefficients, k_i, k are the spring constants, m_i is the vibrating mass, t is the time, $\dot{x} = \mathrm{d}x/\mathrm{d}t$; $i = 1, 2$.

The associated initial conditions are

$$x_i(0) = x_{i0}, \qquad \dot{x}_i(0) = v_{i0}, \quad i = 1, 2. \tag{2}$$

Equations (1) constitute a coupled system of two ordinary differential equations in the two unknown functions $x_1(t)$ and $x_2(t)$, which is difficult to solve in a general form, and so certain simplifying assumptions are made. The theoretical analysis of the vibration problem is conducted for a simplified system (see Figure 1(b)) when the physical parameters characterizing the vibrating system, namely the masses, the external viscous damping coefficients, and the spring constants are identical as follows:

$$c_i = C, \qquad k_i = K, \qquad m_i = m, \quad i = 1, 2.$$

Taking these assumptions into consideration, equations (1) can be rewritten in the form

$$m\ddot{x}_1 + C\dot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + Kx_1 + k(x_1 - x_2) = F_1(t),$$

$$m\ddot{x}_2 + C\dot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + Kx_2 + k(x_2 - x_1) = F_2(t). \tag{3}$$

Further manipulations give

$$\ddot{x}_1 + (C + c)m^{-1}\dot{x}_1 + (K + k)m^{-1}x_1 - cm^{-1}\dot{x}_2 - km^{-1}x_2 = m^{-1}F_1(t),$$

$$\ddot{x}_2 + (C + c)m^{-1}\dot{x}_2 + (K + k)m^{-1}x_2 - cm^{-1}\dot{x}_1 - km^{-1}x_1 = m^{-1}F_2(t). \quad (4)$$

It should be noted that introduction of the new variables being the principal co-ordinates (5, 7, 21), defined as

$$y_1(t) = \sum_{i=1}^{2} x_i(t), \qquad y_2(t) = \sum_{i=1}^{2} a_i x_i(t), \quad a_1 = -a_2 = 1, \quad (5)$$

makes it possible to decouple the differential equations (3). Adding and subtracting equations (4) gives

$$\ddot{y}_1 + Cm^{-1}\dot{y}_1 + Km^{-1}y_1 = f_1(t),$$

$$\ddot{y}_2 + (C + 2c)m^{-1}\dot{y}_2 + (K + 2k)m^{-1}y_2 = f_2(t), \quad (6)$$

where

$$a_1 = -a_2 = 1, \quad f_1(t) = m^{-1}\sum_{i=1}^{2} F_i(t), \quad f_2(t) = m^{-1}\sum_{i=1}^{2} a_i F_i(t). \quad (7)$$

It is seen that the equations of motion are uncoupled, and finally they can be presented in the following form:

$$\ddot{y}_i + 2h_i\dot{y}_1 + \omega_i^2 y_i = f_i(t), \quad i = 1, 2, \quad (8)$$

where

$$h_1 = 0{\cdot}5Cm^{-1}, \quad h_2 = 0{\cdot}5(C + 2c)m^{-1} = h_1 + h_0, \quad h_0 = cm^{-1},$$

$$\omega_1^2 = Km^{-1}, \quad \omega_2^2 = (K + 2k)m^{-1} = \omega_1^2 + \omega_0^2, \quad \omega_0^2 = 2km^{-1}. \quad (9)$$

In equations (9), h_i and ω_i denote the damping coefficients and natural frequencies of the undamped free vibration of the system respectively. Solving equations (8) gives the principal coordinates $y_i(t)$. The unknown solutions of equations (3) $x_i(t)$ are then determined from the relationships

$$x_1(t) = 0{\cdot}5\sum_{i=1}^{2} y_i(t), \quad x_2(t) = 0{\cdot}5\sum_{i=1}^{2} a_i y_i(t), \quad a_1 = -a_2 = 1. \quad (10)$$

3. DAMPED FREE VIBRATION ANALYSIS

For the free vibration analysis of the system shown in Figure 1(c), it is assumed that $F_1(t) = F_2(t) = 0$. Thus, the damped free vibrations are represented by general solutions of the homogeneous equations (8)

$$\ddot{y}_i + 2h_i\dot{y}_i + \omega_i^2 y_i = 0, \quad i = 1, 2. \quad (11)$$

It is well known that these equations have three types of solutions, which depend upon the value of damping [1–22, 42, 43]. The following cases are usually considered:

(1) *Undercritical damping* (*underdamping*): $h_i < \omega_i$,

$$y_i(t) = \mathrm{e}^{-h_i t}[C_i \sin(\omega_{iu} t) + D_i \cos(\omega_{iu} t)], \quad \omega_{iu} = (\omega_i^2 - h_i^2)^{1/2}, \tag{12}$$

where ω_{iu} denotes the damped natural frequency of the system.

An underdamped case (small damping) is important in vibration analysis, because it is the unique case leading to an oscillatory motion. Solution (12) represents the damped free harmonic vibration of the system, which is performed with the frequency ω_{iu} and amplitude decreasing exponentially with time.

(2) *Critical damping*: $h_i = \omega_i$,

$$y_i(t) = \mathrm{e}^{-h_i t}[C_i t + D_i] = \mathrm{e}^{-\omega_i t}[C_i t + D_i]. \tag{13}$$

For a critically damped case (average damping), solution (13) represents an aperiodic motion of the system, which reduces to zero with time.

