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CORRECTED SOLVABILITY CONDITIONS FOR NON-LINEAR ASYMMETRIC VIBRATIONS OF A CIRCULAR PLATE

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An investigation into non-linear asymmetric vibrations of a clamped circular plate under a harmonic excitation is made. We re-examined a primary resonance studied by Sridhar, Mook and Nayfeh, in which the frequency of excitation is near the natural frequency of an asymmetric mode of the plate. We corrected their solvability conditions and found that in the absence of internal resonance, the steady state response can have not only the form of standing wave but also the form of travelling wave, which is a remarkable contrast to their conclusion.

1. INTRODUCTION

A clamped circular plate experiences mid-plane stretching when deflected. The influence of this stretching on the dynamic response increases with the amplitude of the response. This situation can be described with non-linear strain-displacement equations and a linear stress–strain law which give us the dynamic analogue of the von Karman equations with geometric non-linearity. Non-linear dynamic responses of a clamped circular plate subjected to harmonic excitations have been investigated by two approaches. One is to include symmetric vibrations and the other asymmetric vibrations. For symmetric responses, Sridhar *et al.* [1] and Hadian and Nayfeh [2] studied primary resonance of a circular plate with three-mode interaction. Lee and Kim [3] studied combination resonances of the plate. In these studies the steady state response can only have the superposition of standing wave components.

For asymmetric responses, Sridhar *et al.* [4] derived solvability conditions for modal interactions of a clamped circular plate. These conditions are said to be general in the sense of two aspects. First, the conditions include asymmetric vibrations as well as symmetric vibrations. Second, the conditions include all of natural modes. They used these conditions to examine two cases. One is the case of the absence of internal resonance and the other is the case of the internal resonance involving four modes. They concluded that in the absence of internal resonance, the steady state response can only have the form of a standing wave. When the frequency of excitation is near the highest frequency involved in the internal resonance, the steady state response was said to be given by a superposition of the standing wave components of all the modes involved in the internal resonance, or a superposition of the standing wave components of all the lower modes and the travelling

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wave component of the highest mode involved in the internal resonance. However, they did not plot any illustrations on the responses.

In this study, we re-examined the analysis by Sridhar *et al.* [4] to find that they had misderived the solvability conditions in applying the method of multiple scales. We corrected the conditions and found that in the absence of internal resonance, the steady state response can have not only the form of a standing wave but also the form of a travelling wave, which is a remarkable contrast to their conclusion [4].

2. GOVERNING EQUATIONS

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [5]. These equations are simplified to fit the special case of uniform plates, and damping and forcing terms are added. Then the non-dimensionalized equations of motion of a circular plate shown in Figure 1 are given as follows [4]:

$$\frac{\partial^2 w}{\partial t^2} + \nabla^4 w = \varepsilon\left[L(w, F) - 2c\frac{\partial w}{\partial t} + p^*(r, \theta, t)\right], \tag{1}$$

$$\nabla^4 F = \left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right), \tag{2}$$

where

$$\begin{aligned} L(w, F) = &\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) \\ &- 2\left(\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial F}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right), \end{aligned} \tag{3}$$

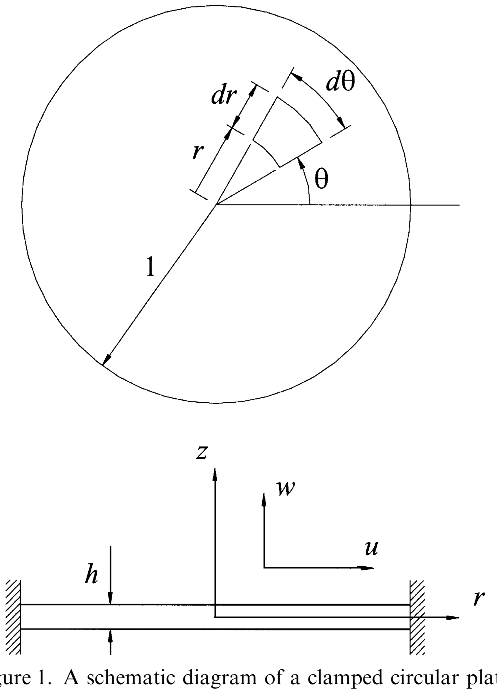

Figure 1. A schematic diagram of a clamped circular plate.

ε is a small parameter given by the Poisson ratio v and the thickness of the plate h, c is the damping coefficient, p^* is the forcing function, w is the deflection of the middle surface, F is the force function which satisfies the in-plane equilibrium conditions (in-plane inertia is neglected), and

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2. \tag{4}$$

The boundary conditions are developed for the plates which are clamped along a circular edge. For all t, and θ,

$$w = 0 \qquad \frac{\partial w}{\partial r} = 0 \quad \text{at } r = 1, \tag{5a, b}$$

$$\frac{\partial^2 F}{\partial r^2} - v\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \quad \text{at } r = 1, \tag{6a}$$

$$\frac{\partial^3 F}{\partial r^3} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{2+v}{r^2}\frac{\partial^3 F}{\partial r \partial \theta^2} - \frac{3+v}{r^3}\frac{\partial^2 F}{\partial \theta^2} = 0 \quad \text{at } r = 1. \tag{6b}$$

In addition, it is necessary to require the solution to be bounded at $r = 0$.

3. SOLUTION

In order to re-examine the analysis by Sridhar *et al.* [4] we expand w and F as follows:

$$w(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j w_j(r, \theta, T_0, T_1, \ldots),$$

$$F(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j F_j(r, \theta, T_0, T_1, \ldots), \tag{7a, b}$$

where $T_n = \varepsilon^n t$.

