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FOURIER REPRESENTATIONS FOR PERIODIC SOLUTIONS OF ODD-PARITY SYSTEMS

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Many important oscillatory dynamical systems are modelled by differential equations which take the form [1]

$$H(x, \dot{x}, \ddot{x}) = \ddot{x} + x + \varepsilon f(x, \dot{x}) = 0, \tag{1}$$

where ε is a positive parameter and $f(x, y)$ is a rational function of its two arguments. An *odd-parity system* is defined to be one for which the following property holds:

$$x \to -x \Rightarrow H(-x, -\dot{x}, -\ddot{x}) = -H(x, \dot{x}, \ddot{x}). \tag{2}$$

Consider now the following two odd-parity systems along with their corresponding perturbation derived solutions (for the case where $0 < \varepsilon \ll 1$) [1, 2]:

$$\ddot{x} + x + \varepsilon x^3 = 0; \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{3}$$

$$x(\theta, \varepsilon) = A \cos\theta + \varepsilon\left(\frac{A^3}{32}\right)(-\cos\theta + \cos 3\theta) + \varepsilon^2\left(\frac{A^5}{1024}\right)(23\cos\theta - 24\cos 3\theta + \cos 5\theta) + O(\varepsilon^3), \tag{4a}$$

$$\theta(\varepsilon, t) \equiv \omega(\varepsilon)t = \left[1 + \varepsilon\left(\frac{3A^2}{8}\right) - \varepsilon^2\left(\frac{21A^4}{256}\right) + O(\varepsilon^3)\right]t \tag{4b}$$

and

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \tag{5}$$

$$x(\theta, \varepsilon) = 2\cos\theta + \left(\frac{\varepsilon}{4}\right)(3\sin\theta - \sin 3\theta) + \left(\frac{\varepsilon^2}{96}\right)(-13\cos\theta + 18\cos 3\theta - 5\cos 5\theta) + O(\varepsilon^3), \tag{6a}$$

$$\theta(\varepsilon, t) \equiv \omega(\varepsilon)t = \left[1 - \frac{\varepsilon^2}{16} + O(\varepsilon^3)\right]t. \tag{6b}$$

Note that for both of these odd-parity systems the perturbation solutions have trigonometric expansions in which only odd multiples of the angular frequencies (ω) appear!

To further illustrate the issue, examine the same situation for a mixed-parity system given by the equation [1]

$$\ddot{x} + x + \varepsilon\alpha x^2 + \varepsilon^2\beta x^3 = 0, \tag{7a}$$

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where

$$\beta = O(1), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{7b}$$

The perturbation derived solution is

$$\begin{aligned} x(\theta, \varepsilon) = & A\cos\theta + \varepsilon\left(\frac{\alpha A^2}{6}\right)(-3 + 2\cos\theta + \cos 2\theta) \\ & + \varepsilon^2\left(\frac{A^3}{3}\right)\left[-\alpha^2 + \left(\frac{174\alpha^2 - 27\beta}{288}\right)\cos\theta \right. \\ & \left. + \left(\frac{\alpha^2}{3}\right)\cos 2\theta + \left(\frac{2\alpha^2 + 3\beta}{32}\right)\cos 3\theta\right] + O(\varepsilon^3), \end{aligned} \tag{8a}$$

$$\theta(\varepsilon, t) \equiv \omega(\varepsilon)t = \left[1 + \varepsilon^2\left(\frac{9\beta - 10\alpha^2}{24}\right)A^2 + O(\varepsilon^3)\right]t. \tag{8b}$$

Observe for this mixed-parity case that both even and odd multiples of the angular frequency (ω) occur.

The main purpose of this Letter-to-the-Editor is to demonstrate the correctness of the following proposition: For odd-parity systems, the Fourier representations only include contributions from terms having odd multiples of the angular frequency. In other words, such systems have periodic solutions which take the form

$$x(t) = \sum_{k=1}^{\infty}[A_k \cos(2k-1)\omega t + B_k \sin(2k-1)\omega t]. \tag{9}$$

To proceed, the following assumptions are needed:

(1) Equation (1) is of odd-parity.

