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ON "A CHECK ON THE ACCURACY OF TIMOSHENKO'S BEAM THEORY"

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1. INTRODUCTION

In a recent article, Renton [1] presented a comparison between the standing wave natural frequency predictions of Timoshenko beam theory (TBT) for a long beam of thin rectangular cross-section, when the flexural mode is sinusoidal in the axial co-ordinate x, and those of a plane stress elastodynamic solution; the latter may be regarded as the exact benchmark. Renton's main concern was the accuracy of the lower of the two frequency predictions of TBT (TBT1), although the results presented in Figure 1 of reference [1] also show the frequency prediction of the so-called second spectrum (TBT2), and re-presents evidence that this mode should be disregarded. Renton concluded that TBT1 may be regarded as accurate when the wavelength exceeds the beam depth, and underestimates the natural frequency by 5·8% when *wavelength/beam depth* is equal to unity, and that simple (Euler–Bernoulli) beam theory overestimates natural frequency by 21·5% even when wavelength is five times the beam depth. The value of the shear coefficient is not stated explicitly in reference [1], but it is clearly $\kappa = \frac{5}{6}$; this value was derived previously by Renton [2] as the limiting case of a more general expression for the rectangular cross-section, when *width/depth* ratio approaches zero, as in plane stress conditions.

In what follows, this accuracy check is repeated using the shear coefficient

$$\kappa = \frac{5(1 + v)}{(6 + 5v)}, \tag{1}$$

an expression which has recently been derived by Hutchinson [3] for the thin rectangle, together with some other published values. In reference [4] it is shown that Hutchinson's general expression for the coefficient is equivalent to that derived by the present author [5, 6] employing two different methods. Included in the comparison is a value derived by Cowper [7] which, from the present author's knowledge of the literature over the past 25 years, is probably the most popular choice, but not the best. Also presented are the predictions of a two-coefficient theory [6], which employs both the above expression (1), and Cowper's. In addition, further comments on the second spectrum, TBT2, are made.

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2. THEORY

The fourth order differential equation of TBT is

$$EI\,\frac{\partial^4 v}{\partial x^4} + \rho A\,\frac{\partial^2 v}{\partial t^2} - \rho I\left(1 + \frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\,\frac{\partial^4 v}{\partial t^4} = 0, \tag{2}$$

where the customary notation is employed. For a standing wave, assume that the transverse displacement is sinusoidal in both axial co-ordinate and time, that is

$$v = \sin \lambda x \sin \omega t \tag{3}$$

and introduce the non-dimensional frequency parameter

$$\gamma = \frac{\rho \omega^2}{G \lambda^2}, \tag{4}$$

both in accordance with the notation of reference [1], then it is straightforward to obtain the two frequency parameters as

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \frac{[(\kappa + E/G + \kappa A/I\lambda^2) \mp \sqrt{(\kappa + E/G + \kappa A/I\lambda^2)^2 - 4\kappa E/G}]}{2}. \tag{5}$$

When the shear coefficient takes the value $\kappa = \frac{5}{6}$, the above reduces to equation (2) of reference [1]; note that wavelength is $2\pi/\lambda$ in this notation, and λ is the wavenumber. As the wavelength approaches zero, so

$$\gamma_1 \to \kappa,\ \gamma_2 \to 2(1 + \nu). \tag{6}$$

For the equivalent travelling wave solution, γ is equal to $(c_p/c_s)^2$ where c_p is the phase velocity, and $c_s = \sqrt{G/\rho}$ is the shear wave velocity.

The exact, plane stress frequency determinant of equation (8) in reference [1] is equivalent to that presented by Cowper [8] in his accuracy assessment, once the notations have been reconciled and, in turn, is the plane stress equivalent of the well-known plane strain Rayleigh–Lamb frequency equation, once the elastic constants have been adjusted to suit plane stress rather than plane strain conditons. As the wavelength approaches zero, so this equation reduces to

$$16(1 - \gamma)\left(1 - \gamma\,\frac{(1 - \nu)}{2}\right) = (2 - \gamma)^4 \tag{7}$$

and again this is equivalent to the well-known equation governing the velocity of Rayleigh surface waves (see, for example, Renton's reference [1, equation (64.42)]), once plane stress conditions have been imposed by replacing the Poisson ratio ν by $\nu'/(1 + \nu')$, and then dropping the primes. For the Poisson ratio $\nu = 0{\cdot}3$, one finds $\gamma_1 = 0{\cdot}83945$.

3. TBT1 COMPARISON

Renton conducted his comparison over the range

$$0{\cdot}5 \leqslant \frac{wavelength}{beam\ \ depth} \leqslant 5$$

and here this range is extended from 0·1 to 10. Figure 1 shows the error in natural frequency prediction employing several different values for the shear coefficient. Except at very short