Substituting equations (7) into equations (1) and (2), and equating coefficients of like powers of ε yields

$$D_0^2 w_0 + \nabla^4 w_0 = 0, \tag{8}$$

$$\nabla^4 F_0 = \left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial \theta}\right)^2 - \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right), \tag{9}$$

$$\begin{aligned} D_0^2 w_1 + \nabla^4 w_1 = & -2D_0 D_1 w_0 - 2cD_0 w_0 + p^* + \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial F_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_0}{\partial \theta^2}\right) \\ & + \frac{\partial^2 F_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right) - 2\left(\frac{1}{r}\frac{\partial^2 F_0}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial F_0}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial \theta}\right), \end{aligned} \tag{10}$$

etc., where $D_n = \partial/\partial T_n$.

Substituting equations (7) into equations (5) and (6), and equating coefficients of like powers of ε, one obtains

$$w_j = 0, \qquad \frac{\partial w_j}{\partial r} = 0 \quad \text{at } r = 1, \tag{11a, b}$$

$$\frac{\partial^2 F_j}{\partial r^2} - \nu\left(\frac{\partial F_j}{\partial r} + \frac{\partial^2 F_j}{\partial \theta^2}\right) = 0 \quad \text{at } r = 1, \tag{12a}$$

$$\frac{\partial^3 F_j}{\partial r^3} + \frac{\partial^2 F_j}{\partial r^2} - \frac{\partial F_j}{\partial r} + (2+\nu)\frac{\partial^3 F_j}{\partial r \partial \theta^2} - (3+\nu)\frac{\partial^2 F_j}{\partial \theta^2} = 0 \quad \text{at } r = 1 \tag{12b}$$

for all j, θ and t. In addition, it is necessary to require w_j and F_j, for all j, to be bounded at $r = 0$.

It follows from equations (8) and (11) that

$$w_0 = \sum_{m=1}^{\infty} \phi_{0m}(r) A_{0m} \mathrm{e}^{\mathrm{i}\omega_{0m} T_0} + \sum_{n,m=1}^{\infty} \phi_{nm}(r)\{A_{nm}\mathrm{e}^{\mathrm{i}(\omega_{nm} T_0 + n\theta)} + B_{nm}\mathrm{e}^{\mathrm{i}(\omega_{nm} T_0 - n\theta)}\} + cc, \tag{13}$$

where the $\phi_{nm}(r)$ are the linear shape functions in the r direction given by

$$\phi_{nm} = \kappa_{nm}\left[\mathrm{J}_n(\eta_{nm} r) - \frac{\mathrm{J}_n(\eta_{nm})}{\mathrm{I}_n(\eta_{nm})}\mathrm{I}_n(n_{nm} r)\right] \tag{14}$$

the κ_{nm} are chosen so that

$$\int_0^1 r\phi_{nm}^2 \, \mathrm{d}r = 1.$$

The function J_n are Bessel function of the first kind, of order n, and the function I_n are modified Bessel function of the first kind, of order n. The η_{nm} are the roots of $\mathrm{I}_n(\eta)\mathrm{J}'_n(\eta) - \mathrm{I}'_n(\eta)\mathrm{J}_n(\eta) = 0, \omega_{nm} = \eta_{nm}^2$, the A_{nm} and the B_{nm} are complex functions of the all T_n for $n \geqslant 1$ which are to be determined from the solvability conditions at the next level of approximation, and cc represents the complex conjugate of the preceding terms. In ϕ_{nm} and ω_{nm}, the first subscript refers to the numbers of nodal diameters and the second subscript refers to the number of nodal circles including the boundary. The first summation of the right-hand side in equation (13) represents a superposition of symmetric standing waves. And the second summation looks a superposition of asymmetric travelling waves, but it contains both travelling and standing waves depending on the relative values of the A_{nm} and B_{nm}. The solution can also be written in the following equivalent form:

$$w_0 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \phi_{nm}(r) u_{nm}(T_0, T_1, \ldots)\mathrm{e}^{\mathrm{i}n\theta}, \tag{15}$$

where

$$u_{nm} = A_{nm}\mathrm{e}^{\mathrm{i}\omega_{nm} T_0} + \bar{B}_{nm}\mathrm{e}^{-\mathrm{i}\omega_{nm} T_0}, \tag{16}$$

$\phi_{-nm} = \phi_{nm}$ and $\omega_{-nm} = \omega_{nm}$. Because w_0 is real,

$$A_{-nm} = B_{nm}. \tag{17}$$

Substituting equation (15) into equation (9) leads to

$$\nabla^4 F_0 = \sum_{n,p=-\infty}^{\infty} \sum_{m,q=1}^{\infty} E(nm, pq) u_{nm} u_{pq} \mathrm{e}^{\mathrm{i}(n+p)\theta}, \tag{18}$$

where

$$E(nm,pq) = -\frac{np}{r^2}\left(\phi'_{nm} - \frac{\phi_{nm}}{r}\right)\left(\phi'_{pq} - \frac{\phi_{pq}}{r}\right) - \frac{1}{2r}(\phi'_{nm}\phi'_{pq})' + \frac{1}{2r^2}(p^2\phi''_{nm}\phi_{pq} + n^2\phi''_{pq}\phi_{nm})$$

and primes denote differentiation with respect to r.

