Journal of Sound and Vibration (2002) **257**(5), 977–984
doi:10.1006/jsvi.5014, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

STRUCTURAL FATIGUE LIFE PREDICTION WITH SYSTEM UNCERTAINTIES

X. Wang and J. Q. Sun

Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, *U.S.A.*

(*Received* 10 *January* 2002, *and in final form* 31 *January* 2002)

1. INTRODUCTION

The prediction of fatigue life with system uncertainties to achieve high reliability and safety of engineering structures is an important task. Uncertainties affecting structural fatigue life come from three main sources [1, 2] : (a) structural parameter uncertainties due to geometry and material properties including density, Young's modulus, damping coefficients, and Poisson ratio; (b) environmental factors including loads, boundary conditions, temperature, and humidity; (c) theoretical assumptions of the idealized modelling. Some uncertainties result in large variance in structural fatigue life while others may have little effect. Such a knowledge is valuable to design engineers. This paper intends to contribute to the accumulation of this knowledge by studying the effect of various uncertainties on structural fatigue life.

Dependent on the available information, the uncertainties can be modelled as an interval, a fuzzy set or a random variable [3–6]. This work will adopt the interval model of uncertainties. In the future, when more information becomes available, we shall consider random and fuzzy models.

Structural fatigue prediction involves two tasks: dynamic analysis and fatigue modelling. Many methods have been developed to study the dynamics of stochastic systems including Monte Carlo simulation, finite element methods, weighted integral method and maximum entropy approach [1, 7–10]. With the wide availability of high-speed computers and efficient algorithms for simulating stochastic processes, Monte Carlo simulation has become an increasingly powerful and popular method [11, 12]. Many researchers believe that Monte Carlo simulation is currently the only universal method that can provide accurate solutions for stochastic mechanics problems involving system stochasticity, large variations of uncertain parameters, etc. [10]. This is the method we shall use in this work.

Fatigue is a topic of long history [13–15]. There are two broad classes of fatigue models: a fracture mechanics-based one and an S–N curve-based one [16–19]. Fatigue is typically a random process, particularly, when the structure is subject to random excitations. This random nature has promoted the probabilistic modelling of fatigue [15, 19, 20]. When the S–N model is adopted, cycle counting schemes are needed to identify fatigue damage events of a random stress history [21–23]. In this paper, we shall make use of the S–N model together with the popular rainflow counting algorithm [14, 24].

In the present paper, we shall consider the structural system with parameter uncertainties subject to Gaussian white noise excitations. The rest of the paper is organized as follows. In section 2, the temporal and spectral response of the stress of a beam is presented. Section 3

0022-460X/02/$35.00

reviews the fatigue theory to be used in the present work. Section 4 discusses the modelling of uncertainties. In section 5, the effect of uncertainties on the structural fatigue of a simply supported beam is studied with Monte Carlo simulation.

2. THE STRESS RESPONSE OF A NOMINAL SYSTEM

Consider a classically damped elastic beam subject to a static in-plane load P and a transverse random point load $g(t)$ with a spectral density function $S_{GG}(\omega)$ acting at a point $x = x_0$. $w(x, t)$ denotes the lateral displacement. The in-plane load P can be due to, for example, the residual or thermal stresses. When the in-plane load is compressive, we assume that its magnitude is less than the critical buckling load $|P| < EI\pi^2/l^2$.

Let $w(x, t) = \sum_{n=1}^{\infty} \phi_n(x) T_n(t)$, where $\phi_n(x)$ is the normalized mode function and $T_n(t)$ is the modal coefficient of the response. The stress at a hot spot (x_{cr}, z_{cr}) can be obtained as

$$\sigma_{xx}(t)\,|_{(x_{cr}, z_{cr})} = \frac{P}{A} + Ez_{cr} \sum_{n=1}^{\infty} T_n(t)\phi''_n(x_{cr}). \tag{1}$$

The in-plane load provides a constant stress which is the mean stress when the excitation is a random process with zero mean. The second part of the stress is the dynamic contribution.

Equation (1) can be rewritten in the following finite summation as

$$\sigma(t) = \sigma_{xx}(t) - \frac{P}{A} = \sum_{n=1}^{N} K_n T_n(t), \tag{2}$$

where $K_n = Ez_{cr}\phi''_n(x_{cr})$, N is the number of modes kept in the solution. Assume that $\sigma(t)$ is a weakly stationary process. The spectral density function of $\sigma(t)$ can be obtained as follows.

$$S_{\sigma\sigma}(\omega) = \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \phi_i(x_0)\phi_j(x_0) S_{GG}(\omega) H_i(\omega) H_j^*(\omega), \tag{3}$$

where $*$ stands for complex conjugate and $H_i(\omega)$ is the frequency response function of the ith modal equation. With $S_{\sigma\sigma}(\omega)$, we can use the algorithms due to Shinozuka to efficiently simulate time histories of the stress [11, 12].