(3) *Overcritical damping* (*overdamping*): $h_i > \omega_i$,

$$y_i(t) = \mathrm{e}^{-h_i t}[C_i \sinh(\omega_{io} t) + D_i \cosh(\omega_{io} t)], \quad \omega_{io} = (h_i^2 - \omega_i^2)^{1/2}. \tag{14}$$

For an overdamped case (large damping), solution (14) described by an aperiodic time function characterizes the motion of the system, which reduces exponentially with time.

Formulating solutions (10) for the damped free vibrations of the system $x_1(t)$ and $x_2(t)$, nine possible cases can be identified. They are listed below.

(1a) $h_1 < \omega_1$, $h_2 < \omega_2$, when $C < 2(Km)^{1/2}$ and $c < [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i \sin(\omega_{iu} t) + B_i \cos(\omega_{iu} t)], \quad x_2(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i \sin(\omega_{iu} t) + B_i \cos(\omega_{iu} t)]\, a_i. \tag{15}$$

(2a) $h_1 < \omega_1$, $h_2 = \omega_2$, when $C < 2(Km)^{1/2}$ and $c = [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u} t) + B_1 \cos(\omega_{1u} t)] + \mathrm{e}^{-h_2 t}[A_2 t + B_2],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u} t) + B_1 \cos(\omega_{1u} t)] - \mathrm{e}^{-h_2 t}[A_2 t + B_2]. \tag{16}$$

(3a) $h_1 < \omega_1$, $h_2 > \omega_2$, when $C < 2(Km)^{1/2}$ and $c > [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u} t) + B_1 \cos(\omega_{1u} t)] + \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u} t) + B_1 \cos(\omega_{1u} t)] - \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)]. \tag{17}$$

(4a) $h_1 = \omega_1$, $h_2 < \omega_2$, when $C = 2(Km)^{1/2}$ and $c < [(K + 2k)m]^{1/2} - (Km)^{1/2}$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 t + B_1] + \mathrm{e}^{-h_2 t}[A_2 \sin(\omega_{2u} t) + B_2 \cos(\omega_{2u} t)],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 t + B_1] - \mathrm{e}^{-h_2 t}[A_2 \sin(\omega_{2u} t) + B_2 \cos(\omega_{2u} t)]. \tag{18}$$

(5a) $h_1 = \omega_1, h_2 = \omega_2$, when $C = 2(Km)^{1/2}$ and $c = [(K + 2k)m]^{1/2} - (Km)^{1/2}$:

$$x_1(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i t + B_i], \qquad x_2(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i t + B_i] a_i. \tag{19}$$

(6a) $h_1 = \omega_1, h_2 > \omega_2$, when $C = 2(Km)^{1/2}$ and $c > [(K + 2k)m]^{1/2} - (Km)^{1/2}$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 t + B_1] + \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 t + B_1] - \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)]. \tag{20}$$

(7a) $h_1 > \omega_1, h_2 < \omega_2$, when $C > 2(Km)^{1/2}$ and $c < [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 \sinh(\omega_{1o} t) + B_1 \cosh(\omega_{1o} t)] + \mathrm{e}^{-h_2 t}[A_2 \sin(\omega_{2u} t) + B_2 \cos(\omega_{2u} t)],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 \sinh(\omega_{1o} t) + B_1 \cosh(\omega_{1o} t)] - \mathrm{e}^{-h_2 t}[A_2 \sin(\omega_{2u} t) + B_2 \cos(\omega_{2u} t)]. \tag{21}$$

(8a) $h_1 > \omega_1, h_2 = \omega_2$, when $C > 2(Km)^{1/2}$ and $c = [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \mathrm{e}^{-h_1 t}[A_1 \sinh(\omega_{1o} t) + B_1 \cosh(\omega_{1o} t) + \mathrm{e}^{-h_2 t}[A_2 t + B_2],$$

$$x_2(t) = \mathrm{e}^{-h_1 t}[A_1 \sinh(\omega_{1o} t) + B_1 \cosh(\omega_{1o} t) - \mathrm{e}^{-h_2 t}[A_2 t + B_2]. \tag{22}$$

(9a) $h_1 > \omega_1, h_2 > \omega_2$, when $C > 2(Km)^{1/2}$ and $c > [(K + 2k)m]^{1/2} - 0{\cdot}5C$:

$$x_1(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i \sinh(\omega_{io} t) + B_i \cosh(\omega_{io} t)],$$

$$x_2(t) = \sum_{i=1}^{2} \mathrm{e}^{-h_i t}[A_i \sinh(\omega_{io} t) + B_i \cosh(\omega_{io} t)] a_i, \tag{23}$$

where $a_1 = -a_2 = 1$,

$$h_1 = 0{\cdot}5Cm^{-1}, \quad h_2 = 0{\cdot}5(C + 2c)m^{-1}, \quad \omega_1 = (Km^{-1})^{1/2}, \quad \omega_2 = [(K + 2k)m^{-1}]^{1/2},$$

$$\omega_{1u} = [Km^{-1} - 0{\cdot}25C^2 m^{-2}]^{1/2}, \quad \omega_{2u} = [(K + 2k)m^{-1} - 0{\cdot}25(C + 2c)^2 m^{-2}]^{1/2},$$

$$\omega_{1o} = [0{\cdot}25C^2 m^{-2} - Km^{-1}]^{1/2}, \quad \omega_{2o} = [0{\cdot}25(C + 2c)^2 m^{-2} - (K + 2k)m^{-1}]^{1/2} \tag{24}$$

It is important to note that all parameters shaping the above solutions are explicitly and directly dependent on the physical parameters characterizing the vibrating system discussed. Applying particular values of C, c, K, k, m, a proper form of solution can be determined without difficulty for any case.