An expansion for F_0 is assumed in the following form:

$$F_0 = \sum_{n=-\infty}^{\infty} U_n(r, T_0, T_1, \ldots)\mathrm{e}^{\mathrm{i}n\theta}. \tag{19}$$

Substituting equation (19) into equation (18), multiplying the result by $\mathrm{e}^{-\mathrm{i}a\theta}$, and integrating from $\theta = 0$ to 2π, we obtain

$$\nabla_a^4 U_a = \sum_{n=-\infty}^{\infty}\sum_{m,q=1}^{\infty} E(nm,pq)u_{nm}u_{pq}, \tag{20}$$

where

$$p = a - n \tag{21}$$

and

$$\nabla_a^4 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{a^2}{r^2}\right]^2.$$

Then U_a is further expanded as

$$U_a = \sum_{n=1}^{\infty} v_{an}(T_0, T_1, \ldots)\psi_{an}(r), \tag{22}$$

where the ψ_{an} are the eigenfunctions of the following problem:

$$(\nabla_a^4 - \xi_{an}^4)\psi_{an} = 0 \quad \text{in } r = [0, 1],$$

where ψ_{an} is bounded at $r = 0$ and, from equations (12),

$$\psi''_{an} - \nu(\psi'_{an} - a^2\psi_{an}) = 0 \quad \text{and} \quad \psi'''_{an} + \psi''_{an} - \psi'_{an} - a^2[(2+\nu)\psi'_{an} - (3+\nu)\psi_{an}] = 0$$

for all θ and t at $r = 1$. It follows that

$$\psi_{an} = \tilde{\kappa}_{an}[J_a(\xi_{an}r) - \tilde{c}_{an}I_a(\xi_{an}r)], \tag{23}$$

where the $\tilde{\kappa}_{an}$ are chosen so that

$$\int_0^1 r\psi_{an}^2\,\mathrm{d}r = 1,$$

$$\tilde{c}_{an} = \frac{[a(a+1)(\nu+1) - \xi_{an}^2]J_a(\xi_{an}) - \xi_{an}(\nu+1)J_{a-1}(\xi_{an})}{[a(a+1)(\nu+1) + \xi_{an}^2]I_a(\xi_{an}) - \xi_{an}(\nu+1)I_{a-1}(\xi_{an})}$$

and ξ_{an} are the roots of

$$a^2(a+1)(\nu+1)[J_a(\xi_{an}) - \tilde{c}_{an}I_a(\xi_{an})] - a^2\xi_{an}(\nu+1)[J_{a-1}(\xi_{an}) - \tilde{c}_{an}I_{a-1}(\xi_{an})] + a\xi_{an}^2[J_a(\xi_{an}) + \tilde{c}_{an}I_a(\xi_{an})] - \xi_{an}^3[J_{a-1}(\xi_{an}) + \tilde{c}_{an}I_{a-1}(\xi_{an})] = 0.$$

Substituting equation (22) into equation (20), multiplying the result by $r\psi_{ab}$, and then integrating from $r = 0$ to 1, one obtains

$$v_{ab}(T_0, T_1, \ldots) = \sum_{n=-\infty}^{\infty} \sum_{m,q=1}^{\infty} G(nm, pa; ab) u_{nm} u_{pq}, \tag{24}$$

where

$$G(nm, pq; ab) = \xi_{ab}^{-4} \int_0^1 r\psi_{ab} E(nm, pq)\, \mathrm{d}r \tag{25}$$

and p, a and n are related according to equation (21). It follows from equations (24), (22) and (19) that

$$F_0 = \sum_{a,n=-\infty}^{\infty} \sum_{b,m,q=1}^{\infty} \psi_{ab} G(nm, pq; ab) u_{nm} u_{pq} \mathrm{e}^{\mathrm{i}a\theta}, \tag{26}$$

where $p = a - n$.

Substituting equations (26) and (15) into equation (10) leads to

$$\begin{aligned} D_0^2 w_1 + \nabla^4 w_1 = & \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} -2\mathrm{i}\omega_{nm} \phi_{nm} [(D_1 A_{nm} + c_{nm} A_{nm}) \mathrm{e}^{\mathrm{i}\omega_{nm} T_0} \\ & - (D_1 \bar{B}_{nm} + c_{nm} \bar{B}_{nm}) \mathrm{e}^{-\mathrm{i}\omega_{nm} T_0})] \mathrm{e}^{\mathrm{i}n\theta} + p^*(r, \theta, t) \\ & + \sum_{a,n,c=-\infty}^{\infty} \sum_{b,m,d,q=1}^{\infty} G(nm, pq; ab) \hat{E}(cd, ab) u_{cd} u_{pq} u_{nm} \mathrm{e}^{\mathrm{i}(a+c)\theta}, \end{aligned} \tag{27}$$

where modal damping has been assumed, p^* has been expanded as

$$p^*(r, \theta, t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} P_{nm} \phi_{nm} \mathrm{e}^{\mathrm{i}(n\theta + \tau_{nm})} \cos \lambda T_0$$

and

$$\begin{aligned} \hat{E}(cd, pq) = & \frac{\phi''_{cd}}{r} \left(\psi'_{ab} - \frac{a^2}{r} \psi_{ab} \right) + \frac{\psi''_{ab}}{r} \left(\phi'_{cd} - \frac{c^2}{r} \phi_{cd} \right) \\ & + \frac{2ac}{r^2} \left(\psi'_{ab} - \frac{1}{r} \psi_{ab} \right) \left(\phi'_{cd} - \frac{1}{r} \phi_{cd} \right). \end{aligned}$$

Because w_1 and w_0 satisfy the same boundary conditions, an expansion for w_1 is assumed in the form

$$w_1 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} H_{nm}(T_0, T_1, \ldots) \phi_{nm} \mathrm{e}^{\mathrm{i}n\theta}. \tag{28}$$

Substituting equation (28) into equation (27), multiplying the result by $r\phi_{kl}(r)\mathrm{e}^{-\mathrm{i}k\theta}$, and integrating the result from $r = 0$ to 1 and $\theta = 0$ to 2π, one obtains

$$\begin{aligned} & D_0^2 H_{kl} + \omega_{kl}^2 H_{kl} \\ & = -2\mathrm{i}\omega_{kl} [(D_1 A_{kl} + c_{kl} A_{kl}) \mathrm{e}^{\mathrm{i}\omega_{kl} T_0} - (D_1 \bar{B}_{kl} + c_{kl} \bar{B}_{kl}) \mathrm{e}^{-\mathrm{i}\omega_{kl} T_0}] + \tfrac{1}{2} P_{kl} \mathrm{e}^{\mathrm{i}\tau_{kl}} [\mathrm{e}^{\mathrm{i}\lambda T_0} + \mathrm{e}^{\mathrm{i}\lambda T_0}] \\ & + \sum_{n,c=-\infty}^{\infty} \sum_{d,m,q=1}^{\infty} \Gamma(kl, cd, nm, pq) \sum_{j=1}^{\infty} S_j \mathrm{e}^{\mathrm{i}\Lambda_j T_0}, \quad k = 0, 1, \ldots, \quad l = 1, 2, \ldots, \end{aligned} \tag{29}$$