(2) The periodic solutions of equation (1) occur about the fixed-point $(\bar{x}, \bar{y}) = (0, 0)$ in the two-dimensional phase-space (x, y), where $y = \dot{x}$.

(3) The periodic solutions of equation (1) are essentially unique [1, 2]. Within this context, essentially unique means that if $x = \phi(t)$ is a non-trivial periodic solution, then for $t_0 > 0$, $z = \phi(t - t_0)$ is also a periodic solution. From the perspective of phase space, the moving point

$$(x(t), y(t)) = (\phi(t), \dot{\phi}(t)) \tag{10}$$

traces out a closed path. Likewise, the moving point

$$(z(t), \dot{z}(t)) = (\phi(t - t_0), \dot{\phi}(t - t_0)) \tag{11}$$

traces out the same closed path, except for being shifted in phase.

Assume that equation (1) has a periodic solution with period T; the corresponding angular frequency is

$$\omega = 2\pi/T \tag{12}$$

and $x(t)$ has the complex Fourier representation [3]

$$x(t) = \sum_{k=1}^{\infty}[a_k \,\mathrm{e}^{ik\omega t} + a_k^* \,\mathrm{e}^{ik\omega t}], \tag{13}$$

where a_k are complex valued coefficients. Now if $x(t)$ is a periodic solution, then so is $z(t)$ defined as

$$z(t) \equiv -x\left(t+\frac{T}{2}\right). \tag{14}$$

This follows from the fact that both $x(t)$ and $-x(t)$ are solutions, and consequently $x(t-t_0)$ and $-x(t-t_0)$ are also periodic solutions. In equation (14), t_0 is taken to be $t_0 = -T/2$. Since $z(t)$ is a solution to equation (1), it follows from uniqueness that

$$z(t) = x(t) \tag{15}$$

or

$$x\left(t+\frac{T}{2}\right) = -x(t). \tag{16}$$

Substituting equation (13) into equation (16) and comparing the coefficients of the two exponential terms gives the relation

$$(-1)^k a_k = -a_k, \tag{17}$$

which allows non-trivial values for the a_k only in $k=$ odd integer. Writing

$$b_m = a_{2m-1}, \quad m = 1, 2, 3, \dots \tag{18}$$

and defining

$$A_m \equiv b_m + b_m^*, \quad B_m \equiv \mathrm{i}(b_m - b_m^*), \tag{19}$$

it follows that for odd-parity systems, the periodic solutions have the Fourier representation

$$x(t) = \sum_{m=1}^{\infty}[A_m \cos(2m-1)\omega t + B_m \sin(2m-1)\omega t]. \tag{20}$$

In other words, only odd multiples of the angular frequency appear.

It should be indicated that the special case of a forced Duffing's equation was studied by Körner [4]. It was concluded that the periodic solution having fundamental angular frequency, ω, took the form given by equation (20). However, the argument given above is general and holds for any odd-parity system having periodic solutions.

The results presented here also can be applied to non-standard odd-parity equations such as [5]

$$\ddot{x} + x + \varepsilon x^{1/3} = 0, \qquad \ddot{x} + x^{1/3} = \varepsilon(1-x^2)\dot{x}. \tag{21, 22}$$

Also, for conservative systems, i.e.,

$$\ddot{x} + x + \varepsilon f(x) = 0, \qquad f(-x) = -f(x), \tag{23}$$

the initial conditions can also be selected such that $B_m = 0$; therefore, no sine terms appear in the Fourier representation.

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REFERENCES

1. R. E. Mickens 1996 *Oscillations in Planar Dynamic Systems*. Singapore: World Scientific.
2. M. Farkas 1994 *Periodic Motions*. New York: Springer-Verlag.
3. A. Zygmund 1995 *Trigonometrical Series*. Dover: New York.
4. T. W. Körner 1988 *Fourier Analysis*. Cambridge University Press: New York, see Chapter 24.
5. R. E. Mickens 2001 *Journal of Sound and Vibration* **246**, 375–378. Oscillations in $x^{4/3}$ potential.