Journal of Sound and Vibration (2002) **258**(4), 677–700
doi:10.1006/jsvi.2002.5145, available online at http://www.idealibrary.com on IDEAL®

PERIODIC SOLUTIONS OF STRONGLY NON-LINEAR OSCILLATORS BY THE MULTIPLE SCALES METHOD

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(*Received* 1 *October* 2001, *and in final form* 28 *January* 2002)

The multiple scales method, developed for the systems with small non-linearities, is extended to the case of strongly non-linear self-excited systems. Two types of non-linearities are considered: quadratic and cubic. The solutions are expressed in terms of Jacobian elliptic functions. Higher order approximations, of solution as well as modulations of amplitude and phase, are derived. Comparisons to numerical simulations are provided and discussed.

1. INTRODUCTION

Throughout the last century, perturbation methods based on circular functions have been successfully developed to accurately determine approximate solutions to weakly non-linear oscillators of the form

$$\ddot{x} + c_1 x = \varepsilon g(\mu, x, \dot{x}), \tag{1}$$

where c_1 is a positive constant, ε is a small positive parameter and g is a polynomial function of its arguments; μ is referred as a control parameter. By now, classical methods, such as harmonic balance (HB) [1], Lindstedt–Poincaré (LP) [1–3], Krylov–Bogolioubov–Mitropolski (KBM) [4, 5], averaging [1, 6] and multiple scales method (MSM) [2] are widely used to obtain approximate periodic solutions of equations of form (1). In the former two methods, one seeks directly a periodic steady state solution, which is assumed *a priori* to occur. On the other hand, the latter three methods yield a set of first order differential equations which describe the slow time evolution on the amplitude and phase of the response. The periodic steady state solution is obtained by setting these amplitude and phase time derivatives to zero. The advantage of these latter methods is that they allow one in a single analysis to study both the steady state responses and their stability. All these methods are now considered to be classical standard tools for the analytical investigations of weakly non-linear systems.

The extension of perturbation methods to strongly non-linear systems, where the unperturbed system is already non-linear, has not received the same attention for at least two reasons [7]. First, analytical solutions for non-linear systems are generally unknown, so that an analytical investigation cannot be carried out. Second, the perturbation schemes themselves become much more difficult to implement.

Recently, special attention has focused on extending the above classical perturbation methods [1–6] by introducing the Jacobian elliptic functions [7–24] for investigating the more general case of the form

$$\ddot{x} + c_1 x + c_2 f(x) = \varepsilon g(\mu, x, \dot{x}), \tag{2}$$

where c_1 and c_2 are fixed constants, $f(x)$ includes cubic or quadratic terms and $g(\mu, x, \dot{x})$ is an arbitrary non-linear function of its arguments. Striking qualitative improvement of the Jacobian over the trigonometric approximation was observed numerically [7–24]. In particular, the use of Jacobian elliptic functions gives an excellent approximation of the periodic orbits, even near the separatrices in the self-excited oscillators (2), just prior to a homoclinic saddle-loop connection. For instance, Barkham and Soudack [8], and Yuste and Bejarano [9] used the (KBM) method to provide approximate solutions of a strongly non-linear oscillator in terms of Jacobian elliptic functions. Also, Margallo *et al.* [10, 11] presented an elliptic HB method using generalized Fourier series and elliptic functions. On the other hand, Coppola and Rand [12, 13] used symbolic computation to implement an averaging method with elliptic functions. Kevorkian and Cole [14] generalized the ideas of two-scale expansions to a strictly non-linear second order equation with solutions that are slowly modulated. A complete mathematical study of Duffing-type equation was conducted by Morozov [15]. He developed a technique which allows the transformation of the perturbed autonomous system to a form suitable for analysis. A shortened system is defined, whose equilibrium states coincide with zeros of the Poincaré–Pontryagin equation.

Recently, the elliptic LP and averaging methods were conducted, respectively, by Belhaq *et al.* [16] and Belhaq and Lakrad [17] to determine analytical approximations of periodic solutions near homoclinicity and to derive analytical criterion of homoclinic bifurcation. The same authors used elliptic HB to approximate the period of solutions to a mixed parity non-linear oscillator [18].

All the methods mentioned above have their own advantages to obtain approximate analytical solutions. However, most of them were implemented to derive first order approximate solutions. Two difficulties arise in performing high order perturbation methods to elliptic functions. First, the definition of secular terms is not straightforward like the circular case. Second, elliptic functions are not closed under integrations (see Table A1). These two points are necessary to calculate higher order approximations of the solution.

To obtain a second order approximation, Coppola [7] formulated an averaging method using the Lie transform method in the cubic case. In a series of papers, Chen and Cheung [17–20] performed an elliptic LP method [19, 20] and derived an elliptic perturbation method [21, 22] based on expanding the amplitude in a power series of ε and imposing to the solutions of higher order to have the same form as the solution of the unperturbed system. On the other hand, Smith [23] performed a multiple scales method using matched inner and outer solutions to construct the second order of the homoclinic solution of the van der Pol–Duffing equation. Cveticanin [24] extended the HB method, KB method and the elliptic perturbation method to the case of complex strongly non-linear differential equations. Leung and Zhang [25] extended the normal form method to study the asymptotic solutions of cubic non-linear terms. More recently, Belhaq and Lakrad [26] formulated the multiple scales method with elliptic functions for the case of $f(x) = x^3$.