where

$$\Gamma(kl, cd, nm, pq) = \sum_{b=1}^{\infty} G(nm, pq; ab) \int_0^1 r\phi_{kl} \hat{E}(cd, ab)\, \mathrm{d}r, \tag{30a}$$

$$a = k - c, \qquad p = k - c - n, \tag{30b, c}$$

Λ_j are frequency combinations, and S_j are functions of A_{nm} and B_{nm}. Both Λ_j and S_j are listed in Appendix A. Up to now, the result may be said to be the same as one by Sridhar *et al*. [4] if we ignore several misprints in reference [4].

Eliminating the secular terms (the coefficients of $e^{\pm i\omega_{kl}T_0}$) from the right-hand sides of equation (29), we obtain the following solvability conditions:

$$-2i\omega_{kl}(D_1A_{kl} + c_{kl}A_{kl}) + A_{kl}\left\{\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl}A_{kl}\bar{A}_{kl}\right\}$$

$$+ 2(1-\delta_{k0})B_{kl}\left\{\sum_{m=1}^{\infty}\hat{\gamma}_{klkm}A_{km}\bar{B}_{km} - \hat{\gamma}_{klkl}A_{kl}\bar{B}_{kl}\right\} + N_{kl}^A + R_{kl}^A = 0, \tag{31a}$$

$$2i\omega_{kl}(D_1\bar{B}_{kl} + c_{kl}\bar{B}_{kl}) + \bar{B}_{kl}\left\{\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl}B_{kl}\bar{B}_{kl}\right\}$$

$$+ 2(1-\delta_{k0})\bar{A}_{kl}\left\{\sum_{m=1}^{\infty}\hat{\gamma}_{klkm}A_{km}\bar{B}_{km} - \hat{\gamma}_{klkl}A_{kl}\bar{B}_{kl}\right\} + N_{kl}^B + R_{kl}^B = 0, \tag{31b}$$

where δ_{k0} are the Kronecker delta, $R_{kl}^{A,B}$ are terms due to internal resonances, if any, $N_{kl}^{A,B}$ are terms due to the external excitation, if any, and γ_{klnm} and $\hat{\gamma}_{klkm}$ are constants given in Appendix A. It is noted that these solvability conditions are different from those by Sridhar *et al.* [4]. Terms including expressions $A_{kl}\bar{A}_{kl}$, $B_{kl}\bar{B}_{kl}$ and $2(1-\delta_{k0})$ in equation (31) are added to their solvability conditions. We can only conjecture two possible ways how this deviation happens. First, they might fail to collect all of the secular terms from equation (29). Second, the misprints might influence seriously the solvability conditions.

4. STEADY STATE RESPONSES

In this study, we consider a primary resonance in the absence of internal resonance. The frequency of excitation λ is near natural frequency ω_{fg}. We introduce a detuning parameter, σ, defined as follows:

$$\lambda = \omega_{fg} + \hat{\sigma}, \qquad \hat{\sigma} = \varepsilon\sigma, \tag{32a, b}$$

$$N_{fg}^A = \tfrac{1}{2}P_{fg}e^{i(\sigma T_1 + \tau_{nm})}, \qquad N_{fg}^B = \tfrac{1}{2}P_{fg}e^{-(i\sigma T_1 - \tau_{nm})} \tag{33a, b}$$

and

$$N_{kl}^{A,B} = 0 \quad \text{for } kl \neq fg. \tag{33c}$$

Next we let

$$A_{nm} = \tfrac{1}{2}a_{nm}e^{i\alpha_{nm}}, \qquad B_{nm} = \tfrac{1}{2}b_{nm}e^{i\beta_{nm}}, \tag{34a, b}$$

where $a_{nm}, b_{nm}, \alpha_{nm}$ and β_{nm} are real functions of T_1. Substituting equations (33) and (34) into (31) and separating the result into real imaginary parts yields

$$\omega_{kl}(a_{kl}{}' + c_{kl}a_{kl}) - \tfrac{1}{4}(1-\delta_{k0})b_{kl}\hat{s}_{kl}^s - \tfrac{1}{2}\delta_{kf}\delta_{lg}P_{fg}\sin\mu_{fg}^a = 0, \tag{35a}$$

$$\omega_{kl}(b_{kl}{}' + c_{kl}b_{kl}) + \tfrac{1}{4}(1-\delta_{k0})a_{kl}\hat{s}_{kl}^s - \tfrac{1}{2}\delta_{kf}\delta_{lg}P_{fg}\sin\mu_{fg}^b = 0, \tag{35b}$$

$$\omega_{kl}a_{kl}\alpha_{kl}' + \tfrac{1}{8}a_{kl}(s_{kl} - \gamma_{klkl}a_{kl}^2) + \tfrac{1}{4}(1-\delta_{k0})b_{kl}\hat{s}_{kl}^c + \tfrac{1}{2}\delta_{kf}\delta_{lg}P_{fg}\cos\mu_{fg}^a = 0, \tag{35c}$$

$$\omega_{kl}b_{kl}\beta_{kl}' + \tfrac{1}{8}b_{kl}(s_{kl} - \gamma_{klkl}b_{kl}^2) + \tfrac{1}{4}(1-\delta_{k0})a_{kl}\hat{s}_{kl}^c + \tfrac{1}{2}\delta_{kf}\delta_{lg}P_{fg}\cos\mu_{fg}^b = 0, \tag{35d}$$

where

$$s_{kl} = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(a_{nm}^2 + b_{nm}^2), \tag{36a}$$

$$\hat{s}_{kl}^s = \sum_{m=1}^{\infty}\hat{\gamma}_{klkm}a_{km}b_{km}\sin(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{36b}$$

$$\hat{s}_{kl}^c = \sum_{m=1}^{\infty}(1-\delta_{ml})\hat{\gamma}_{klkm}a_{km}b_{km}\cos(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{36c}$$

$$\mu_{fg}^a = \sigma T_1 + \tau_{fg} - \alpha_{fg}, \qquad \mu_{fg}^b = \sigma T_1 - \tau_{fg} - \beta_{fg}. \tag{37a, b}$$

