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LETTERS TO THE EDITOR

CONSTRUCTION AND ANALYSIS OF A NON-STANDARD FINITE DIFFERENCE SCHEME FOR THE BURGERS–FISHER EQUATION

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The main purpose of this Letter is to construct a non-standard finite difference scheme and study its associated properties for the Burgers–Fisher partial differential equation [1]

$$u_t + auu_x = Du_{xx} + \lambda u(1-u), \tag{1}$$

where (a, D, λ) are non-negative parameters. This equation, with $\lambda = 0$, has been used to investigate sound waves in a viscous medium by Lighthill [2]. However, it was originally introduced by Burgers [3] to model one-dimensional turbulence and can also be applied to waves in fluid-filled viscous elastic tubes and magnetohydrodynamic waves in a medium with finite electrical conductivity [4]. With all three parameters positive, equation (1) corresponds to Burgers equation having non-linear reaction. An alternative view of equation (1) is to consider it as a modified Fisher equation [5]

$$u_t = Du_{xx} + \lambda u(1-u), \tag{2}$$

that has non-linear advection, i.e., auu_x. In summary, the Burgers–Fisher equation includes the effects of non-linear advection, linear diffusion, and non-linear logistic reaction.

It should be noted that equation (1), for special values for its three parameters (a, D, λ), reduces to six equations that play fundamental roles in the mathematical modelling of a range of important physical and engineering phenomena. For example, $\lambda = 0$, gives the Burgers equation [3, 4]; $a = 0$ is the Fisher equation [4, 5]; $D = 0$ is the diffusionless Burgers equation with non-linear reaction [6]; $a = 0$ and $D = 0$ correspond to Logistic ordinary differential equation [5]; $a = 0$ and $\lambda = 0$ are the linear diffusion equation; and, $D = 0$ and $\lambda = 0$ are the diffusionless Burgers equation. Consequently, any finite difference scheme constructed for equation (1) also provides corresponding discrete models for the above indicated other six ordinary and partial differential equations.

Before proceeding with the construction of the non-standard numerical scheme for equation (1), a brief summary of its significant mathematical properties will be given. The reason why this is being done is to make sure that the non-standard finite difference scheme [7] to be derived has these properties, otherwise, numerical instabilities will occur [8]. First note that equation (1) has two fixed-points or constant solutions,

$$\bar{u}^{(1)} = 0, \quad \bar{u}^{(2)} = 1. \tag{3}$$

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The first fixed-point is linearly unstable, while the second is linearly stable [1, 4, 5]. A second very important pair of properties are related to the positivity of the solutions and their boundedness [5, 8, 9], i.e.,

$$0 \leqslant u(x, 0) \leqslant 1 \Rightarrow 0 \leqslant u(x, t) \leqslant 1, \quad t > 0. \tag{4}$$

Third, the Burgers–Fisher equation has travelling wave solutions which take the form

$$u(x, t) = f(x - ct), \quad z = x - ct, \tag{5}$$

where $f(z)$ has a second derivative and the speed of propagation, c, has the minimum value [5]

$$c \geqslant 2\sqrt{D\lambda} \tag{6}$$

Observe that the speed of propagation depends on both the diffusion coefficient, D, and the "reaction rate", λ. An increase in either causes the minimum speed of the travelling wave to become greater. From a physics point of view, this conclusion is reasonable; for example, an increasing λ corresponds to increasing the rate of reaction and, consequently, the increased chemical interactions force the reaction to speed up. Similar remarks apply to increasing D.

Based on the previous work of Mickens on non-standard finite difference schemes [7, 8] and the enforcement of a positivity condition [9, 10], the following discrete model is selected for equation (1):

$$\frac{u_m^{k+1} - u_m^k}{\Delta t} + \left(\frac{a}{2}\right)\left[\frac{(u_m^k)^2 - (u_{m-1}^k)^2}{\Delta x}\right] = D\left[\frac{u_{m+1}^k - 2u_m^k + u_{m-1}^k}{(\Delta x)^2}\right] + \lambda u_m^k - \lambda u_m^k u_m^{k+1}, \tag{7}$$

where

$$x \to x_m = (\Delta x)m, \quad t \to t_k = (\Delta t)k, \quad u(x, t) \to u(x_m, t_k) \simeq u_m^k. \tag{8}$$

Note that this scheme has the following features: (1) the first order time derivative is replaced by a forward-Euler form; (2) the non-linear advection term is written in conservative form, i.e., $uu_x = (u^2/2)_x$, and a backward-Euler representation is then used for the first order space derivative; (3) a central difference scheme replaces the second order space derivative in the diffusion term; (4) the non-linear u^2 term, in the reaction expression, is modelled non-locally, i.e.,

$$u^2 \to u_m^k u_m^{k+1}. \tag{9}$$

The detailed reasons as to why these particular selections were made can be located in the papers of Mickens [8–10].