The main goal of this paper is to present a generalization and an extension of the elliptic multiple scales method proposed in reference [26]. Hence, the principle of the proposed method is given for both quadratic and cubic $f(x)$. See section 2. For illustration, in section 3 we include explicit calculations for both quadratic and cubic non-linearities $f(x)$. Comparisons to numerical simulations are also provided. A summary is given in section 4.

2. FORMULATION OF THE METHOD

Consider the strongly non-linear oscillator given by equation (2). One main feature of the phase portrait [27] of the unperturbed system, i.e., $\varepsilon = 0$, is the appearance of families of periodic orbits which are symmetric with respect to the x-axis. Typically, such families terminate either (a) at an equilibrium point (center), or (b) at an invariant set consisting of a saddle and one or two homoclinic (orbits asymptotic to this saddle). Observe that this unperturbed equation is both conservative (in fact, Hamiltonian) and time reversible (that is, if $x(t)$ is a solution then so is $x(-t)$).

Indeed, the unperturbed system has periodic solutions whenever the potential

$$V(x) = \int_0^x f(s)\,\mathrm{d}s \tag{3}$$

is concave in some interval $a < x < b$, and we restrict our attention to oscillations in this interval [15].

In the present paper, we are interested in the case where $f(x) = x^2$ or $f(x) = x^3$. However, under perturbation, i.e., for $\varepsilon \neq 0$, it is possible that a periodic solution survives (as a limit cycle). Usually, perturbation methods are used to approximate this solution from the corresponding periodic solution of the unperturbed equation.

In what follows, the multiple scales method (MSM) is adapted to Jacobi elliptic functions. It is worth noting that MSM is characterized by the introduction of independent scales of time and consequently the transformation of the ordinary differential equations to a set of partial differential equations. Hence, the solution of equation (2) and the scales of time are expressed in terms of ε as follows:

$$x(t; \varepsilon, N) = \sum_{i=0}^{N} \varepsilon^i x_i(T_0, T_1, \ldots, T_N) + \mathcal{O}(\varepsilon^{N+1}), \tag{4}$$

$$T_i = \varepsilon^i t. \tag{5}$$

Here $\mathcal{O}$ is the order symbol which measures the relative order of magnitude of various quantities. We say that $h(\varepsilon) = \mathcal{O}(\varepsilon^N)$ if $h(\varepsilon)/\varepsilon^N$ is bounded when $\varepsilon \to 0$.

The scales of time T_i $(i = 0, \ldots, N)$ are considered independent and get longer as the integer i increases. Thus, T_0 is a fast time scale on which the main oscillatory behavior occurs and T_i (where $i > 0$) are slow time scales characterizing modulations of amplitudes and phases. Using the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{i=0}^{N} \varepsilon^i D_i, \tag{6}$$

where $D_i = \partial/\partial T_i$. The functions $x_i(T_0, \ldots, T_i)$ are assumed to be periodic. Thus, the derivatives of x are expressed as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{i=0}^{N} \varepsilon^i \sum_{l=0}^{i} (D_l x_{i-l}) + \mathcal{O}(\varepsilon^{N+1}), \tag{7}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \sum_{i=0}^{N} \varepsilon^i \left[\sum_{j=0}^{i} D_j \left(\sum_{l=0}^{i-j} D_l x_{i-j-l} \right) \right] + \mathcal{O}(\varepsilon^{N+1}) \tag{8}$$

and so

$$\left(\sum_{i=0}^{N} x_i \right)^l = \sum_{j_0+j_1+\cdots+j_N=l} \frac{l!}{j_0! j_1! \cdots j_N!} x_0^{j_0} x_1^{j_1} \cdots x_N^{j_N}, \tag{9}$$

where

$$\sum_{j_1+j_2+\cdots+j_N=l} = \sum_{j_1} \cdots \sum_{j_N} \quad \text{such that } j_1 + j_2 + \cdots + j_N = l. \tag{10}$$

As $f(x)$ is a polynomial function, then

$$f(x) = \left(\sum_{j=0}^{N} \varepsilon^j x_j \right)^l = \sum_{j_0+\cdots+j_N=l} \frac{l!}{j_0! \cdots j_N!} \varepsilon^{\sum_{k=0}^{N} k j_k} x_0^{j_0} \cdots x_N^{j_N} + \mathcal{O}(\varepsilon^{1+\sum_{k=0}^{N} k j_k}). \tag{11}$$

In fact, $g(\mu, x, \dot{x})$ can be written as

$$g(\mu, x, \dot{x}) = \sum_{m=0}^{M} \mu_m x^{p_m} \dot{x}^{q_m}, \tag{12}$$

$$\begin{aligned} x^p \dot{x}^q = &\left[\sum_{j_0+\cdots+j_N=p} \frac{p!}{j_0! \cdots j_N!} \prod_{k=0}^{N} (\varepsilon^k x_k)^{j_k} \right] \\ &\left[\sum_{j_0+\cdots+j_N=q} \frac{q!}{j_0! \cdots j_N!} \prod_{k=0}^{N} \left(\varepsilon^k \sum_{l=0}^{k} (D_l x_{k-l}) \right)^{j_k} \right]. \end{aligned} \tag{13}$$