The above results are for the nominal system. When the parameters of the system are uncertain, these results provide a basis for studying the effect of the uncertainties. Because the system parameters appear in the solution in a very non-linear fashion, analytical methods for studying the effect of uncertainties are quite limited. Monte Carlo simulation will be used subsequently.

3. ESTIMATE OF FATIGUE LIFE

The material fatigue property can be characterized by the well-known S–N curve defined as

$$N_F = Ks^{-b}, \tag{4}$$

where s is the stress amplitude, N_F is the number of the cycles to failure, and K and b are material constants. For the stress response with a large mean relative to its fluctuations, the effect of the mean stress has to be considered in the fatigue life prediction. An equivalent stress amplitude including the effect of the mean stress on fatigue life can be developed. There have been several such models [22, 23]. The one adopted in this paper is Goodman's model which states

$$\frac{s}{s_{eq}} + \frac{s_m}{s_f} = 1, \tag{5}$$

where s is the real stress amplitude, s_{eq} is the equivalent stress amplitude, s_m is the mean stress, and s_f is a material constant related to the true fracture strength.

For the high-cycle fatigue, the Palmgren–Miner linear damage accumulation rule can be applied [21, 22]. Denote the probability density function (PDF) of the stress range as $\rho(s)$. Then, the average damage is given by

$$D = N_c \int_0^\infty \frac{\rho(s)}{N_F}\,\mathrm{d}s = N_c E\left(\frac{1}{N_F}\right) = N_c E\left(\frac{s^b}{K}\right), \tag{6}$$

where N_c is the random number of cycles in the stress process which can be obtained, for example, by the rainflow counting scheme [14, 22, 24]. N_F is the number of cycles to fatigue failure at the stress range level s.

The mean fatigue life in cycles N_r is reached when $D = 1$,

$$N_r = \frac{1}{E(s^b/K)} = \frac{K}{E(s^b)}. \tag{7}$$

Because fatigue is a random process, a sufficiently large number of the stress samples should be used to count the damage events and to calculate the fatigue life. In practice, on-line rainflow counting algorithm can be utilized to monitor if the structure is approaching its designed fatigue life [18, 25].

With the linear damage accumulation assumption, the fatigue life in cycles can be converted into that in time [26],

$$T_r = \frac{L_t}{N_r E(s^b/K)}, \tag{8}$$

where T_r is the fatigue life in time. L_t is the total time duration of the stress history samples used for cycle counting.

4. UNCERTAINTY MODELLING

We now consider a beam made of 2014-T6 aluminum. The nominal parameters of the beam are $E = 7{\cdot}3087 \times 10^{10}$ Pa, $\rho = 2{\cdot}77 \times 10^3$ kg/m^3, $h = 0{\cdot}008$ m, $l = 0{\cdot}38$ m, $b = 0{\cdot}05$ m, $\xi_1 = 0{\cdot}05$, $P = 0$ N.

We shall consider the uncertainties of these parameters: Young's modulus E, the first mode damping ratio ξ_1, length l, thickness h and in-plane load P. These parameters directly or indirectly affect the stress response of the structure.

The geometrical parameters of structures are often given by their nominal values plus and minus the tolerance. For a batch of products, the available information about length and thickness would be their minimum and maximum values. This uncertainty of parameters can be modelled as an interval variable, which implies that the parameter has a uniform distribution in the interval range. In the following, we assume that the uncertain parameters are interval variables with a uniform distribution. The effect of the probability distribution of the parameter uncertainties on fatigue damage will be studied in the future.

In the following numerical study, all the intervals of the parameter uncertainties are selected to center around the nominal values with a $\pm 6\%$ variation, except the in-plane load whose interval is chosen to be $\pm 50\%$ of the critical buckling load.

5. NUMERICAL EXAMPLES

The effect of uncertainties on the mean and variance of the fatigue life of a simply supported elastic beam is studied. In order to investigate the individual effect of each parameter, we treat one parameter to be uncertain at a time. Other parameters assume their nominal values. Extensive numerical studies have been done. Only a portion of the results are presented in the following. The number of modes for transverse motion is taken to be 10. The external excitation is a Gaussian white noise with a constant spectral density function $S_{GG}(\omega) = 3{\cdot}5\,\mathrm{N}^2\,\mathrm{s/rad}$ acting at the point $x_0 = 1/\sqrt{2}\,l$.