The unknown integration constants A_i, B_i ($i = 1, 2$) are received from the assumed initial conditions (2). As is well known, a T-d.o.f. system executes two types of free vibrations (motions): synchronous and asynchronous. The synchronous vibration ($a_1 = 1 > 0$) corresponds to the first mode shape of vibration, and is characterized by the following parameters: $h_1, \omega_1, \omega_{1o}, \omega_{1u}$. These quantities are not functions of a viscous damping coefficient c and spring constant k. Simplifying assumptions can be introduced that cause the system to vibrate as a whole without any relative motion between two masses. This implies that the middle spring is not deformed, and the middle damper does not work. The asynchronous vibration ($a_2 = -1 < 0$) corresponds to the second mode shape of vibration. The parameters characterizing this motion $h_2, \omega_2, \omega_{2o}, \omega_{2u}$ are greater than those describing the fundamental (synchronous) motion. The displacements of both masses in the corresponding motions are identical.

4. DAMPED FORCED VIBRATION ANALYSIS

The damped forced vibrations of the system for the simplifying assumptions are governed by the non-homogeneous differential equations (8) and are represented by their particular solutions. Three possible forms [3, 6, 7, 42, 43] depending on the values of damping coefficients must be considered:
(1) *Undercritical damping*: $h_i < \omega_i$,

$$y_i(t) = \omega_{iu}^{-1} \int_0^t f_i(s) \mathrm{e}^{-h_i(t-s)} \sin[\omega_{iu}(t-s)]\, \mathrm{d}s, \quad \omega_{iu} = (\omega_i^2 - h_i^2)^{1/2}. \tag{25}$$

(2) *Critical damping*: $h_i = \omega_i$,

$$y_i(t) = \int_0^t f_i(s) \mathrm{e}^{-h_i(t-s)} (t-s)\, \mathrm{d}s. \tag{26}$$

(3) *Overcritical damping*: $h_i > \omega_i$,

$$y_i(t) = \omega_{io}^{-1} \int_0^t f_i(s) \mathrm{e}^{-h_i(t-s)} \sinh[\omega_{io}(t-s)\, \mathrm{d}s, \quad \omega_{io} = (h_i^2 - \omega_i^2)^{1/2}. \tag{27}$$

Setting solutions (6) for the damped forced vibrations of a T-d.o.f. system subjected to arbitrary exciting forces, nine possible cases can be identified. They are listed below.
(1b) $h_1 < \omega_1$, $h_2 < \omega_2$:

$$x_1(t) = 0{\cdot}5 \sum_{i=1}^{2} \omega_{iu}^{-1} \int_0^t f_i(s) \mathrm{e}^{-h_i(t-s)} \sin[\omega_{iu}(t-s)]\, \mathrm{d}s,$$

$$x_2(t) = 0{\cdot}5 \sum_{i=1}^{2} a_i \omega_{iu}^{-1} \int_0^t f_i(s) \mathrm{e}^{-h_i(t-s)} \sin[\omega_{iu}(t-s)]\, \mathrm{d}s, \tag{28}$$

(2b) $h_1 < \omega_1$, $h_2 = \omega_2$:

$$x_1(t) = 0{\cdot}5 \left\{ \omega_{1u}^{-1} \int_0^t f_1(s) \mathrm{e}^{-h_1(t-s)} \sin[\omega_{1u}(t-s)]\, \mathrm{d}s + \int_0^t f_2(s) \mathrm{e}^{-h_2(t-s)} (t-s)\, \mathrm{d}s \right\},$$

$$x_2(t) = 0{\cdot}5 \left\{ \omega_{1u}^{-1} \int_0^t f_1(s) \mathrm{e}^{-h_1(t-s)} \sin[\omega_{1u}(t-s)]\, \mathrm{d}s - \int_0^t f_2(s) \mathrm{e}^{-h_2(t-s)} (t-s)\, \mathrm{d}s \right\}. \tag{29}$$

(3b) $h_1 < \omega_1$, $h_2 > \omega_2$:

$$x_1(t) = 0{\cdot}5 \left\{ \omega_{1u}^{-1} \int_0^t f_1(s) \mathrm{e}^{-h_1(t-s)} \sin[\omega_{1u}(t-s)]\, \mathrm{d}s + \omega_{2o}^{-1} \int_0^t f_2(s) \mathrm{e}^{-h_2(t-s)} \sinh[\omega_{2o}(t-s)]\, \mathrm{d}s \right\},$$

$$x_2(t) = 0{\cdot}5 \left\{ \omega_{1u}^{-1} \int_0^t f_1(s) \mathrm{e}^{-h_1(t-s)} \sin[\omega_{1u}(t-s)]\, \mathrm{d}s - \omega_{2o}^{-1} \int_0^t f_2(s) \mathrm{e}^{-h_2(t-s)} \sinh[\omega_{2o}(t-s)]\, \mathrm{d}s \right\}. \tag{30}$$

(4b) $h_1 = \omega_1, h_2 < \omega_2$:

$$x_1(t) = 0{\cdot}5\left\{\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}(t-s)\,\mathrm{d}s + \omega_{2o}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sin[\omega_{2u}(t-s)]\,\mathrm{d}s\right\},$$

$$x_2(t) = 0{\cdot}5\left\{\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}(t-s)\,\mathrm{d}s - \omega_{2o}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sin[\omega_{2u}(t-s)]\,\mathrm{d}s\right\}. \quad (31)$$