Substituting equations (4)–(13) into equation (2), and equating coefficients of like powers of ε leads to the following equations:
order $\mathcal{O}(\varepsilon^0)$:

$$D_0^2 x_0 + c_1 x_0 + c_2 f(x_0) = 0, \tag{14}$$

order $\mathcal{O}(\varepsilon^1)$:

$$D_0^2 x_1 + c_1 x_1 + c_2 \frac{\partial f(x_0)}{\partial x} x_1 = -2 D_0 D_1 x_0 + g(\mu, x_0, D_0 x_0), \tag{15}$$

order $\mathcal{O}(\varepsilon^i)$:

$$D_0^2 x_i + c_1 x_i + c_2 l x_0^{l-1} x_i = -2(D_0 D_i x_0) - D_0\left(\sum_{l=1}^{i-1} D_l x_{i-l}\right) - \sum_{j=1}^{i-1} D_j\left(\sum_{l=0}^{i-j} D_l x_{i-j-l}\right)$$
$$- c_2 \sum_{j_0+\cdots+j_{i-1}=l} \frac{l!}{j_0!\cdots J_{i-1}!}\prod_{k=0}^{i-1} x_k^{j_k}$$
$$+ \sum_{m=0}^{M} \mu_m\left[\sum_{j_{m_0}+\cdots+j_{m_N}=p_m} \frac{p_m!}{j_{m_0}!\cdots j_{m_N}!}\prod_{k=0}^{N} x_k^{j_{m_k}}\right]$$
$$\left[\sum_{h_{m_0}+\cdots+h_{m_N}=q_m} \frac{q_m!}{h_{m_0}!\cdots h_{m_N}!}\prod_{k=0}^{N}\left(\sum_{l=0}^{k}(D_l x_{k-l})^{j_k}\right)\right], \tag{16}$$

where

$$\sum_{k=0}^{N} k j_k = i, \quad j_{k>i} = 0, \tag{17}$$

$$\left(\sum_{k=0}^{N} k j_{mk}\right)\left(\sum_{k=0}^{N} k h_{mk}\right) = i - 1. \tag{18}$$

The $\mathcal{O}(\varepsilon^0)$ order equation (14) has an exact analytical solution. It is expressed in terms of Jacobian elliptic functions, in the following form:

$$x_0 = F(A(T_1, T_2, \ldots, T_i)\,\mathrm{ep}(\omega T_0 + \Phi(T_1, T_2, \ldots, T_i), k)). \tag{19}$$

Here $\mathrm{ep}(.\,,k)$ is one convenient Jacobian elliptic function ($\mathrm{sn}(.\,,k)$, $\mathrm{cn}(.\,,k)$ or $\mathrm{dn}(.\,,k)$), $F(.)$ is a polynomial function of its arguments, and k the modulus of the elliptic function. The quantities A, ω and Φ are, respectively, the amplitude, the frequency and the phase. A survey of elliptic function properties is given in Appendix A. The arguments $(.\,,k)$ to ep will be suppressed throughout this section.

In our case, the function $F(.)$ can be expressed as follows:

$$F = A(\mathrm{ep}) \quad \text{for } f(x) = x^3, \tag{20}$$

$$F = A_1(\mathrm{ep})^2 + A_2 \quad \text{for } f(x) = x^2. \tag{21}$$

Let the primes denote the derivatives of elliptic function with respect to its argument $u = \omega T_0 + \Phi(T_1, T_2, \ldots, T_i)$. For a given (ep), $(\mathrm{ep})''$ can be written as

$$(\mathrm{ep})'' = [\alpha(k)(\mathrm{ep}) + \beta(k)(\mathrm{ep})^3]. \tag{22}$$

Here $\alpha(k)$ and $\beta(k)$ are functions of the modulus k. On the other hand, $(\mathrm{ep})'$ can be written as

$$(\mathrm{ep})' = \gamma(k)(\mathrm{ep}_1)(\mathrm{ep}_2), \tag{23}$$

where $\gamma(k)$ is a function of k, and (ep_1) and (ep_2) are other two elliptic functions, which are different from (ep) (see Table 1), given by

$$(\mathrm{ep}_1)^2 = n(k) + m(k)(\mathrm{ep})^2, \tag{24}$$

$$(\mathrm{ep}_2)^2 = l(k) + h(k)(\mathrm{ep})^2. \tag{25}$$

TABLE 1

Terms of relations and derivatives of the elliptic functions with respect to the arguments

ep	ep_1	ep_2	$\alpha(k)$	$\beta(k)$	$\gamma(k)$	$n(k)$	$m(k)$	$l(k)$	$h(k)$
(cn)	(sn)	(dn)	$2k^2-1$	$-2k^2$	-1	1	-1	$1-k^2$	k^2
(sn)	(cn)	(dn)	$-1-k^2$	$2k^2$	1	1	-1	1	$-k^2$
(dn)	(sn)	(cn)	$2-k^2$	-2	$-k^2$	$1/k^2$	$-1/k^2$	$-(1-k^2)/k^2$	$1/k^2$

Substituting equation (19) and its derivatives into equation (14) and equating coefficients of like powers of (ep) to zero yields

$$\text{for } f(x)=x^3,\quad \alpha(k)=-\frac{c_1}{\omega^2},\quad \beta(k)=-\frac{c_2A^2}{\omega^2}, \tag{26}$$

$$\begin{aligned}&\text{for } f(x)=x^2,\quad 2A_1\omega^2\gamma^2(k)n(k)l(k)+c_1A_2+c_2A_2^2=0,\\ &2A_1\omega^2[\gamma^2(k)(n(k)h(k)+m(k)l(k))+\alpha(k)]+c_1A_1+2c_2A_1A_2=0,\\ &2A_1\omega^2[\gamma^2(k)m(k)h(k)+\beta(k)]+c_2A_1^2=0.\end{aligned} \tag{27}$$

The explicit dependence of k and ω on A will be suppressed in the determination of higher orders approximation of the periodic perturbed solutions, i.e., they depend only on the stationary amplitude and not on its modulations.