Figures 1–5 show all the numerical results of fatigue life prediction. The fatigue life is presented with its mean and $\pm$ standard deviation denoted by the vertical bars. Figures 6 and 7 show the sensitivity of fatigue life to the uncertainties. In all the simulations, 50 time histories of the stress with 2^{18} points in each record are used to calculate the response statistics.

Figures 1–5 indicate that the mean of fatigue life increases significantly with in-plane tensile stress and thickness, slightly with the first mode damping ratio, and it decreases significantly with in-plane compressive stress, changes little with uncertainties in length and decreases slightly with Young's modulus. These trends can also be seen clearly from Figures 6 and 7.

From Figures 1 to 5, it is seen that beams with larger tensile in-plane load or thickness than the nominal values have larger variance of the fatigue life excited by the same white noise, while the length, Young's modulus and the first mode damping ratio have a small effect on the propagation of the variance of the excitation force.

Because fatigue is a random process, the difference in the fatigue life can be due to either the changes of the parameters or the variance of the excitation. In Figures 1 and 4, the changes of the fatigue life is larger than their standard deviation. This seems to suggest that the uncertainties in the parameters, namely, in-plane load and thickness, have a more dominant effect than the variance of the excitation. On the other hand, in Figures 2, 3 and 5,

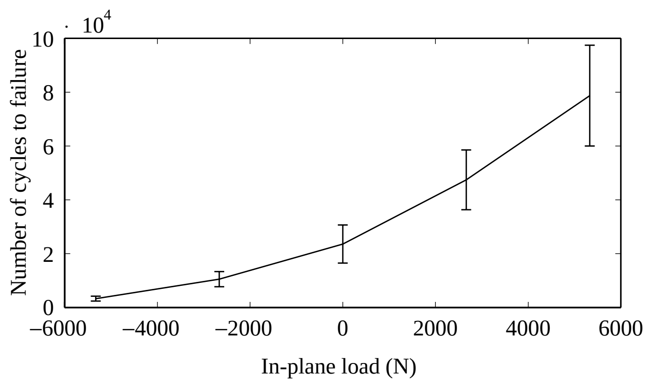

Figure 1. Influence of the in-plane load on the structural fatigue. Vertical bars stand for the standard deviation.

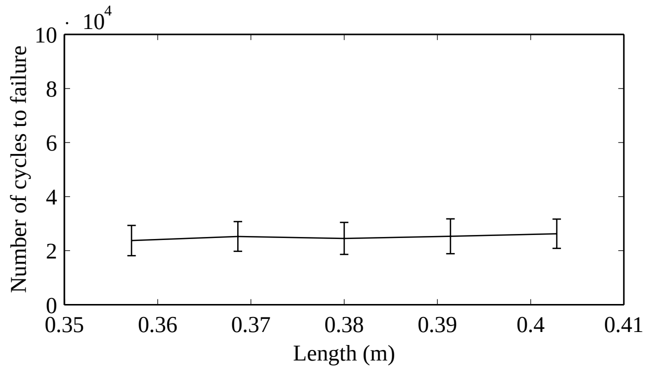

Figure 2. Influence of the length on the structural fatigue. Vertical bars stand for the standard deviation.

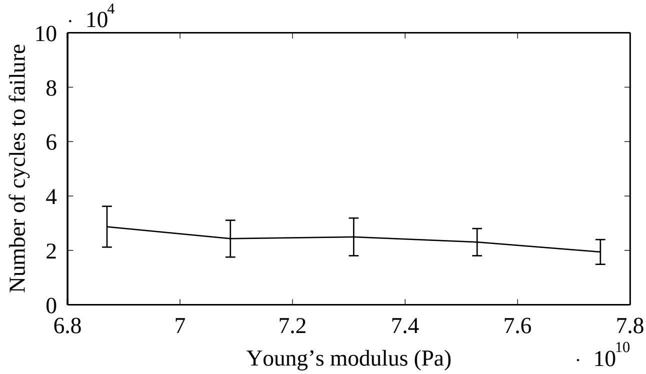

Figure 3. Influence of Young's modulus on the structural fatigue. Vertical bars stand for the standard deviation.