(5b) $h_1 = \omega_1, h_2 = \omega_2$:

$$x_1(t) = 0{\cdot}5\sum_{i=1}^{2}\int_0^t f_i(s)\mathrm{e}^{-h_i(t-s)}(t-s)\,\mathrm{d}s, \quad x_2(t) = 0{\cdot}5\sum_{i=1}^{2} a_i\int_0^t f_i(s)\mathrm{e}^{-h_i(t-s)}(t-s)\,\mathrm{d}s. \quad (32)$$

(6b) $h_1 = \omega_1, h_2 > \omega_2$:

$$x_1(t) = 0{\cdot}5\left\{\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}(t-s)\,\mathrm{d}s + \omega_{2o}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sinh[\omega_{2o}(t-s)]\,\mathrm{d}s\right\},$$

$$x_2(t) = 0{\cdot}5\left\{\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}(t-s)\,\mathrm{d}s - \omega_{2o}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sinh[\omega_{2o}(t-s)]\,\mathrm{d}s\right\}. \quad (33)$$

(7b) $h_1 > \omega_1, h_2 < \omega_2$:

$$x_1(t) = 0{\cdot}5\left\{\omega_{1o}^{-1}\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}\sinh[\omega_{1o}(t-s)]\,\mathrm{d}s + \omega_{2u}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sin[\omega_{2u}(t-s)]\,\mathrm{d}s\right\},$$

$$x_2(t) = 0{\cdot}5\left\{\omega_{1o}^{-1}\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}\sinh[\omega_{1o}(t-s)]\,\mathrm{d}s - \omega_{2u}^{-1}\int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}\sin[\omega_{2u}(t-s)]\,\mathrm{d}s\right\}. \quad (34)$$

(8b) $h_1 > \omega_1, h_2 = \omega_2$:

$$x_1(t) = 0{\cdot}5\left\{\omega_{1o}^{-1}\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}\sinh[\omega_{1o}(t-s)]\,\mathrm{d}s + \int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}(t-s)\,\mathrm{d}s\right\},$$

$$x_2(t) = 0{\cdot}5\left\{\omega_{1o}^{-1}\int_0^t f_1(s)\mathrm{e}^{-h_1(t-s)}\sinh[\omega_{1o}(t-s)]\,\mathrm{d}s - \int_0^t f_2(s)\mathrm{e}^{-h_2(t-s)}(t-s)\,\mathrm{d}s\right\}. \quad (35)$$

(9b) $h_1 > \omega_1, h_2 > \omega_2$:

$$x_1(t) = 0{\cdot}5\sum_{i=1}^{2}\omega_{io}^{-1}\int_0^t f_i(s)\mathrm{e}^{-h_i(t-s)}\sinh[\omega_{io}(t-s)]\mathrm{d}s,$$

$$x_2(t) = 0{\cdot}5\sum_{i=1}^{2} a_i\omega_{io}^{-1}\int_0^t f_i(s)\mathrm{e}^{-h_i(t-s)}\sinh[\omega_{io}(t-s)]\mathrm{d}s, \quad (36)$$

where $a_1 = -a_2 = 1$. All parameters in these solutions are described in corresponding relationships (24).

Both above solutions (28–36) for the forced responses and solutions (15–23) for the free motions are sufficiently versatile and general to allow a simple formulation of solutions for other particular variants of the system and for exciting forces as they are arbitrary time functions.

5. ILLUSTRATIVE EXAMPLE

As an illustrative example, the problem of damped free and forced responses of a T-d.o.f. system is solved on the assumption that $c_i = C = c$ and $k_i = K = k$, which simplifies the calculations. Considering expressions (15–23), it is shown that in this case the solutions for a free motion of types (1a), (2a), (3a), (6a), (9a), (equations (15–17, 20, 23)) can be determined in five intervals depending on the mutual relation between the value of viscous damping coefficient c and spring constant k. The damping coefficients and natural frequencies of undamped free vibration needed in formulating the solutions and in further analysis are evaluated from relations (9)

$$h_1 = 0{\cdot}5cm^{-1}, \quad h_2 = 1{\cdot}5cm^{-1} = 3h_1, \quad \omega_1^2 = km^{-1}, \quad \omega_1 = (km^{-1})^{1/2} = \omega,$$

$$\omega_2^2 = 3km^{-1} = 3\omega_1^2, \quad \omega_2 = \sqrt{3}(km^{-1})^{1/2} = \sqrt{3}\omega_1 = \sqrt{3}\omega.$$

The solutions of the problem $x_1(t)$ and $x_2(t)$ corresponding to the respective intervals are presented below.

(1a) If $0 < c < c_{2c}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 < \omega_2$:

$$x_{1,2}(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u}t) + B_1 \cos(\omega_{1u}t)] \pm \mathrm{e}^{-h_2 t}[A_2 \sin(\omega_{2u}t) + B_2 \cos(\omega_{2u}t)], \quad (37)$$

where if for example, $c = 0{\cdot}5(km)^{1/2}$,

$$h_1 = 0{\cdot}25(km^{-1})^{1/2} = 0{\cdot}25\omega < \omega_1, \quad h_2 = 0{\cdot}75(km^{-1})^{1/2} = 0{\cdot}75\omega < \omega_2,$$

$$\omega_{1u} = 0{\cdot}25\sqrt{15}(km^{-1})^{1/2} \cong 0{\cdot}97\omega, \quad \omega_{2u} = 0{\cdot}25\sqrt{39}(km^{-1})^{1/2} \cong 1{\cdot}56\omega.$$