The first order equation (15) can be rewritten as follows:

$$\omega^2x_1''+c_1x_1+c_2\frac{\partial f(x_0)}{\partial x}x_1=-2\omega(D_1x_0')+g(\mu,x_0,\omega x_0'), \tag{28}$$

where

$$D_1x_0'=(D_1A)\left[(\text{ep})'\frac{\partial F}{\partial \text{ep}\,\partial A}\right]+(D_1\Phi)x_0'', \tag{29}$$

$$x_0''=(\text{ep})''\frac{\partial F}{\partial \text{ep}}+(\text{ep}')^2\frac{\partial^2F}{\partial \text{ep}^2}. \tag{30}$$

It is worth noting that the homogeneous equation of equation (28) has x_0' as a solution. Multiplying both sides of equation (28) by x_0' and then integrating the equation, we obtain

$$\begin{aligned}&[\omega^2(x_0'.x_1'-x_1.x_0'')]_0^u+\int_0^u x_1\left[\omega^2x_0'''+c_1x_0'+c_2\frac{\partial f(x_0)}{\partial x}x_0'\right]\mathrm{d}\sigma\\ &=\int_0^u[-2\omega(D_1x_0')+g(\mu,x_0,\omega x_0')]x_0'\,\mathrm{d}\sigma.\end{aligned} \tag{31}$$

Differentiating equation (14) with respect to u leads to

$$\omega^2x_0'''+c_1x_0'+c_2\frac{\partial f(x_0)}{\partial x}x_0'=0. \tag{32}$$

Then equation (31) becomes

$$\omega^2(x_0'x_1'-x_1x_0'')|_0^u=\int_0^u[-2\omega(D_1x_0')+g(\mu,x_0,\omega x_0')]x_0'\,\mathrm{d}\sigma. \tag{33}$$

Note that x_0 is a periodic function with a real period $T(k)$ (in the normal range of k, i.e., $0\leqslant k^2\leqslant 1$, $T(k)$ is $4K[k]$ for sn and cn or $2K[k]$ for dn, $K[k]$ is the first kind complete elliptic integral). The functions x_0' and x_0'' are also periodic functions with the same period $T(k)$.

It is worth noting that, in the normal range of k, for the case of $f(x) = x^2$, the period $T(k) = 2K(k)$ and for $f(x) = x^3$ the period $T(k)$ depends on the (ep).

Assume that x_1 is also periodic function with the period $T(k)$. Then letting $u = T(k)$ in equation (33) gives

$$\int_0^{T(k)} [-2\omega(D_1 x_0') + g(\mu, x_0, x_0')].x_0' \, \mathrm{d}\sigma = 0. \tag{34}$$

The term $x_0'.x_0''$ is an odd function for any x_0 expressed in terms of a polynomial function of elliptic functions as in equation (19). Using the independence of time scales, then, the term related to $(D_1 \Phi)$ will vanish under integration over one period. Hence, from equation (34), the modulation equation of the amplitude, over one period, is given by

$$(D_1 A) = \frac{\int_0^{T(k)} g(\mu, x_0, \omega x_0').x_0' \, \mathrm{d}u}{2\omega \int_0^{T(k)} [(\mathrm{ep})' \partial^2 F/\partial \mathrm{ep}\, \partial A] x_0' \, \mathrm{d}u}. \tag{35}$$

It is worth noting that, in the case of $f(x) = x^3$, equation (35) takes into account only terms of $g(\mu, x, \dot{x})$ of the form $x^{2p}.\dot{x}^{2q+1}$ for all ep. In addition, it takes into account also terms of the form $x^{2p+1}\dot{x}^{2q+1}$ for ep = dn. In the case $f(x) = x^2$, only terms of the form $\dot{x}^{2q+1}$ (independent of the power relative to x) are taken into account (here p and q are integers).

A necessary periodicity condition is given by the vanishing of the amplitude modulation $(D_1 A) = 0$, i.e., the right-hand side of equation (35) has a non-zero solution. This condition arises, in a mathematically rigorous way, from Melnikov's approach for bifurcation of periodic or homoclinic orbits [16].

On the other hand, using a canonical transformation and the averaging method, Morozov [15] deduced a formula similar to the numerator in equation (35), that he called "the generating function". Specially, he reduced the problem of estimating the number of limit cycles to estimating the number of real zeros of

$$I_1(k) = \int_0^{T(k)} g(\mu, x_0, \omega x_0') x_0' \, \mathrm{d}u = 0. \tag{36}$$

For a rigorous discussion of the relation between the zeros of the averaged system and the original ODE see reference [15].

A particular solution of equation (28) with initial conditions $x_0'(0) = 0$, $x_1(0) = 0$, $x_1'(0) = 0$, respectively, can be expressed as

$$x_1(u) = \frac{x_0'(u)}{\omega^2} \int_0^u \frac{1}{x_0'^2(\sigma_1)} \left\{ \int_0^{\sigma_1} x_0' [g(\mu, x_0, \omega x_0') - 2\omega(D_1 \Phi) x_0''] \, \mathrm{d}\sigma_2 \right\} \mathrm{d}\sigma_1. \tag{37}$$

However, secular terms could be produced by x_0'' in the bracket on the right-hand side of equation (37). Indeed, integrating the last term of equation (37) leads to

$$x_0'(u) \int_0^u \frac{1}{x_0'^2} \left[\int_0^{\sigma_1} \frac{2(D_1 \Phi)}{\omega} x_0 x_0'' \, \mathrm{d}\sigma_2 \right] \mathrm{d}\sigma_1 = \frac{(D_1 \Phi)}{\omega} x_0'(u) u. \tag{38}$$

Here the term $x_0' u$ tends to infinity as $u \to \infty$. However, in order that equation (4) remains a uniformly valid expansion, $|x_1/x_0|$ should be bounded for all u (Lighthill condition). Hence, the modulation of the phase $(D_1 \Phi)$ is chosen to kill all terms that produce secular terms, i.e., to eliminate the parameter related to u. In particular, it is chosen to eliminate the coefficient of x_0'' in the bracket on the right-hand side of equation (37).