Terms including $\hat{s}$ in the system of equations (35), and terms of $\gamma_{klkl}a_{kl}^2$ and $\gamma_{klkl}b_{kl}^2$, respectively, in equations (35c) and (35d) make system (35) different from the corresponding system by Sridhar *et al.* [4]. Terms including $\hat{s}$ have something to do with internal resonance. Since in this study we consider the case of no internal resonance, these terms do not affect the result at all. Terms of $\gamma_{klkl}a_{kl}^2$ and $\gamma_{klkl}b_{kl}^2$, therefore, are the effective difference between both systems by us and Sridhar *et al.* [4].

Each equilibrium solution $(a_{kl}' = b_{kl}' = \mu_{fg}^{a'} = \mu_{fg}^{b'} = 0)$ of the system of autonomous ordinary differential equations (35) is corresponding to a steady state response. It follows immediately from equations (35) that

$$a_{kl} = b_{kl} = 0 \quad \text{for } kl \neq fg$$

and that neither a_{fg} and b_{fg} can be zero. Thus, the steady state solution is given by equations (35) which can be rewritten as

$$\omega_{fg}c_{fg} = \frac{P_{fg}}{2a_{fg}}\sin\mu_{fg}^a, \qquad \omega_{fg}\sigma + \tfrac{1}{8}\gamma_{fgfg}(a_{fg}^2 + 2b_{fg}^2) = -\frac{P_{fg}}{2a_{fg}}\cos\mu_{fg}^a, \tag{38a, b}$$

$$\omega_{fg}c_{fg} = \frac{P_{fg}}{2b_{fg}}\sin\mu_{fg}^b, \qquad \omega_{fg}\sigma + \tfrac{1}{8}\gamma_{fgfg}(2a_{fg}^2 + b_{fg}^2) = -\frac{P_{fg}}{2b_{fg}}\cos\mu_{fg}^b. \tag{39a, b}$$

Instead of $2b_{fg}^2$ and $2a_{fg}^2$, respectively, in equations (38b) and (39b), Sridhar *et al.* [4] obtained b_{fg}^2 and a_{fg}^2, respectively, in their corresponding equations.

Using equations (37), (34) and (13), one can write the steady state forced responses as

$$w = \phi_{fg}\{a_{fg}\cos(\lambda t - \mu_{fg}^a + f\theta + \tau_{fg}) + b_{fg}\cos(\lambda t - \mu_{fg}^b - f\theta - \tau_{fg})\} + O(\varepsilon), \tag{40}$$

which is a superposition of two travelling waves rotating clockwise and counterclockwise, respectively. Form (40) can also be written as follows:

$$w = Z_1\cos(\lambda t + \zeta_1)\phi_{fg}\cos f\theta + Z_2\cos(\lambda t + \zeta_2)\phi_{fg}\sin f\theta + O(\varepsilon), \tag{41}$$

which is the superposition of two standing waves. The constants Z_1, Z_2, ζ_1 and ζ_2 are given in Appendix A.

It depends on the relations between $a_{fg}, b_{fg}, \mu^a_{fg}$, and μ^b_{fg} whether form (40) turns out to be standing or travelling wave. When $a_{fg} = b_{fg}$ and $\mu^a_{fg} = \mu^b_{fg}$, form (40) can be reduced to the standing wave of the form

$$w = 2\phi_{fg} a_{fg} \cos(\lambda t - \mu^a_{fg}) \cos(f\theta + \tau_{fg}) + O(\varepsilon), \tag{42}$$

which is similar to the natural mode corresponding to ω_{fg}. Form (40) gives travelling wave otherwise.

5. NUMERICAL RESULTS

For a numerical example we consider the case of $f = 1$ and $g = 1$. The corresponding mode has one nodal diameter and no other nodal circle but the boundary. In Figure 2 the amplitudes a_{11} and b_{11} are plotted as functions of detuning parameter $\hat{\sigma} = \varepsilon\sigma$ when $\omega_{11} = 21{\cdot}2604$ [6], $v = 1/3$, $\varepsilon = 0{\cdot}001067$, $\varepsilon c = 0{\cdot}01$, $\varepsilon P_{11} = 4$ and $\tau_{11} = 0$. Branches SS1, US1, US2 and SS2 represent the standing waves, while branches ST1, UT1, and UT2 represent travelling waves. Solid and dotted lines denote, respectively, stable and unstable responses. Except for the instability of branch US1, the response in the form of standing wave is the response of Duffing oscillator. The stable response in the form of travelling wave, $\{ST1_A, ST1_B\}$ represents $\{a_{11}, b_{11}\}$ or $\{b_{11}, a_{11}\}$. When $\hat{\sigma} < \hat{\sigma}_1$ and $\hat{\sigma}_1 < \hat{\sigma} < \hat{\sigma}_2$, respectively, standing and travelling waves coexist in reality. While standing and travelling waves coexist when $\hat{\sigma}_2 < \hat{\sigma} < \hat{\sigma}_3$, standing wave only exists when $\hat{\sigma} > \hat{\sigma}_3$. This result is remarkably different from one by Sridhar *et al.* [4]. They expected that the response is in the form of standing wave, which is the response of Duffing oscillator. We believe that this difference comes from the correction of solvability conditions.