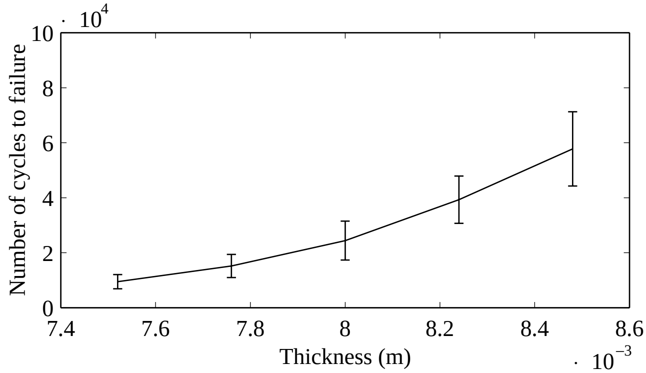

Figure 4. Influence of the thickness on the structural fatigue. Vertical bars stand for the standard deviation.

the changes of the fatigue life are of the same order as the standard deviation. In this case, it is difficult to distinguish various effects from that due to the random excitation.

Figures 6 and 7 show that the fatigue life is most sensitive to the in-plane load and thickness, moderately sensitive to the first mode damping ratio, and least sensitive to the length and Young's modulus.

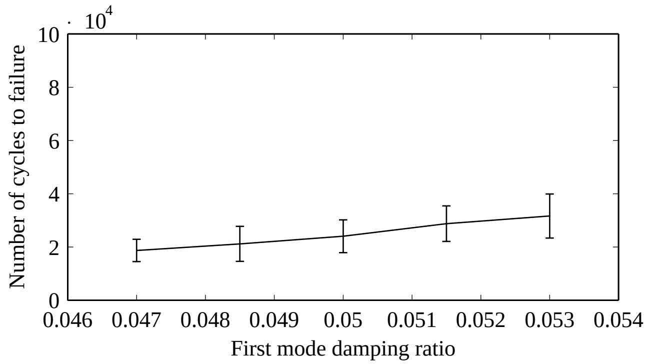

Figure 5. Influence of the first mode damping ratio on the structural fatigue.

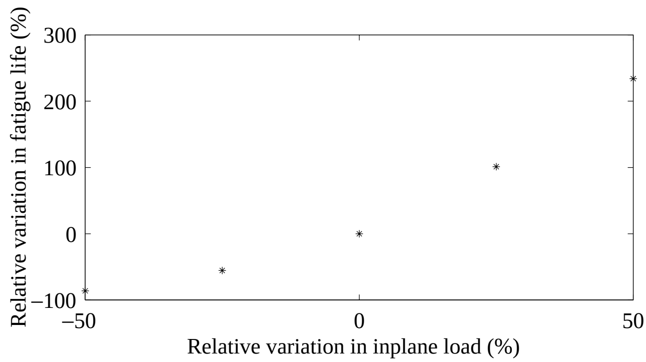

Figure 6. Sensitivity of fatigue life to the uncertainty of in-plane load. * stands for the data points.

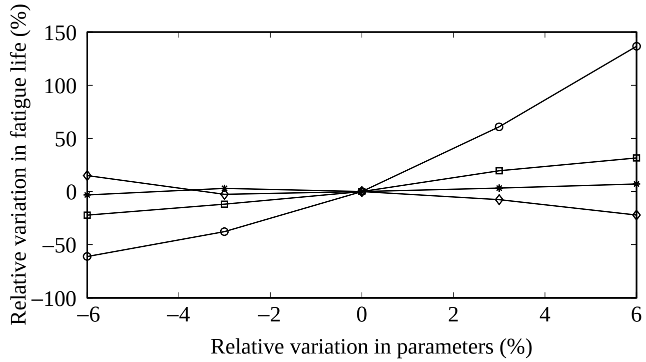

Figure 7. Comparision of sensitivity of the fatigue life to various uncertainties: (○, thickness; □ first mode damping ratio; ∗, length; ◇, Young's modulus).

It is well known that tensile mean stress shortens the fatigue life and compressive mean stress elongates it in constant fully reversed deterministic stress cyclic experiments [21]. However, Figures 1 and 6 show that the fatigue life increases with tensile in-plane load. In the present problem, the in-plane load has both static and dynamic effects on the fatigue life.

The static effect, which is the first term on the right-hand side of equation (1), is still the same as observed in reference [21]. However, the dynamic effect reduces the response level of the structure by increasing the effective stiffness, and thereby increases the fatigue life. In the case of the simply supported beam, the dynamic effect dominates the static one.

6. CONCLUSIONS

Structural fatigue life of an elastic beam with uncertainties modelled as interval variables is studied using Monte Carlo simulation. Some interesting trends are observed. The increase of in-plane load, thickness and the first mode damping ratio from their nominal values helps elongate the mean of the fatigue life, while the deviation of the length has little effect on the fatigue life, and the mean of the fatigue life decreases slightly with Young's modulus. The fatigue life is more sensitive to the in-plane load and thickness than the other parameters. Such a knowledge will prove to be beneficial to structural engineers in their design work.

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