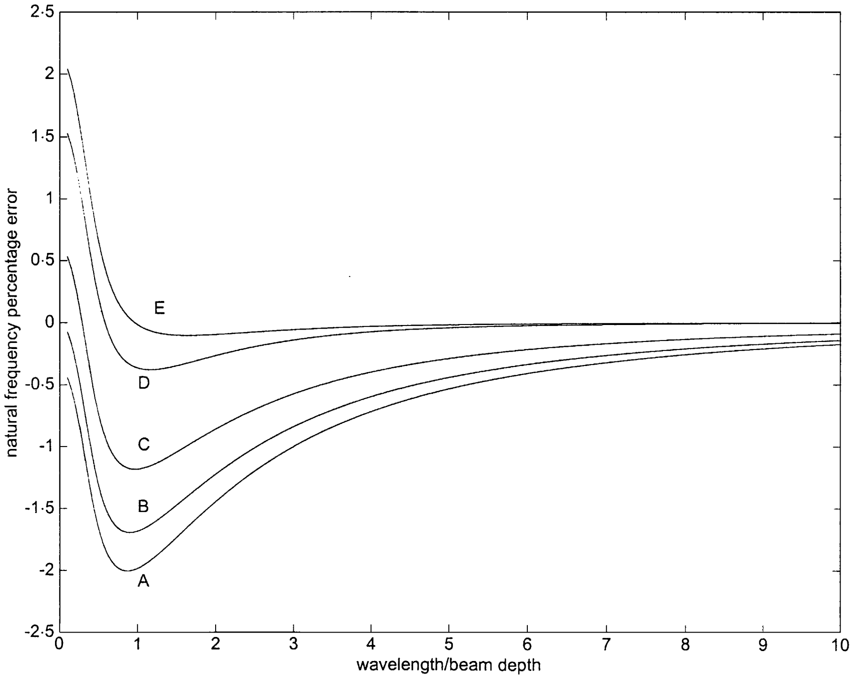

Figure 1. Percentage error in natural frequency prediction. A: $\kappa = \frac{5}{6}$, B: $\kappa = 0{\cdot}83945$, C: $\kappa = 10(1 + \nu)/(12 + 11\nu)$, D: $\kappa = 5(1 + \nu)/(6 + 5\nu)$, E: two coefficient theory $\kappa_1 = 10(1 + \nu)/(12 + 11\nu)$, $\kappa_3 = 5(1 + \nu)/(6 + 5\nu)$.

wavelength, all of these values lead to an underestimate, and each shows a turning point in their error characteristic when the wavelength is approximately equal to the depth of the beam. The value $\kappa = \frac{5}{6}$ leads to a maximum underestimate of 2%, which suggests that Renton's conclusion of a 5·8% underestimate is too pessimistic. Cowper's value $\kappa = 10(1 + \nu)/(12 + 11\nu)$ leads to a maximum underestimate of about 1·2%, and this is in agreement with the error estimate in Figure 2 of reference [6]. The value $\kappa = 5(1 + \nu)/(6 + 5\nu)$ gives a maximum underestimate of about 0·4%. As wavelength becomes shorter, so travelling waves approach the velocity of Rayleigh surface waves; to achieve agreement at this extreme, when one would hardly describe the vibration as flexural, requires use of the value $\kappa = 0{\cdot}83945$ (for the Poisson ratio $\nu = 0{\cdot}3$). As can be seen this choice produces the smallest error for *wavelength/beam depth* = 0·1, but for longer wavelengths gives an error characteristic very close to that of Renton's value. Figure 2 of reference [6] also shows the error prediction from the two-coefficient beam theory

$$EI\,\frac{\partial^4 v}{\partial x^4} + \rho A\,\frac{\partial^2 v}{\partial t^2} - \rho I\left(1 + \frac{E}{\kappa_3 G}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa_1 G}\,\frac{\partial^4 v}{\partial t^4} = 0 \tag{8}$$

in which $\kappa_3 = 5(1 + v)/(6 + 5v)$ and κ_1 is Cowper's value. This error is also shown in Figure 1 of the present note and is seen to provide a maximum underestimate of about 0·1%. Finally, which shear coefficient is best? TBT was constructed originally in order to deal with the inadequacies of Euler–Bernoulli theory (nicely shown in Figure 1 of reference [1]) at short, but not extremely short, wavelength. One would not normally expect an approximate beam theory to predict behaviour accurately when the wavelength becomes shorter than the beam depth, when depth-wise modes of vibration becomes a possibility. Thus in order to answer this question, it is reasonable to restrict the range to *wavelength/beam depth* $\geqslant 1$. For a single coefficient theory that given by equation (1) is clearly the best, although better accuracy can be achieved with a two-coefficient theory. On the other hand, the other coefficients provide better agreement as *wavelength/beam depth* $\to 0$. In particular, the value $\kappa = 0{\cdot}83945$ provides zero error at the limit of zero wavelength; but why should one ask TBT to predict the Rayleigh surface wave velocity, when that velocity can be determined by other means?

4. SECOND SPECTRUM, TBT2

Referring to Figure 1 of reference [1], note that the TBT2 prediction, denoted by the symbol ⓣ2, provides good agreement with the second branch predictions of the exact theory, denoted ②, at the longer wavelength range, for example when (*wavelength/beam depth*) $\geqslant 3$, but provides better agreement with the third exact branch, denoted ③, at shorter wavelength, for example when *wavelength/beam depth* $\leqslant 1$. Such behaviour, which is also displayed by the so-called w_2-mode of Mindlin plate theory [9], is one reason why these second branch predictions should be disregarded. Figure 1 of reference [1] does not show the myriad of other modes of vibration, for example longitudinal, to which the TBT2 prediction also provides good agreement at these short wavelengths.

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