The elliptic functions (sn, cn, dn) are not closed under integration with respect to u (see Table A2 in Appendix A) because of the appearance of logarithmic functions. These latters are complex in general.

As two integrations are performed to obtain $x_1(u)$, there will be eventually such terms. To overcome this, inherent and inevitable, difficulty only the real part is considered.

Indeed, the perturbation function g commands the existence of secular terms. The phase Φ is chosen to kill the remaining secular terms which were not taken into account by the vanishing of the amplitude modulation equation (19). The phase Φ will eliminate the secular terms which are orthogonal to the solution $x_0'(u)$.

Thus, a non-zero phase is to be expected whenever g contains terms of the form $\dot{x}^{2p}$ in both the cases $f(x) = x^3$ and x^2. This fact will be illustrated in applications.

The ith order equation (16) can be written as

$$\omega^2 x_i'' + c_1 x_i + c_2 l x_0^{l-1} x_i = -2\omega(D_i x_0') + G(\mu, c_2, \omega, T_0, \ldots, T_{i-1}), \tag{39}$$

where

$$D_i x_0' = (D_i A)\left[(\mathrm{ep})' \frac{\partial F}{\partial \mathrm{ep}\, \partial A}\right] + (D_i \Phi) x_0'', \tag{40}$$

$$\mu = \mu(\mu_1, \ldots, \mu_M), \tag{41}$$

$$\begin{aligned} G(\mu, c_2, \omega, T_0, \ldots, T_{i-1}) = & - D_0\left(\sum_{l=1}^{i-1} D_l x_{i-l}\right) - \sum_{j=1}^{i-1} D_j\left(\sum_{l=0}^{i-j} D_l x_{i-j-l}\right) \\ & - c_2 \sum_{j_0+\cdots+j_{i-1}=l} \frac{l!}{j_0!\cdots J_{i-1}!} \prod_{k=0}^{i-1} x_k^{j_k} \\ & + \sum_{m=0}^{M} \mu_m \left[\sum_{j_{m_0}+\cdots+j_{m_N}=p_m} \frac{p_m!}{j_{m_0}!\cdots j_{m_N}!} \prod_{k=0}^{N} x_k^{j_{m_k}}\right] \\ & \times \left[\sum_{h_{m_0}+\cdots+h_{m_N}=q_m} \frac{q_m!}{h_{m_0}!\cdots h_{m_N}!} \prod_{k=0}^{N}\left(\sum_{l=0}^{k} (D_l x_{k-l})^{j_k}\right)\right]. \end{aligned} \tag{42}$$

The solution x_i and the modulation of the amplitude and the phase, with respect to T_i (where i is an integer greater than 1), are obtained using the same steps as the order $\mathcal{O}(\varepsilon^1)$. Hence, the modulation equation of the amplitude, over one period, is given by

$$(D_i A) = \frac{\int_0^T G(\mu, c_2, \omega, T_0, \ldots, T_i) x_0' \,\mathrm{d}u}{2\omega \int_0^T [(\mathrm{ep})' \partial^2 F / \partial(\mathrm{ep}) \partial A] x_0' \,\mathrm{d}u}. \tag{43}$$

The vanishing of the amplitude modulation (43) eliminates the secular terms in the direction of x_0'. A particular solution of equation (39) with initial conditions $x_0'(0) = 0,\ x_i(0) = 0,\ x_i'(0) = 0$, respectively, can be expressed as

$$\begin{aligned} x_i(u) = & \frac{x_0'(u)}{\omega^2} \int_0^u \frac{1}{x_0'^2(\sigma_1)} \\ & \left\{ \int_0^{\sigma_1} x_0' [G(\mu, c_2, \omega, T_0, \ldots, T_{i-1}) - 2\omega (D_i \Phi) x_0''] \,\mathrm{d}\sigma_2 \right\} \mathrm{d}\sigma_1. \end{aligned} \tag{44}$$

The modulation of the phase $(D_i \Phi)$ is chosen to kill all secular terms orthogonal to x_0'.

It is worth noting that the modulations of the phase play the same part as the expansion of the frequency in power series of ε in the LP method [19, 20]. On the other hand, the proposed MSM computes the modulations of the amplitude, which enables to handle the transient motion. Thus, the proposed MSM can be viewed, as in the case when circular functions are used, as a generalization of the elliptic LP method.

3. APPLICATIONS

3.1. CUBIC NON-LINEARITY

In this section we deal with the case $f(x) = x^3$. The unperturbed system contains the lowest order symmetric non-linear perturbation to the linear oscillator. The perturbed oscillator is described by

$$\ddot{x} + c_1 x + c_2 x^3 = \varepsilon g(\mu, x, \dot{x}). \tag{45}$$

We will assume that c_1 and c_2 are not both zero. For an exhaustive discussion see reference [15].