Inspection of equation (7) shows that it is linear in u_m^{k+1} and solving for it gives the expression:

$$[1 + (\lambda\Delta t)u_m^k]u_m^{k+1} = u_m^k\left[1 - 2DR - \left(\frac{a\beta}{2}\right)u_m^k\right] + \left(\frac{a\beta}{2}\right)(u_{m-1}^k)^2 + 2DR\left(\frac{u_{m+1}^k + u_{m-1}^k}{2}\right) + (\lambda\Delta t)u_m^k, \tag{10}$$

where β and R are defined as

$$\beta \equiv \frac{\Delta t}{\Delta x}, \quad R \equiv \frac{\Delta t}{(\Delta x)^2}. \tag{11}$$

The discrete version of the positivity condition is

$$0 \leqslant u_m^k \Rightarrow 0 \leqslant u_m^{k+1} \quad \text{fixed } k \quad \text{all } m. \tag{12}$$

It follows that the condition of equation (12) is satisfied if

$$1 - 2DR - \left(\frac{a\beta}{2}\right) u_m^k \geqslant 0. \tag{13}$$

If further, $0 \leqslant u_m^k \leqslant 1$, then this inequality can be written as

$$2DR + \left(\frac{a\beta}{2}\right) \leqslant 1. \tag{14}$$

Using the definitions of β and R, the following bound is placed on the time step-size, Δt, if Δx, the space step-size is given:

$$\Delta t \leqslant \frac{2(\Delta x)^2}{4D + a\Delta x}. \tag{15}$$

Previous work by Mickens [6], for the case where $D = 0$, shows that a more restricted bound must be used, namely,

$$\Delta t \leqslant \frac{(\Delta x)^2}{4D + a\Delta x}, \tag{16}$$

and where

$$\beta \equiv \frac{\Delta t}{\Delta x} \leqslant \frac{\Delta x}{4D + a\Delta x}, \quad R \equiv \frac{\Delta t}{(\Delta x)^2} \leqslant \frac{1}{4D + a\Delta x}. \tag{17}$$

Equation (10) can be rewritten as

$$u_m^{k+1} = \frac{u_m^k[(1 - 2DR) - (a\beta/2)u_m^k] + (a\beta/2)(u_{m-1}^k)^2 + DR(u_{m+1}^k + u_{m-1}^k) + (\lambda\Delta t)u_m^k}{1 + (\lambda\Delta t)u_m^k} \tag{18}$$

A rather direct calculation allows the following boundedness condition to be proven (see Appendix A for this result):

$$\begin{matrix} 0 \leqslant u_m^k \leqslant 1 \\ k \text{ fixed all } m \end{matrix} \Rightarrow \begin{matrix} 0 \leqslant u_m^{k+1} \leqslant 1, \\ k \text{ fixed all } m. \end{matrix} \tag{19}$$

This is the discrete equivalence of the results given by equation (4). These results, of course, depend on having the conditions listed in equations (16) and (17) hold.

To test the above derived non-standard finite difference scheme for the Burgers–Fisher partial differential equation, the following initial-value problem was studied:

$$u(x, 0) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x > 1. \end{cases} \tag{20}$$

Numerical solutions for $t > 0$ were obtained for a variety of parameter values (a, D, λ), and γ, $0 < \gamma \leqslant 1$, where γ appears as

$$\Delta t = \frac{\gamma(\Delta x)^2}{4D + a\Delta x}, \quad \beta = \frac{\gamma \Delta x}{4D + a\Delta x}, \quad R = \frac{\gamma}{4D + a\Delta x}. \tag{21}$$

This inclusion of γ allowed the study of how the numerical solutions varied, if at all, when the time/space step-size relation became more restricted. The following is a summary of these results:

(1) For $0 < \gamma \leqslant 1$, the numerical solution had the same general properties as the expected exact solutions to the Burgers–Fisher equation [1, 5], namely, the numerical solution is smooth, monotonically decreasing, and bounded by one; see Figure 1.