(2a) If $c = c_{2c}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 = \omega_2$:

$$x_{1,2}(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u}t) + B_1(\omega_{1u}t)] \pm \mathrm{e}^{-h_2 t}[A_2 t + B_2], \quad (38)$$

where

$$h_1 = \sqrt{3}(km^{-1})^{1/2}/3 \cong 0{\cdot}58\omega < \omega_1, \quad h_2 = \sqrt{3}(km^{-1})^{1/2} = \sqrt{3}\omega \cong 1{\cdot}73\omega = \omega_2,$$

$$\omega_{1u} = (2/3)^{1/2}(km^{-1})^{1/2} \cong 0{\cdot}82\omega.$$

(3a) If $c_{2c} < c < c_{1c}$, $c_{1c} = 2(km)^{1/2}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 > \omega_2$:

$$x_{1,2}(t) = \mathrm{e}^{-h_1 t}[A_1 \sin(\omega_{1u}t) + B_1 \cos(\omega_{1u}t)] \pm \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o}t) + B_2 \cosh(\omega_{2o}t)], \quad (39)$$

where if for example $c = 1{\cdot}5(km)^{1/2}$,

$$h_1 = 0{\cdot}75(km^{-1})^{1/2} = 0{\cdot}75\omega < \omega_1, \quad h_2 = 2{\cdot}25(km^{-1})^{1/2} = 2{\cdot}25\omega > \omega_2,$$

$$\omega_{1u} = 0{\cdot}25\sqrt{7}(km^{-1})^{1/2} \cong 0{\cdot}66\omega, \quad \omega_{2o} = 0{\cdot}25\sqrt{33}(km^{-1})^{1/2} \cong 1{\cdot}44\omega.$$

(6a) If $c = c_{1c}$, $c_{1c} = 2(km)^{1/2}$, then $h_1 = \omega_1$, $h_2 > \omega_2$:

$$x_{1,2}(t) = \mathrm{e}^{-h_1 t}[A_1 t + B_1] \pm \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)], \tag{40}$$

where

$$h_1 = (km^{-1})^{1/2} = \omega = \omega_1, \quad h_2 = 3(km^{-1})^{1/2} = 3\omega > \omega_2, \quad \omega_{2o} = \sqrt{6}(km^{-1})^{1/2} \cong 2{\cdot}45\omega.$$

(9a) If $c > c_{1c}$, $c_{1c} = 2(km)^{1/2}$, then $h_1 > \omega_1$, $h_2 > \omega_2$:

$$x_{1,2}(t) = \mathrm{e}^{-h_1 t}[A_1 \sinh(\omega_{1o} t) + B_1 \cosh(\omega_{1o} t)] \pm \mathrm{e}^{-h_2 t}[A_2 \sinh(\omega_{2o} t) + B_2 \cosh(\omega_{2o} t)], \tag{41}$$

where if for example $c = 2{\cdot}5(km)^{1/2}$,

$$h_1 = 1{\cdot}25(km^{-1})^{1/2} = 1{\cdot}25\omega > \omega_1, \quad h_2 = 3{\cdot}75(km^{-1})^{1/2} = 3{\cdot}75\omega > \omega_2,$$

$$\omega_{1o} = 0{\cdot}75(km^{-1})^{1/2} = 0{\cdot}75\omega, \quad \omega_{2o} = 0{\cdot}25\sqrt{177}(km^{-1})^{1/2} \cong 3{\cdot}33\omega.$$

The results obtained show the evident influence of the magnitude of damping on the form of solutions for the damped free vibrations of a T-d.o.f. system. In the case considered, there are five different possible forms of solutions according to the value of the viscous damping coefficient c. The masses perform both synchronous and asynchronous component motions in the damped harmonic vibrations for little damping or aperiodic motions for average and large damping, and the solutions are the combinations of time functions expressing the damped harmonic vibrations and damped aperiodic motions. It should be noted that the influence of damping is greater for the asynchronous motion than for the synchronous one. The asynchronous motion is more strongly damped, since the middle damper does not work in the synchronous motion, and $h_2 > h_1$ (9). In this connection, the interval of existence of damped harmonic vibrations for the synchronous motion is always greater than that for the asynchronous motion, because $c_{1c} = 2(km)^{1/2} > c_{2c} = 2\sqrt{3}(km)^{1/2}/3$.

The sample problem of damped forced response is solved for a single harmonic force. The exciting force $F_1(t) = F\sin(pt)$, where F and p are the amplitude and frequency of the harmonic force, respectively, is subjected to the first mass (see Figure 1(b)), and the second mass is unloaded, i.e., $F_2(t) = 0$. On the basis of relations (25–27), the particular solutions $y_i(t)$ found for the harmonic excitation have the forms

(1) *Undercritical damping*: $h_i < \omega_i$,

$$y_i(t) = H_i \sin(pt) + K_i \cos(pt) + \mathrm{e}^{-h_i t}[M_i \sin(\omega_{iu} t) + N_i \cos(\omega_{iu} t)], \tag{42}$$

where

$$H_i = F(\omega_i^2 - p^2)\{m[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1}, \quad \omega_{iu} = (\omega_i^2 - h_i^2)^{1/2},$$

$$K_i = -2Fh_i p\{m[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1} = -N_i,$$

$$M_i = -Fp(\omega_{iu}^2 - h_i^2 - p^2)\{m\omega_{iu}[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1}. \tag{43}$$

