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REPLY TO: DISCUSSION ON "FREE VIBRATIONS OF BEAMS WITH GENERAL BOUNDARY CONDITIONS"

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The author would like to thank Dr Zhou for his interest and comments.

1. First of all, I am sorry for not being able to give a complete review of the relevant investigations such as references [1, 2]. However, the author cannot agree to Dr Zhou's claim that the technique proposed in reference [3] is essentially the same as the one he previously used. Although he started with expressing the beam displacement as the superposition of a Fourier series and a polynomial function, that is,

$$w(x) = \sum_{m=1}^{\infty} A_m \sin m\pi x + c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad 0<x<1, \tag{1}$$

there are probably two flaws related to his derivations. First, it is incorrect to rewrite equation (1) as

$$w(x) = \sum_{m=1}^{\infty} A_m (\sin m\pi x + c_{m0} + c_{m1} x + c_{m2} x^2 + c_{m3} x^3), \tag{2}$$

where

$$c_{mn} = c_n / A_m \quad (n = 0, 1, 2, 3). \tag{3}$$

This can be easily seen from substitution of equation (3) into equation (2):

$$w(x) = \sum_{m=1}^{\infty} (A_m \sin m\pi x + c_0 + c_1 x + c_2 x^2 + c_3 x^3) \tag{4}$$

which is clearly not the same as equation (1).

Secondly, since the series expansion of the displacement, equation (1), is defined over an open region, $0<x<1$ (excluding the two end points, 0 and 1), equation (1) cannot be automatically substituted into the boundary conditions to determine the unknown coefficients without first knowing that the Fourier series and its relevant derivatives are actually converging at the end points.

Based on the above arguments, especially the first one, it can be said that the Fourier series expansion, equation (1), is not actually used in references [1, 2]. However, this should not deny that the static beam functions are a set of useful trial functions that satisfy the general beam boundary conditions. Besides, amending polynomials to sinusoidal functions to satisfy a specific boundary condition has long been a viable practice in using the Ritz or Rayleigh–Ritz method.

The technique used in reference [3] is essentially an improved Fourier series method. There the auxiliary function, $p(x)$, is specifically introduced to overcome the potential

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discontinuity (convergence) problems of (the Fourier expansions of) the displacement and its derivatives at the boundary points. Regardless boundary conditions, the auxiliary function is required to satisfy

$$p'''(0) = w'''(0), \quad p'''(L) = w'''(L), \tag{5, 6}$$

$$p'(0) = w'(0), \quad p'(L) = w'(L), \tag{7, 8}$$

for a cosine series expansion; and

$$p''(0) = w''(0), \quad p''(L) = w''(L), \tag{9, 10}$$

$$p(0) = w(0), \quad p(L) = w(L), \tag{11, 12}$$

for a sine series expansion.

Also, it must be pointed out that although a simple polynomial was specifically used in reference [3], the auxiliary function $p(x)$ can actually be *any* continuous function defined over [0, L]. This implies that there is a large (theoretically, an infinite) number of possible choices for the auxiliary function. For each specific $p(x)$, one is able to obtain a corresponding set of trial functions. Therefore, what is described in reference [3] is actually a general technique for widening the application and/or improving the accuracy and convergence of the traditional Fourier series method, rather than finding a set of trial functions.

2. In Dr Zhou's letter, it is claimed that if a beam is subject to a series of cosine static loads, the static beam functions will be right in the form given in reference [3]. However, it is not clear why for free vibration the expression of the displacement (or mode shapes) is dictated by that of the static loads. A related question is then whether the modes obtained from the static beam functions (sine, cosine or both) are only those that spatially conform to the given load pattern. In addition, why will fourth order polynomials be used in the static (cosine) beam functions instead of the polynomials no more than the third order as explicitly mentioned in reference [1]? A fourth order polynomial is no longer a general solution of the beam equation. However, it is clear from reference [3] that if the auxiliary function is a polynomial, the convergence speed of a cosine series expansion cannot be fully achieved without including the fourth power term or higher.

3. For a completely free beam, the H matrix will become singular. However, this is not actually a serious problem. In numerical calculations, the free–free boundary condition can be considered as a special case, when the stiffnesses of restraining springs are very small in comparison with the rigidity of the beam. As shown in Table 1, the singularity problem can be easily overcome by artificially adding a sufficiently soft spring to the beam. Since the Fourier series method is proposed as a unified solution for all boundary

TABLE 1

Frequency parameters, $\mu = (L^2\omega\sqrt{\rho A/D})^{1/2}$, for a free–free beam

Mode	$\mu = (L^2\omega\sqrt{\rho A/D})^{1/2}$				
	$\hat{K}_0 L = 10^{-4}$	$\hat{k}_1 L^3 = 10^{-4}$	$\hat{K}_0 L = 10^{-6}$	$\hat{k}_1 L^3 = 10^{-6}$	Exact [4]
1	4·730064	4·730044	4·730043	4·730043	4·730041
2	7·85323	7·853218	7·853218	7·853217	7·853205
3	10·99565	10·99564	10·99564	10·99564	10·99561
4	14·13725	14·13724	14·13724	14·13724	14·13717
5	17·2789	17·27889	17·27889	17·27889	17·27876

conditions, it seems not absolutely necessary to develop a different formulation solely for the free–free boundary condition, especially in view that the exact beam functions are already well studied and readily available.

4. As aforementioned, the static beam functions represent a (complete) set of trial functions that satisfy the general beam boundary conditions. Although the solution obtained from a complete set is generally known to converge from above in the Rayleigh–Ritz method, its convergence speed may not be always easily estimated. However, the convergence theorems for Fourier series expansions have been well established in mathematics [5] and they set up the foundation for the work described in reference [3]. A more thorough discussion about the convergence of the sine and cosine series expansions (of the beam displacement) can be found in reference [6]. It is concluded that for a generally supported beam, the cosine and sine series solutions are, respectively, converging according to m^{-2} and m^{-1}. However, for the cases when a beam is simply supported with only rotational restraints, the convergence speed of the sine series solution can be greatly increased to m^{-3}. Similarly, for beams that are not allowed to rotate at each end, the cosine series will be converging at a speed of m^{-4}. Several numerical examples are presented there to verify the conclusions.

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