(2) The numerical scheme was also tested for values of γ in the interval $1 < \gamma \leqslant 2$. The calculated numerical solutions again had the correct behavior. In fact, it works for γ slightly above the value 2, e.g., $\gamma = 2{\cdot}05$. However, for $\gamma = 2{\cdot}051$, the numerical solution oscillates and the boundedness condition is violated since values of $u > 1$ were obtained; see Figure 2.

(3) The threshold speed condition given by equation (6) was verified. This was done by fixing D at the value $D = 1$ and running the simulation for values of λ up to 100.

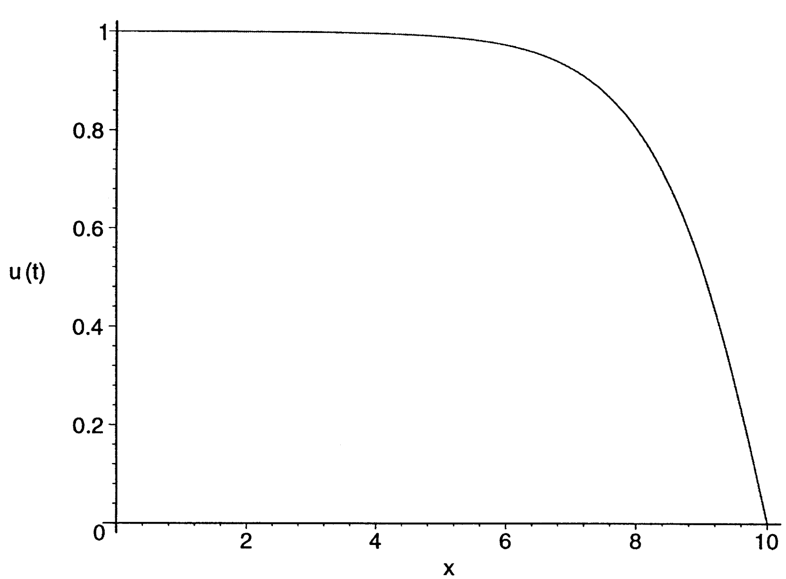

Figure 1. Plot of $u(x, 25)$ for $a = D = \lambda = 1$ with $\Delta x = 0{\cdot}1$.

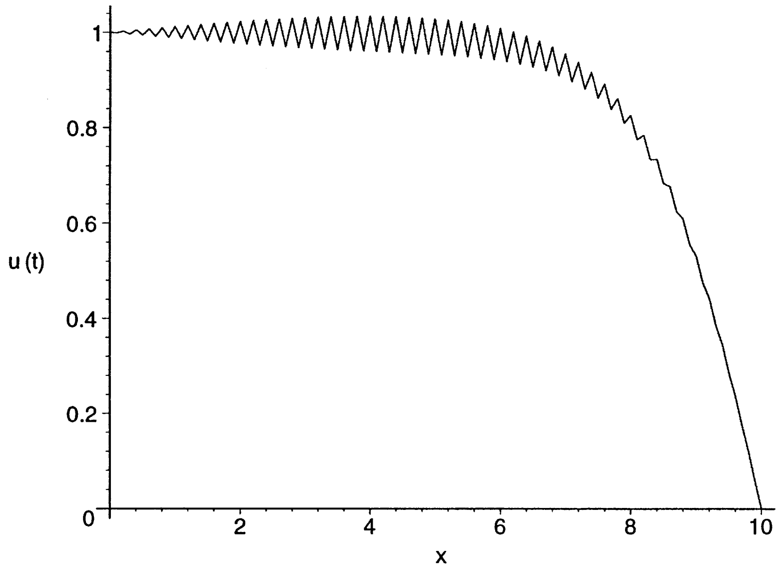

Figure 2. Plot of $u(x, 51{\cdot}275)$ for $a = D = \lambda = 1$, $\gamma = 2{\cdot}051$, and $\Delta x = 0{\cdot}1$.

Since equation (6) is symmetric in λ and D, the obtained solutions have general validity. As expected, an increase in the value of λD led to the appropriate increase in the speed of the waveform [4, 5].

A large number of simulations were carried out for various values of the parameters (a, D, λ, γ). In all cases, the numerical results were consistent with the above indicated theoretical predictions. Figures 1 and 2 are but a small sample of the performed numerical experiments.