Using Table 1 and equation (26), the unperturbed solution, for any c_1 or c_2, can be expressed as

$$x = A\,\mathrm{cn}(u, k), \tag{46}$$

where

$$k^2 = \frac{c_2 A^2}{2\omega^2}, \quad \omega^2 = c_1 + c_2 A^2, \quad u = \omega t + \Phi, \tag{47}$$

here both A and Φ may be complex. To apply MSM, we restrict our attention to the part of the $(x, \dot{x})$ phase space containing closed orbits (i.e., unperturbed motions that are periodic). On these orbits both A and Φ are real and ω is non-negative. Table 2 lists the range of k values and the real period $T(k)$ in u of $\mathrm{cn}(u, k)$ for the closed orbits in Figure 1.

Equations (14) and (15), expressing the zero and the first orders in ε, become

order $\mathcal{O}(\varepsilon^0)$

$$D_0^2 x_0 + c_1 x_0 + c_2 x_0^3 = 0, \tag{48}$$

order $\mathcal{O}(\varepsilon^1)$

$$D_0^2 x_1 + c_1 x_1 + 3c_2 x_0^2 x_1 = -2D_0 D_1 x_0 + g(\mu, x_0, D_0 x_0). \tag{49}$$

TABLE 2

Parameter values for periodic motions in the unperturbed system

c_1	c_2	Region	k^2	$T(k)$
>0	>0	Entire space	$0 \leqslant k^2 \leqslant 1/2$	$2\pi \leqslant T(k) \leqslant 4K(\sqrt{1/2})$
	$=0$	Entire space	0	2π
	<0	Separatrix	$-\infty$	∞
		Inside separatrix	$-\infty < k^2 \leqslant 0$	$\infty > T(k) \geqslant 2\pi$
$=0$	>0	Entire space	$1/2$	$4K(\sqrt{1/2})$
<0	>0	Outside separatrix	$1/2 \leqslant k^2 \leqslant 1$	$4K(\sqrt{1/2}) \leqslant T(k) < \infty$
		Separatrix	1	∞
		Inside separatrix	$1 < k^2 \leqslant \infty$	$\infty > T(k) \geqslant 0$

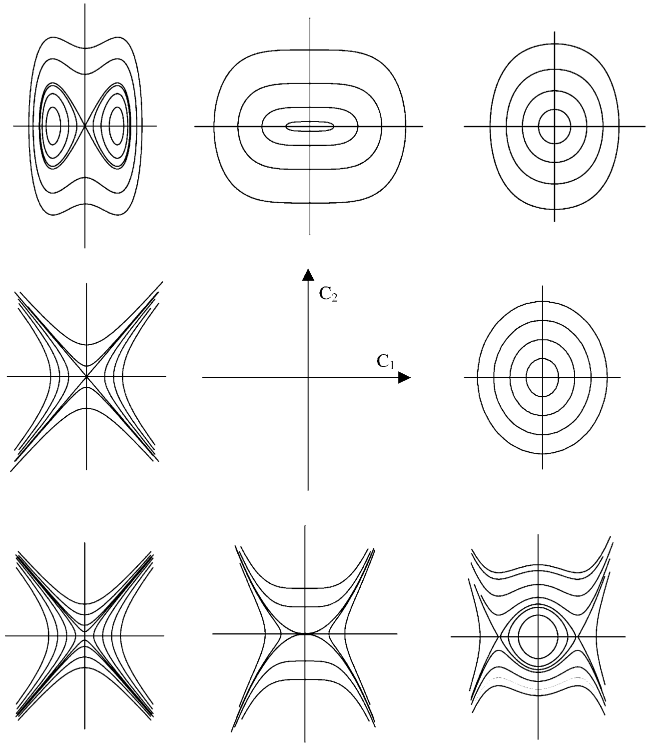

Figure 1. Orbits in the $(x, \dot{x})$ phase plane for the unperturbed Duffing oscillator. Each portrait shows a typical diagram for the region of the (c_1, c_2) parameter space upon which it lies. In each portrait, x is plotted horizontally, $\dot{x}$ is plotted vertically.

The solution of equation (48) is given by

$$x_0 = A(T_1, T_2, \ldots)\mathrm{cn}(\omega T_0 + \Phi(T_1, T_2, \ldots), k). \tag{50}$$

Following equation (35), the modulation equation of amplitude is given by

$$(D_1 A) = \frac{\int_0^{4K(k)} g(\mu, x_0, \omega x_0')x_0' \,\mathrm{d}u}{2\omega \int_0^{4K(k)} (\mathrm{cn})' x_0' \,\mathrm{d}u}. \tag{51}$$

The modulation of the phase and the solution x_1 are computed following the discussions given in section 2.

3.1.1. *Useful transformations*

It is worth noting that the previous developments are valid for any given c_1 and c_2. However, we can distinguish three cases.

Case 1: $c_1 \geqslant 0$ *and* $c_2 \geqslant 0$. All the previous developments are used for k in the normal range, i.e., $0 \leqslant k^2 \leqslant 1$.

Case 2: $c_1 < 0$ *and* $c_2 > 0$. In this case $k^2 > 1$. Consequently, the reciprocal modulus transformations (RMT) listed in Table A2 are used, specially the following transformations are made:

$$u \to ku, \quad k \to \frac{1}{k}, \quad K(k) \to \frac{1}{k} K\left(\frac{1}{k}\right), \tag{52}$$

$$E(k) \to k\left(E\left(\frac{1}{k}\right) + \frac{k'^2}{k^2} K\left(\frac{1}{k}\right)\right), \quad Z(u,k) \to kZ\left(ku, \frac{1}{k}\right). \tag{53}$$

The period is given by $T(k) = 4K(1/k)/k$. Alternatively, we could have taken $x_0 = A\,\mathrm{dn}(u,k)$ and did the same developments as done with $\mathrm{cn}(u,k)$ in order to work in the normal range of k.