Considering the case of no internal resonance as the case of 1:1 internal resonance between two modes having shapes of $\phi_{fg} \cos f\theta$ and $\phi_{fg} \sin f\theta$ corresponding to one natural frequency ω_{fg}, Nayfeh and Vakakis [7] observed the coexistence of subharmonic standing and travelling waves in the case of subharmonic resonance. We believe that their result supports the validity of our observation.

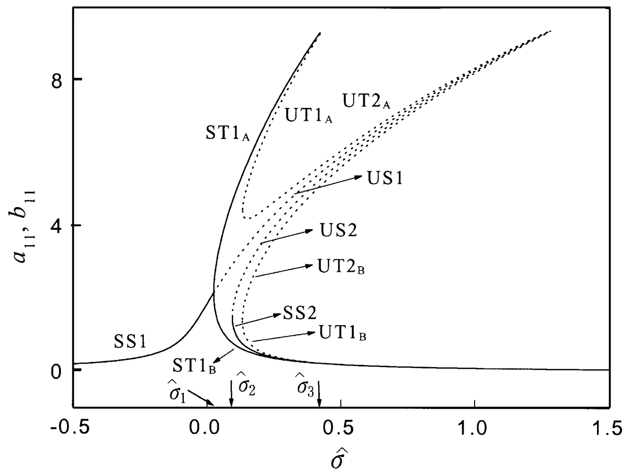

Figure 2. Variations of the amplitudes with detuning parameter $\hat{\sigma} = \varepsilon\sigma$ when $\varepsilon P_{11} = 4$. ——, stable; – – –, unstable.

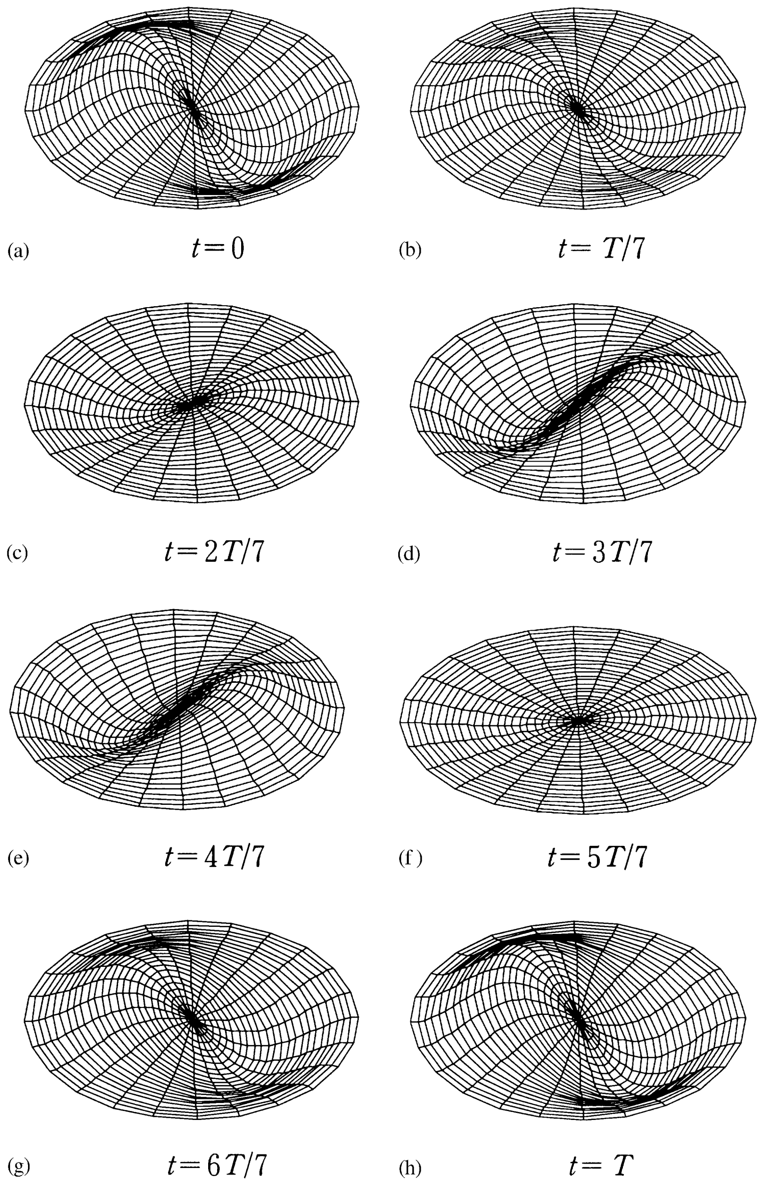

Figure 3. Deflections of the circular plate for one period of excitation ($T = 2\pi/\lambda$) when $a_{11} = 1{\cdot}1608$, $b_{11} = 1{\cdot}1608$, $\mu^a_{11} = 3{\cdot}0179$, $\mu^b_{11} = 3{\cdot}0179$, $\omega_{11} = 21{\cdot}2604$, $\hat{\sigma} = 0{\cdot}1$ and $\tau_{11} = 0$. A standing wave ($a_{11} = b_{11}$).

In order to show the deflection of the plate we consider the case of $\hat{\sigma} = 0{\cdot}1$, in which there exist three stable responses (one is a standing wave and two are travelling waves). The initial condition determines which deflection is to be realized. Figures 3–5 represent deflections corresponding to the stable responses of the plate for one period of excitation $T(= 2\pi/\lambda)$. A standing wave ($a_{11} = b_{11}$) is shown in Figure 3(a–h), in each of which we can see a nodal line at 5 min past 7 o'clock. Figures 4 and 5 represent travelling waves, which are rotating clock-wise ($a_{11} > b_{11}$) and counterclockwise ($a_{11} < b_{11}$) respectively. It is noted that the dominant amplitude (a_{11} or b_{11}) determines the direction of the rotation.