(2) *Critical damping*: $h_i = \omega_i$,

$$y_i(t) = H_i \sin(pt) + K_i \cos(pt) + \mathrm{e}^{-h_i t}[M_i t + N_i], \tag{44}$$

where

$$H_i = F(\omega_i^2 - p^2)m^{-1}(\omega_i^2 + p^2)^{-2}, \quad M_i = Fpm^{-1}(\omega_i^2 + p^2)^{-1},$$

$$K_i = -2Fh_i pm^{-1}(\omega_i^2 + p^2)^{-2} = -N_i. \tag{45}$$

(3) *Overcritical damping*: $h_i > \omega_i$,

$$y_i(t) = H_i \sin(pt) + K_i \cos(pt) + \mathrm{e}^{-h_i t}[M_i \sinh(\omega_{io}t) + N_i \cosh(\omega_{io}t)], \tag{46}$$

where

$$H_i = F(\omega_i^2 - p^2)\{m[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1}, \quad \omega_{io} = (h_i^2 - \omega_i^2)^{1/2},$$

$$K_i = -2Fh_i p\{m[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1} = -N_i,$$

$$M_i = Fp(\omega_{io}^2 + h_i^2 + p^2)\{m\omega_{io}[(\omega_i^2 - p^2)^2 + 4h_i^2 p^2]\}^{-1}, \quad i = 1, 2. \tag{47}$$

Applying the above expressions, damped forced responses of the system are formulated in five analogous intervals, as for free motions. The following solutions of type (1b), (2b), (3b), (6b), (9b), (equations (28–30, 33, 36)) for $x_1(t)$ and $x_2(t)$ are obtained in corresponding intervals with respect to the values of damping coefficient c:

(1b) If $0 < c < c_{2c}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 < \omega_2$:

$$x_1(t) = A_1 \sin(pt - \varphi_1) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] + \mathrm{e}^{-h_2 t}[M_2 \sin(\omega_{2u}t) + N_2 \cos(\omega_{2u}t)]\},$$

$$x_2(t) = A_2 \sin(pt - \varphi_2) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] - \mathrm{e}^{-h_2 t}[M_2 \sin(\omega_{2u}t) + N_2 \cos(\omega_{2u}t)]\}. \tag{48}$$

(2b) If $c = c_{2c}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 = \omega_2$:

$$x_1(t) = A_1 \sin(pt - \varphi_1) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] + \mathrm{e}^{-h_2 t}[M_2 t + N_2]\},$$

$$x_2(t) = A_2 \sin(pt - \varphi_2) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] - \mathrm{e}^{-h_2 t}[M_2 t + N_2]\}. \tag{49}$$

(3b) If $c_{2c} < c < c_{1c}$, $c_{1c} = 2(km)^{1/2}$, $c_{2c} = 2\sqrt{3}(km)^{1/2}/3 \cong 1{\cdot}16(km)^{1/2}$, then $h_1 < \omega_1$, $h_2 > \omega_2$:

$$x_1(t) = A_1 \sin(pt - \varphi_1) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] + \mathrm{e}^{-h_2 t}[M_2 \sinh(\omega_{2o}t) + N_2 \cosh(\omega_{2o}t)]\},$$

$$x_2(t) = A_2 \sin(pt - \varphi_2) + 0{\cdot}5\{\mathrm{e}^{-h_1 t}[M_1 \sin(\omega_{1u}t) + N_1 \cos(\omega_{1u}t)] - \mathrm{e}^{-h_2 t}[M_2 \sinh(\omega_{2o}t) + N_2 \cosh(\omega_{2o}t)]\}. \tag{50}$$

(6b) If $c = c_{1c}$, $c_{1c} = 2(km)^{1/2}$, then $h_1 = \omega_1$, $h_2 > \omega_2$:

$$x_1(t) = A_1 \sin(pt - \varphi_1) + 0{\cdot}5\{e^{-h_1 t}[M_1 t + N_1] + e^{-h_2 t}[M_2 \sinh(\omega_{2o} t) + N_2 \cosh(\omega_{2o} t)]\},$$

$$x_2(t) = A_2 \sin(pt - \varphi_2) + 0{\cdot}5\{e^{-h_1 t}[M_1 t + N_1] - e^{-h_2 t}[M_2 \sinh(\omega_{2o} t) + N_2 \cosh(\omega_{2o} t)]\}. \tag{51}$$

(9b) If $c > c_{1c}$, $c_{1c} = 2(km)^{1/2}$, then $h_1 > \omega_1$, $h_2 > \omega_2$:

$$x_1(t) = A_1 \sin(pt - \varphi_1) + 0{\cdot}5\{e^{-h_1 t}[M_1 \sinh(\omega_{1o} t) + N_1 \cosh(\omega_{1o} t)] + e^{-h_2 t}[M_2 \sinh(\omega_{2o} t) + N_2 \cosh(\omega_{2o} t)]\},$$

$$x_2(t) = A_2 \sin(pt - \varphi_2) + 0{\cdot}5\{e^{-h_1 t}[M_1 \sinh(\omega_{1o} t) + N_1 \cosh(\omega_{1o} t)] - e^{-h_2 t}[M_2 \sinh(\omega_{2o} t) + N_2 \cosh(\omega_{2o} t)]\}, \tag{52}$$

where

$$A_1 = 0{\cdot}5\sqrt{(H_1 + H_2)^2 + (K_1 + K_2)^2}, \quad \tan\varphi_1 = (K_1 + K_2)/(H_1 + H_2),$$

$$A_2 = 0{\cdot}5\sqrt{(H_1 - H_2)^2 + (K_1 - K_2)^2}, \quad \tan\varphi_2 = (K_1 - K_2)/(H_1 - H_2). \tag{53}$$

Particular constants H_i, K_i, M_i, N_i can be calculated using relations (43, 45, 47) according to the value of damping coefficient c for corresponding intervals.