In conclusion, a non-standard finite difference scheme was constructed for the Burgers–Fisher partial differential equation. This equation can model sound waves in a viscous medium with logistic reaction. However, more complex functional forms may be used. For example, a reaction term $\lambda u^2(1 - u)$ could be substituted for the logistic expression. The resulting equation would then describe (among many possible applications) sound waves in a viscous medium having combustion dynamics [10, 11]. This new scheme has the correct fixed-points, satisfies both the positivity and boundedness conditions of equation (1), and is easy to implement for obtaining numerical solutions since the scheme is effectively explicit (see equation (18)). The validity of the scheme depends on the inequalities stated in equations (16) and (17), i.e., once Δx is selected, then Δt must satisfy equation (16). Numerical studies indicate that the derived non-standard scheme provides excellent numerical solutions.

A future project is to study the Burgers–Fisher equation having a non-linear diffusion term. This equation takes the form

$$u_t + auu_x = D(uu_x)_x + \lambda u(1 - u). \tag{22}$$

It will be of interest to see if positivity and boundedness conditions can be made to hold for a possible non-standard finite difference scheme.

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APPENDIX A: BOUNDEDNESS CONDITION

The boundedness condition given by equation (19) can be derived by use of the following arguments. First, note that if $0 \leqslant u_m^k \leqslant 1$ for fixed k and all values of m, then

$$0 \leqslant \left(\frac{a\beta}{2}\right)(u_{m-1}^k)^2 + DR(u_{m+1}^k + u_{m-1}^k) \leqslant \left(\frac{a\beta}{2}\right) + 2DR. \tag{A.1}$$

Second, define $y(v)$ as

$$y(v) = v(c_1 - c_2 v), \tag{A.2}$$

where

$$v = u_m^k, \quad c_1 = 1 - 2DR, \quad c_2 = \frac{a\beta}{2}. \tag{A.3}$$

Then the maximum of $y(v)$ occurs at

$$\bar{v} = \frac{c_1}{2c_2} = \frac{1 - 2DR}{a\beta}, \tag{A.4}$$

or using equation (21),

$$\bar{v} = \frac{1}{\gamma} + \frac{2D(2-\gamma)}{a\gamma\Delta x} > 1. \tag{A.5}$$

The last inequality follows from $0 < \gamma \leqslant 1$, $D > 0$, $a > 0$, and $\Delta x > 0$. Now the function $y(v)$ is zero at $v = 0$ and monotonically increases to its maximum value $y(\bar{v})$. Since $\bar{v} > 1$ and further since $v = u_m^k$ is restricted to values between zero and one (because $0 \leqslant u_m^k \leqslant 1$), the maximum of $y(v)$ over the interval $0 \leqslant v \leqslant 1$ occurs at the endpoint, $v = 1$, *i.e.*,

$$\max_{0 \leqslant v \leqslant 1} y(v) = y(1) = c_1 - c_2. \tag{A.6}$$

Consequently, it can be concluded that

$$\max_{\substack{0 \leqslant u_m^k \leqslant 1 \\ k \text{ fixed} \\ \text{all } m}} u_m^k\left[(1-2DR) - \left(\frac{a\beta}{2}\right)u_m^k\right] = 1 - 2DR - \left(\frac{a\beta}{2}\right), \tag{A.7}$$

and

$$\begin{aligned} &u_m^k\left[(1-2DR) - \left(\frac{a\beta}{2}\right)u_m^k\right] + \left(\frac{a\beta}{2}\right)(u_{m-1}^k)^2 + DR(u_{m+1}^k + u_{m-1}^k) + (\lambda\Delta t)u_m^k \\ &\leqslant 1 - 2DR - \left(\frac{a\beta}{2}\right) + \left(\frac{a\beta}{2}\right) + 2DR + (\lambda\Delta t)u_m^k = 1 + (\lambda\Delta t)u_m^k. \end{aligned} \tag{A.8}$$

Dividing both sides of equation (A.8) by $[1 + (\lambda\Delta t)u_m^k]$ gives

$$\frac{u_m^k[(1-2DR) - (a\beta/2)u_m^k] + (a\beta/2)(u_{m-1}^k)^2 + DR(u_{m+1}^k + u_{m-1}^k) + (\lambda\Delta t)u_m^k}{1 + (\lambda\Delta t)u_m^k} \leqslant 1. \tag{A.9}$$

However, the left side is just u_m^{k+1}, i.e., see equation (18). Hence, it follows that the inequality relations of equation (19) are correct.