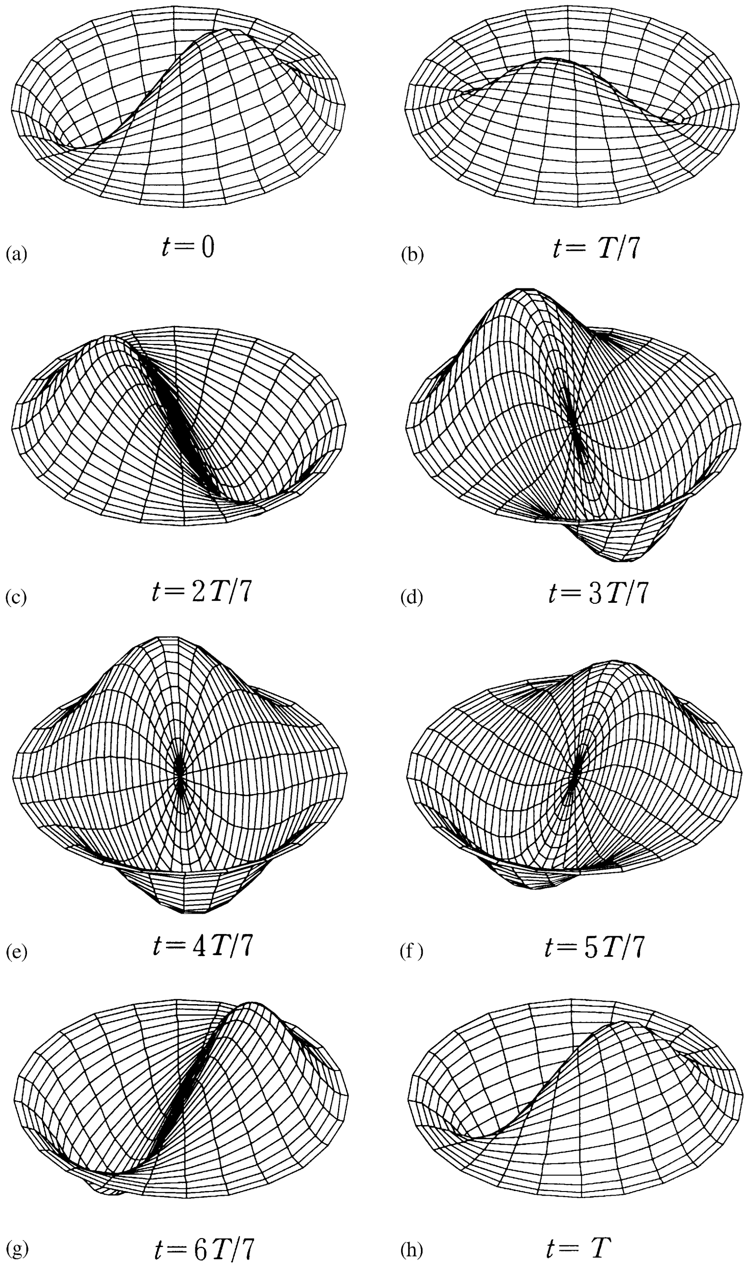

Figure 4. Deflections of the circular plate for one period of excitation ($T = 2\pi/\lambda$) when $a_{11} = 4{\cdot}7974$, $b_{11} = 0{\cdot}7464$, $\mu_{11}^{a} = 0{\cdot}5352$, $\mu_{11}^{b} = 0{\cdot}07943$, $\omega_{11} = 21{\cdot}2604$, $\hat{\sigma} = 0{\cdot}1$ and $\tau_{11} = 0$. A travelling wave ($a_{11} > b_{11}$).

In these figures we can see that the period of deflection is the same as the one of excitation, which means the response of a primary resonance.

6. CONCLUSIONS

In order to investigate non-linear asymmetric vibrations of a clamped circular plate under a harmonic excitation, we examine a primary resonance, in which the frequency of