Each of solutions (48–52) is composed of two fundamental parts. The first part $A_i \sin(pt - \varphi_i)$, $(i = 1, 2)$ represents the steady state forced response of the system, and the other denotes the damped free vibration or non-oscillatory damped free motion produced by the application of the exciting force. This part of the harmonic response is transient. The transient free motion dies out after some time due to damping. Then, the forced vibrations are established. Finally, neglecting the transient terms in relationships (48–52), the steady state damped forced vibrations of a T-d.o.f. system can be presented in the following known form:

$$x_1(t) = A_1 \sin(pt - \varphi_1), \quad x_2(t) = A_2 \sin(pt - \varphi_2), \tag{54}$$

where

$$A_1 = \frac{F[(\omega_1^2 + \omega_2^2 - 2p^2)^2 + 4(h_1 + h_2)^2 p^2]^{1/2}}{2m\{[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2][(\omega_2^2 - p^2)^2 + 4h_2^2 p^2]\}^{1/2}},$$

$$A_2 = \frac{F[(\omega_1^2 - \omega_2^2)^2 + 4(h_1 - h_2)^2 p^2]^{1/2}}{2m\{[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2][(\omega_2^2 - p^2)^2 + 4h_2^2 p^2]\}^{1/2}},$$

$$\tan\varphi_1 = \frac{2h_1 p[(\omega_2^2 - p^2)^2 + 4h_2^2 p^2] + 2h_2 p[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2]}{(\omega_1^2 - p^2)[(\omega_2^2 - p^2)^2 + 4h_2^2 p^2] + (\omega_2^2 - p^2)[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2]},$$

$$\tan\varphi_2 = \frac{2h_1 p[(\omega_2^2 - p^2)^2 + 4h_2^2 p^2] - 2h_2 p[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2]}{(\omega_1^2 - p^2)[(\omega_2^2 - p^2)^2 + 4h_2^2 p^2] - (\omega_2^2 - p^2)[(\omega_1^2 - p^2)^2 + 4h_1^2 p^2]}.$$

A_i and φ_i $(i = 1, 2)$ denote amplitudes and phase angles of the steady state harmonic vibrations of the system respectively. The form of solutions (54) is valid for the whole interval of values of damping coefficient c. Generally, it is not dependent upon the magnitude of damping existing in the vibratory system considered.

6. CONCLUSIONS

In this paper, the damped vibration theory of a two-degree-of-freedom discrete system is developed. A classical linear spring–mass–damper model with arbitrary viscous damping under certain simplifying assumptions concerning physical parameters characterizing the system is discussed. Its motion is described by a set of two coupled non-homogeneous ordinary differential equations. The introduction of principal co-ordinates leads to the decoupling of the differential equations of motion. Solving them, exact analytical solutions for damped free and forced vibrations of the system due to arbitrary exciting forces are determined without any difficulty. The solutions can have nine possible forms depending on the mutual relations between viscous damping coefficients and spring constants. It is relevant to note that all coefficients shaping the solutions obtained are explicitly and directly expressed in terms of the physical parameters characterizing the vibrating system discussed. Free motions are described by the combinations of time functions expressing both the damped harmonic vibration (for undercritical damping cases) and damped aperiodic motion (for critical and overcritical damping cases). While underdamped cases are usually only significant in the study of mechanical vibrations, it seems however that all possible solutions presented in this paper allow a better understanding of the vibration phenomena occurring in an arbitrarily damped T-d.o.f. system. Moreover, a T-d.o.f. forced vibration analysis is of great technical importance because of the possibility of its wide application in the design of different types of discrete dynamic absorbers. General vibration theory derived here makes it possible to easily find solutions for other particular variants of this simplified system and for arbitrary exciting forces. The results obtained can also be a base for the formulation of vibration solutions for more general T-d.o.f. system shown in Figure 1(a), whose motion is governed by the fundamental general equations (1).

REFERENCES

1. J. P. DEN HARTOG 1956 *Mechanical Vibrations*. New York: McGraw-Hill.
2. S. ZIEMBA 1957 *Vibration Analysis*, Vol. I. Warsaw: PWN (in Polish).
3. S. ZIEMBA 1959 *Vibration Analysis*, Vol. II. Warsaw: PWN (in Polish).
4. R. E. D. BISHOP and D. C. JOHNSON 1960 *The Mechanics of Vibration*. Cambridge: Cambridge University Press.
5. F. S. TSE, I. E. MORSE and R. T. HINKLE 1963 *Mechanical Vibrations*. Boston: Allyn and Bacon.
6. R. SOLECKI and M. SZYMKIEWICZ 1964 *Rodlike and Surfacelike Systems. Dynamical Calculations*. Warsaw: Arkady (in Polish).
7. S. KALISKI 1966 *Vibrations and Waves in Solids*. Warsaw: IPPT PAN (in Polish).
8. J. C. SNOWDON 1968 *Vibrations and Shock in Damped Mechanical Systems*. New York: John Wiley.
9. W. NOWACKI 1972 *Dynamics of Structures*. Warsaw: Arkady (in Polish).
10. W. T. THOMSON 1972 *Theory of Vibrations with Applications*. New York: Prentice-Hall.
11. S. P. TIMOSHENKO, D. H. YOUNG and W. WEAVER JR 1974 *Vibration Problems in Engineering*. New York: John Wiley.
12. K. PISZCZEK and J. WALCZAK 1975 *Vibrations in Mechanical Engineering*. Warsaw: PWN (in Polish).
13. Z. OSIŃSKI 1978 *Vibration Theory*. Warsaw: PWN (in Polish).
14. J. B. HUNT 1979 *Dynamic Vibration Absorbers*. London: Mechanical Engineering Publications Ltd.
15. R. R. CRAIG JR 1981 *Structural Dynamics*. New York: John Wiley.