Case 3: $c_1 > 0$ *and* $c_2 < 0$. In this case $k^2 < 0$. The imaginary modulus transformations (IMT) listed in Table A2 are used, specially the following transformations are made:

$$u \to k'u = \hat{u}, \quad k^2 \to -\frac{k^2}{k'^2} = \hat{k}^2. \tag{54}$$

The period is given by $T(k) = 4K(\hat{k})$. Alternatively, we could have taken $x_0 = A\,\mathrm{sn}(u,k)$ and did the same developments as done with $\mathrm{cn}(u,k)$ to work in the normal range of k.

Note that the transformations listed above allow mainly to extract the real part of the solutions.

It is worth pointing out that explicit computations, given in the examples, are performed using the symbolic language Mathematica [28].

3.1.2. *Examples*

The three cases related to the signs of c_1 and c_2 will be illustrated through three applications. Consider the generalized van der Pol oscillator with cubic non-linearity in the form

$$\ddot{x} + c_1 x + c_2 x^3 = \varepsilon[(\mu_1 + \mu_2 x^2)\dot{x} + \mu_3 x]. \tag{55}$$

Here $g(\mu, x, \dot{x}) = (\mu_1 + \mu_2 x^2)\dot{x} + \mu_3 x$. Through the discussions in section 2, about the role of g in generating secular terms, the secular terms caused by the two first terms of g (since they are of the form $\dot{x}^{2p+1}$, with $p = 0$ in our case) are killed by the vanishing of the amplitude modulation equation. Those related to the third term of g (since it is of the form $\dot{x}^{2p}$, with $p = 0$ in our case) will be taken into account by the modulation of the phase.

The modulation equation of amplitude (51) is given by

$$(D_1 A) = \frac{A}{2}\left[\mu_1 + \mu_2 A^2 \frac{I_2^K}{I_1^K}\right], \tag{56}$$

where

$$I_1^K = \int_0^{4K} (\mathrm{sn})^2 (\mathrm{dn})^2 \,\mathrm{d}u = \frac{4K}{3k^2}\left[k'^2 + (2k^2 - 1)\frac{E}{K}\right], \tag{57}$$

$$I_2^K = \int_0^{4K} (\mathrm{sn})^2 (\mathrm{cn})^2 (\mathrm{dn})^2 \,\mathrm{d}u = \frac{4K}{15k^4}\left[k'^2(k^2 - 2) + 2(k^4 + k'^2)\frac{E}{K}\right]. \tag{58}$$

Consequently, the stationary amplitude is written as

$$A^2 = -\frac{\mu_1}{\mu_2}\frac{I_1^K}{I_2^K}. \tag{59}$$

For oscillator (55), the term related to μ_3 leads to a secular term, thus

$$(D_1\Phi) = -\mu_3 \frac{E}{2k'^2 K\omega}. \tag{60}$$

The expression of x_1 is given in connection with equation (37) by

$$x_1(u) = \frac{x_0'}{\omega}\int_0^u \frac{\mu_1 I_1(\sigma_1) + \mu_2 A^2 I_2(\sigma_1)}{\mathrm{sn}^2(\sigma_1)\mathrm{dn}^2(\sigma_1)}\,\mathrm{d}\sigma_1 - \frac{x_0'}{\omega^2}\frac{\mu_3}{2k'^2}\left[Z(u) - k^2(\mathrm{sn})\frac{(\mathrm{cn})}{(\mathrm{dn})}\right], \tag{61}$$

where

$$I_1(\sigma_1) = \int_0^{\sigma_1} \mathrm{sn}^2(\sigma_2)\mathrm{dn}^2(\sigma_2)\,\mathrm{d}\sigma_2, \tag{62}$$

$$I_2(\sigma_1) = \int_0^{\sigma_1} \mathrm{sn}^2(\sigma_2)\mathrm{cn}^2(\sigma_2)\mathrm{dn}^2(\sigma_2)\,\mathrm{d}\sigma_2. \tag{63}$$

The computation of I_1 and I_2 leads to

$$I_1(\sigma_1) = \frac{1}{3k^2}[(2k^2-1)E(\sigma_1) + k'^2\sigma_1 - k^2\,\mathrm{sn}(\sigma_1)\mathrm{cn}(\sigma_1)\mathrm{dn}(\sigma_1)], \tag{64}$$

$$\begin{aligned} I_2(\sigma_1) = {}& \frac{1}{15k^4}[k'^2(k^2-2)\sigma_1 + 2(k^4+k'^2)E(\sigma_1) \\ &+ k^2\,\mathrm{sn}(\sigma_1)\mathrm{cn}(\sigma_1)\mathrm{dn}(\sigma_1)(3k^2\,\mathrm{sn}^2(\sigma_1) - 1 - k^2)]. \end{aligned} \tag{65}$$