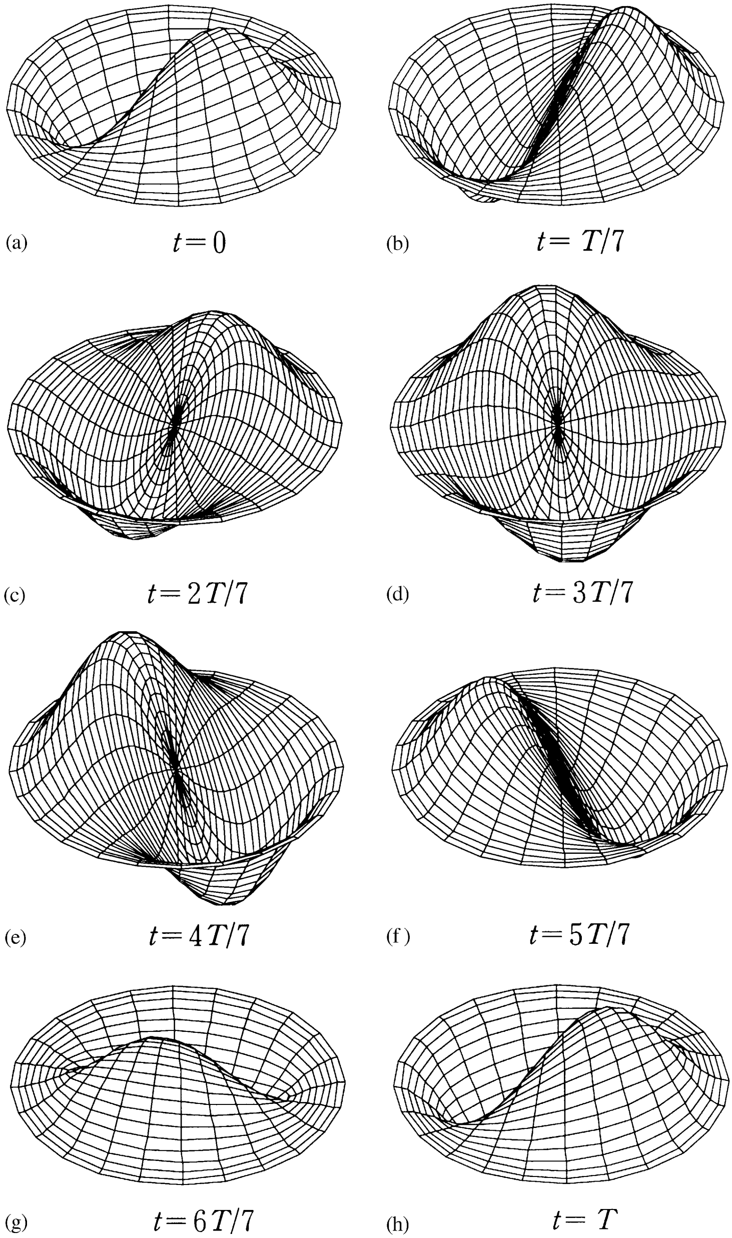

Figure 5. Deflections of the circular plate for one period of excitation ($T = 2\pi/\lambda$) when $a_{11} = 0{\cdot}7464$, $b_{11} = 4{\cdot}7974$, $\mu^a_{11} = 0{\cdot}07943$, $\mu^b_{11} = 0{\cdot}5352$ $\omega_{11} = 21{\cdot}2604$, $\hat{\sigma} = 0{\cdot}1$ and $\tau_{11} = 0$. A travelling wave ($a_{11} < b_{11}$).

excitation is near the natural frequency of an asymmetric mode of the plate. We re-examined the analysis by Sridhar *et al.* [4] to correct their solvability conditions and to find that in the absence of internal resonance, the steady state response can have not only the form of a standing wave but also the form of a travelling wave, which is a remarkable contrast to their conclusion.

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REFERENCES

1. S. Sridhar, D. T. Mook and A. H. Nayfeh 1975 *Journal of Sound and Vibration* **41**, 359–373. Non-linear resonances in the forced responses of plates. Part I: symmetric responses of circular plates.
2. J. Hadian and A. H. Nayfeh 1990 *Journal of Sound and Vibration* **142**, 279–292. Modal interaction in circular plates.
3. W. K. Lee and C. H. Kim 1995 *American Society of Mechanical Engineers Journal of Applied Mechanics* **62**, 1015–1022. Combination resonances of a circular plate with three-mode interaction.
4. S. Sridhar, D. T. Mook and A. H. Nayfeh 1978 *Journal of Sound and Vibration* **59**, 159–170. Non-linear resonances in the forced responses of plates. Part II: asymmetric responses of circular plates.
5. G. J. Efstathiades 1971 *Journal of Sound and Vibration* **16**, 231–253. A new approach to the large-deflection vibrations of imperfect circular disks using Galerkin's procedure.
6. A. W. Leissa 1969 *NASA-SP*-160. Vibration of plates.
7. T. A. Nayfeh and A. F. Vakakis 1994 *International Journal of Non-linear Mechanics* **29**, 233–245. Subharmonic travelling waves in a geometrically non-linear circular plate.

APPENDIX A

Coefficients S_j and frequency combinations Λ_j in equations (29) are given in Table A1.

Table A1

j	S_j	Λ_j
1	$A_{cd}A_{nm}A_{pq}$	$\omega_{cd}+\omega_{nm}+\omega_{pq}$
2	$A_{cd}A_{nm}\bar{B}_{pq}$	$\omega_{cd}+\omega_{nm}-\omega_{pq}$
3	$A_{cd}\bar{B}_{nm}A_{pq}$	$\omega_{cd}-\omega_{nm}+\omega_{pq}$
4	$\bar{B}_{cd}A_{nm}A_{pq}$	$-\omega_{cd}+\omega_{nm}+\omega_{pq}$
5	$\bar{B}_{cd}\bar{B}_{nm}\bar{B}_{pq}$	$-\omega_{cd}-\omega_{nm}-\omega_{pq}$
6	$\bar{B}_{cd}\bar{B}_{nm}A_{pq}$	$-\omega_{cd}-\omega_{nm}+\omega_{pq}$
7	$\bar{B}_{cd}A_{nm}\bar{B}_{pq}$	$-\omega_{cd}+\omega_{nm}-\omega_{pq}$
8	$A_{cd}\bar{B}_{nm}\bar{B}_{pq}$	$\omega_{cd}-\omega_{nm}-\omega_{pq}$

$$\gamma_{klnm}=\Gamma(kl,kl,nm,-nm)+\Gamma(kl,-nm,kl,nm)+\Gamma(kl,nm,-nm,kl),$$
$$\hat{\gamma}_{klkm}=\Gamma(kl,km,km,-kl)+\Gamma(kl,-kl,km,km)+\Gamma(kl,km,-kl,km),$$

$$Z_1=\sqrt{a_{11}^2+b_{11}^2+2a_{11}b_{11}\cos(\mu_{11}^a-\mu_{11}^b-2\tau_{11})},$$
$$Z_2=\sqrt{a_{11}^2+b_{11}^2-2a_{11}b_{11}\cos(\mu_{11}^a-\mu_{11}^b-2\tau_{11})},$$

$$\tan\zeta_1=\frac{a_{11}\sin(\mu_{11}^a-\tau_{11})+b_{11}\sin(\mu_{11}^b+\tau_{11})}{a_{11}\cos(\mu_{11}^a-\tau_{11})+b_{11}\cos(\mu_{11}^b+\tau_{11})},$$
$$\tan\zeta_2=\frac{-a_{11}\cos(\mu_{11}^a-\tau_{11})+b_{11}\cos(\mu_{11}^b+\tau_{11})}{a_{11}\sin(\mu_{11}^a-\tau_{11})-b_{11}\sin(\mu_{11}^b+\tau_{11})}.$$