16. Z. Parszewski 1982 *Vibrations and Dynamics of Machines*. Warsaw: WNT (in Polish).
17. R. Gutowski and W. A. Swietlicki 1986 *Dynamics and Vibrations of Mechanical Systems*. Warsaw: PWN (in Polish).
18. Z. Osiński 1986 *Damping of Mechanical Vibrations*. Warsaw: PWN (in Polish).
19. J. Giergiel 1990 *Damping of Mechanical Vibrations*. Warsaw: PWN (in Polish).
20. B. G. Korenev and L. M. Reznikov 1993 *Dynamic Vibration Absorbers. Theory and Technical Applications*. Chichester: John Wiley.
21. S. S. Rao 1995 *Mechanical Vibrations*. Reading, Massachusetts: Addison-Wesley.
22. Z. Osiński 1998 *Damping of Vibrations*. Rotterdam, Netherlands: A. A. Balkema.
23. Z. Oniszczuk 1997 *Vibration Analysis of the Compound Continuous Systems with Elastic Constraints*. Rzeszów: Publishing House of Rzeszów University of Technology (in Polish).
24. Z. Oniszczuk 2000 *Journal of Sound and Vibration* **232**, 367–386. Transverse vibrations of elastically connected double-string complex system, Part II: forced vibrations.
25. Z. Oniszczuk 2000 *Machine Dynamics Problems* **24**, 2, 81–94. Dynamic vibration absorption in complex continuous systems.
26. E. V. Wilms and R. B. Pinkney 1991 *Journal of Sound and Vibration* **148**, 533–534. The finite residual motion of a damped two-degree-of-freedom vibrating system.
27. N. G. Stephen 1992 *Journal of Sound and Vibration* **156**, 557. Intuitive approach to "the finite residual motion of a damped two-degree-of-freedom vibrating system".
28. M. Gürgöze 1994 *Journal of Sound and Vibration* **169**, 409–410. Another solution to "the finite residual motion of a damped two-degree-of-freedom vibrating system".
29. Q. K. Liu and J. E. Sneckenberger 1994 *Journal of Sound and Vibration* **177**, 43–55. Free vibration analysis of linear vibrating deficient systems with arbitrary damping.
30. A. G. Thompson 1981 *Journal of Sound and Vibration* **77**, 403–415. Optimum tuning and damping of a dynamic vibration absorber applied to a force excited and damped primary system.
31. H. Sekiguchi and T. Asami 1984 *Bulletin of the Japan Society of Mechanical Engineers* **27**, 2939–2846. Theory of vibration isolation of a system with two degrees of freedom (1st report, motion excitation).
32. T. Asami, H. Sekiguchi and R. Nishiuchi 1986 *Bulletin of the Japan Society of Mechanical Engineers* **29**, 4297–4305. Theory of vibration isolation of a system with two degrees of freedom (2nd report, the case where a harmonic force acts on the mass located far from the supporting foundation).
33. A. F. Vakakis and S. A. Paipetis 1986 *Journal of Sound and Vibration* **105**, 49–60. The effect of a viscously damped dynamic absorber on a linear multi-degree-of-freedom system.
34. U. V. Tambe and H. R. Srirangarajan 1986 *Journal of Sound and Vibration* **110**, 359–362. Optimum Lanchester damper for a non-linear main system.
35. I. N. Jordanov and B. I. Cheshankov 1988 *Journal of Sound and Vibration* **123**, 157–170. Optimal design of linear and non-linear dynamic vibration absorbers.
36. U. V. Tambe, G. Chandrasekarappa and H. R. Srirangarajan 1988 *Journal of Sound and Vibration* **124**, 381–384. Voigt dynamic vibration absorber.
37. T. Dahlberg 1989 *Journal of Sound and Vibration* **132**, 518–522. On optimal use of the mass of a dynamic vibration absorber.
38. L. H. Van Berkel and S. E. Semercigil 1991 *Journal of Sound and Vibration* **150**, 322–329. Tuned vibration absorber for secondary structures.
39. N. Olgac and B. Holm-hansen 1994 *Journal of Sound and Vibration* **176**, 93–104. A novel active vibration absorption technique: delayed resonator.
40. A. Tondl and R. Nabergoj 2000 *Journal of Sound and Vibration* **234**, 611–624. Dynamic absorbers for an externally excited pendulum.
41. T. Aida, T. Aso, K. Takeshita, T. Takiuchi and T. Fujii 2001 *Journal of Sound and Vibration* **242**, 333–353. Improvement of the structure damping performance by interconnection.
42. E. Kamke 1971 *Differentialgleichungen*: *Lösungsmethoden und Lösungen*. New York: Chelsea (in German).
43. G. Birkhoff and G.-C. Rota 1989 *Ordinary Differential Equations*. New York: John Wiley.