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STEP-SIZE DEPENDENCE OF THE PERIOD FOR A FORWARD-EULER SCHEME OF THE VAN DER POL EQUATION

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Recently, Mickens and Gumel [1] studied the numerical solutions of a non-standard finite-difference scheme [2] for the van der Pol differential equation

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \qquad \varepsilon > 0. \tag{1}$$

These results were compared to those obtained by use of the standard forward-Euler method applied to the system equations form of equation (1), i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \varepsilon(1 - x^2)y. \tag{2}$$

From the numerical work, it was found that the Euler method, for a given value of ε, gave periods for the limit-cycle behavior that increased as the step size, $\Delta t = h$, increased. The purpose of this note is to show mathematically that this is a general feature of the forward-Euler scheme when used to numerically integrate equation (1).

The forward-Euler finite-difference scheme for equation (1) is [2]

$$\frac{x_{k+1} - x_k}{h} = y_k, \qquad \frac{y_{k+1} - y_k}{h} = -x_k + \varepsilon(1 - x_k^2)y_k, \tag{3}$$

where h is the time step size Δt; $t_k = hk$, where k is an integer; and x_k and y_k are approximations to the exact solutions of equation (1), i.e.,

$$x_k \simeq x(t_k), \qquad y_k \simeq y(t_k). \tag{4}$$

Note that if y_k is eliminated, then the two first order difference equations become the following single second order equation:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + x_{k-1} = \varepsilon(1 - x_{k-1}^2)\left(\frac{x_k - x_{k-1}}{h}\right) \tag{5}$$

which can be rewritten in the form

$$x_{k+1} - (2 + \varepsilon h)x_k + (1 + \varepsilon h + h^2)x_{k-1} = (\varepsilon h)(x_{k-1})^3 - (\varepsilon h)x_k(x_{k-1})^2. \tag{6}$$

In the calculations to come, it is assumed that the magnitudes of h and ε satisfy the inequality

$$0 < h \ll \varepsilon \ll 1. \tag{7}$$

The periodic solutions to equations (5) or (6) are to be studied by using the method of harmonic balance as modified for non-linear, second order difference equations. The full

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details as to how this should be done is provided in the publication of Mickens [3]. To proceed, the assumed approximation to the solution is taken as

$$x_k \simeq A\cos(\omega t_k), \tag{8}$$

where the amplitude A and angular frequency ω are to be calculated by the following procedure:

(1) The result of equation (8) is substituted into equation (6) and the resulting expression is expanded into a finite number of trigonometric functions; doing this gives

$$\begin{aligned} &H_1(A,\omega,\varepsilon,h)\cos(\omega t_k) + H_2(A,\omega,\varepsilon,h)\sin(\omega t_k) \\ &+ (\text{higher order harmonics}) = 0. \end{aligned} \tag{9}$$

(2) Next, the coefficients H_1 and H_2 are set equal to zero, i.e.,

$$H_1(A,\omega,\varepsilon,h) = 0, \qquad H_2(A,\omega,\varepsilon,H) = 0, \tag{10}$$

and these two equations are solved for A and ω in terms of ε and h.

(3) The substitution of these values for A and ω into equation (8) gives the required approximation to the limit-cycle solution of equations (5) or (6).

To carry out the above procedure, the following trigonometric relations are useful:

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\cos\theta_2, \tag{11a}$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2, \tag{11b}$$

$$(\cos\theta)^3 = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta, \qquad (\sin\theta)^3 = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta, \tag{11c, d}$$

$$\sin\theta_1\cos\theta_2 = \tfrac{1}{2}\sin(\theta_1+\theta_2) + \tfrac{1}{2}\sin(\theta_2-\theta_1). \tag{11e}$$

Also, it should be indicated that

$$x_{k\pm1} \simeq A\cos(\omega t_k \pm \omega h). \tag{12}$$

After a large amount of both algebraic and trigonometric manipulation, H_1 and H_2 are found to be the expressions

$$H_1 \equiv \left[2 + h^2 + \varepsilon h - \left(\frac{3\varepsilon h A^2}{4}\right)\right]\cos\beta - (2+\varepsilon h) + (\varepsilon h A^2)\left[\frac{2+\cos(2\beta)}{4}\right], \tag{13}$$

$$H_2 \equiv \left[h^2 + \varepsilon h - \left(\frac{3\varepsilon h A^2}{4}\right)\right]\sin\beta + \left(\frac{\varepsilon h A^2}{4}\right)\sin(2\beta), \tag{14}$$

where

$$\beta \equiv \omega h. \tag{15}$$

To further continue with the calculation, it must be understood that the approximation to the solution of equation (6), as given by equation (8), is valid only to terms of order ε [3]. Consequently, the angular frequency takes the form

$$\omega = \omega(\varepsilon, h, A) = \omega_0 + \varepsilon\omega_1(h, A) + O(\varepsilon^2), \tag{16}$$

where it is indicated that ω_0 should not depend on h and A, while ω_1 does depend on them. Further, it should be clear that $\omega_0 = 1$. This follows from the fact that in the limits [2]

$$h \to 0, \quad k \to \infty, \quad hk = t = \text{fixed}, \tag{17}$$

equation (5) reduces to the van der Pol differential equation and its first order perturbation solution is [4]

$$x(t) \simeq 2\cos t. \tag{18}$$

Setting $H_1(A, \omega, \varepsilon, h)$ and $H_2(A, \omega, \varepsilon, h)$ equal to zero and using the relations given in equations (7) and (16), the following results are obtained, respectively, from $H_1 = 0$ and $H_2 = 0$:

$$\omega_1 \simeq -\left(\frac{hA^2}{8}\right), \qquad A \simeq 2. \tag{19a, b}$$

Thus, $\omega_1 \simeq -(h/2)$ and ω is

$$\omega \simeq 1 - \varepsilon\left(\frac{h}{2}\right). \tag{20}$$

Since the period is $T = 2\pi/\omega$, it follows that to the same order of approximation

$$T \simeq 2\pi\left(1 + \frac{\varepsilon h}{2}\right), \tag{21}$$

and a first approximation to the (periodic) limit-cycle solution to equation (5) is

$$x_k \simeq 2\cos\left[\left(1 - \frac{\varepsilon h}{2}\right)t_k\right]. \tag{22}$$

An examination of equation (21) shows, under the conditions given in equation (7), that the period increases with an increase in the magnitude of the step size. This is precisely the result found by Mickens and Gumel [1] in their numerical integration of the van der Pol differential equation using the forward-Euler method on the system form of this equation, i.e., see equation (3). While this property was derived as a result of the application of a perturbation procedure, it may be expected to have a more general validity. This follows from the observation that perturbation results often provide the correct qualitative behavior of a phenomena even when the expansion parameter is large [4].

The above discussion also indicates that the period of the limit cycle of the van der Pol differential equation, determined from (5) or (6), takes the form

$$T(\varepsilon, h) = T(\varepsilon, 0) + h\bar{T}(\varepsilon, h), \tag{23}$$

where $T(\varepsilon, 0)$ is the period function calculated using perturbation theory [4]. The second term, on the right side, is the contribution coming from the influence of the finite magnitude of the step size, $h = \Delta t$. Consequently, for any non-zero value of the step size, the period determined from the numerical solution will differ from the actual period of oscillations of the original van der Pol differential equation. This is a fundamental difficulty that plagues all numerical integration techniques.

In summary, a mathematical explanation has been provided to explain the numerically derived behavior of the period for the van der Pol differential equation integrated using a forward-Euler method.

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