Integrating equation (61) and taking into account equation (59), the solution x_1 is given by

$$\begin{aligned} x_1(u) = {}& -\frac{A}{\omega}(\mathrm{sn})(\mathrm{dn})\Big[R_1(\mathrm{sn})^2 + R_2\ln(\Theta(u)) + R_3 Z(u) \\ & \left[R_4 Z(u) + K(k)(\mathrm{cn})\left(k'^2\frac{(\mathrm{dn})}{(\mathrm{sn})} + k^4\frac{(\mathrm{sn})}{(\mathrm{dn})}\right)\right] \\ & - R_2\ln(\Theta(0)) - R_3 k'^2 K(k)\left(1 - \frac{E(k)}{K(k)}\right)\Big] \\ & + \frac{\mu_3}{2k'^2\omega^2}A(\mathrm{sn})(\mathrm{dn})\left[Z(u) - k^2(\mathrm{sn})\frac{(\mathrm{cn})}{(\mathrm{dn})}\right], \end{aligned} \tag{66}$$

where

$$R_1 = \mu_1\frac{(k^2-k'^2)^2}{6k'^2} - \mu_2 A^2\frac{(k^4+k'^2)(k'^2-k^2)}{15k^2k'^2}, \tag{67}$$

$$R_2 = \mu_2\frac{A^2}{5k^2}, \quad R_3 = 3k^2\mu_1 M, \quad R_4 = -\frac{(2k^2-1)}{2}K(k), \tag{68}$$

$$M = \frac{1}{3k^2[k'^2(k^2-2)K(k) + 2(k^4+k'^2)E(k)]}. \tag{69}$$

Here $\Theta(u)$ is the theta function which is an even periodic function with period $2K(k)$, and $Z(u)$ is the Jacobian zeta function which is odd with period $2K(k)$ (see Appendix A).

Thus, the solution up to second order of ε is given as

$$x(t) = x_0(t) + \varepsilon x_1(t) + \mathcal{O}(\varepsilon^2). \tag{70}$$

This solution is valid for all c_1 and c_2 (both not zero), through the use of the IMT and the RMT given in Table A2.

To illustrate case 1, we consider the following equation:

$$\ddot{x} + x^3 = \varepsilon(1 - x^2)\dot{x}, \tag{71}$$

where $c_1 = 0$, $c_2 = 1$, $\mu_1 = 1$ and $\mu_2 = -1$. From equations (47) and (59), we have $\omega = A = 1{\cdot}9098$ and $k = \sqrt{0{\cdot}5}$. The approximated solution x is given by equation (70).

In Figure 2, and in all the subsequent figures, comparisons between the limit cycle phase portraits $(x, \dot{x})$, for the cases $\varepsilon = 0{\cdot}1$, $0{\cdot}4$ and 1 are given. The following conventions are adopted: the first order elliptic method is represented by a continuous line, the MSM up to second order by dashed lines and the numerical integration using Runge–Kutta (RK) method by dots.

To illustrate case 2, we consider the equation

$$\ddot{x} - x + x^3 = \varepsilon(-1 + 1{\cdot}1x^2)\dot{x}, \tag{72}$$

where $c_1 = -1$, $c_2 = 1$, $\mu_1 = -1$ and $\mu_2 = 1{\cdot}1$. For this case, we have two types of limit cycles (see Figure 1, for $c_1 < 0$ and $c_2 > 0$) deduced from equations (47) and (59). The first one is called the outer limit cycle and has a modulus k in the normal range. The unknowns are given by $k = \sqrt{0{\cdot}747608}$, $A = 1{\cdot}73762$ and $\omega = 1{\cdot}42103$. The second one is called the inner limit cycle (since it is contained in the outer one) and has a modulus greater than one. The unknowns are given by $k = \sqrt{1{\cdot}14029}$, $A = 1{\cdot}3345$ and $\omega = 0{\cdot}8836$. For this latter case, the RMT are used to compute the solution x given by equation (70). In fact, equation (72) is Z_2 symmetric, consequently, it is invariant under a rotation by π. This means that there exists a third limit cycle related to the inner one (with the same amplitude).

In Figure 3, the inner limit cycle is plotted and the same comparisons as the first application are given. Figure 4 deals with the outer limit cycle.

Case 3 is illustrated by the following equation:

$$\ddot{x} + 2x - 0{\cdot}4x^3 = \varepsilon(0{\cdot}4 - x^2)\dot{x}, \tag{73}$$

where $c_1 = 2$, $c_2 = -0{\cdot}4$, $\mu_1 = 0{\cdot}4$, $\mu_2 = -1$ and $\mu_3 = 0$. For this case we have $\omega = 1{\cdot}15878$, $A = 1{\cdot}28183$ and $k^2 = -0{\cdot}244729$. As the modulus k is imaginary, the IMT are used to compute the solution x given in equation (70).

In Figure 5, the limit cycle portraits for different ε are shown.

3.2. QUADRATIC NONLINEARITY

In this section, general equation (2) is assumed to be of the following form:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon g(\mu, x, \dot{x}). \tag{74}$$

The unperturbed system has two equilibria $(0, 0)$ and $(c_1/c_2, 0)$. Their stability depends on c_1 and c_2. Equations (14) and (15) become

$$D_0^2 x_0 + c_1 x_0 + c_2 x_0^2 = 0, \tag{75}$$

$$D_0^2 x_1 + c_1 x_1 + 2c_2 x_0 x_1 = -2(D_0 D_1 x_0) + g(\mu, x_0, D_0 x_0). \tag{76}$$

We first solve unperturbed system (75) with the ansatz,

$$x_0 = A_1(T_1, T_2, \ldots)\mathrm{cn}^2(\omega t + \Phi(T_1, T_2, \ldots)) + A_2. \tag{77}$$

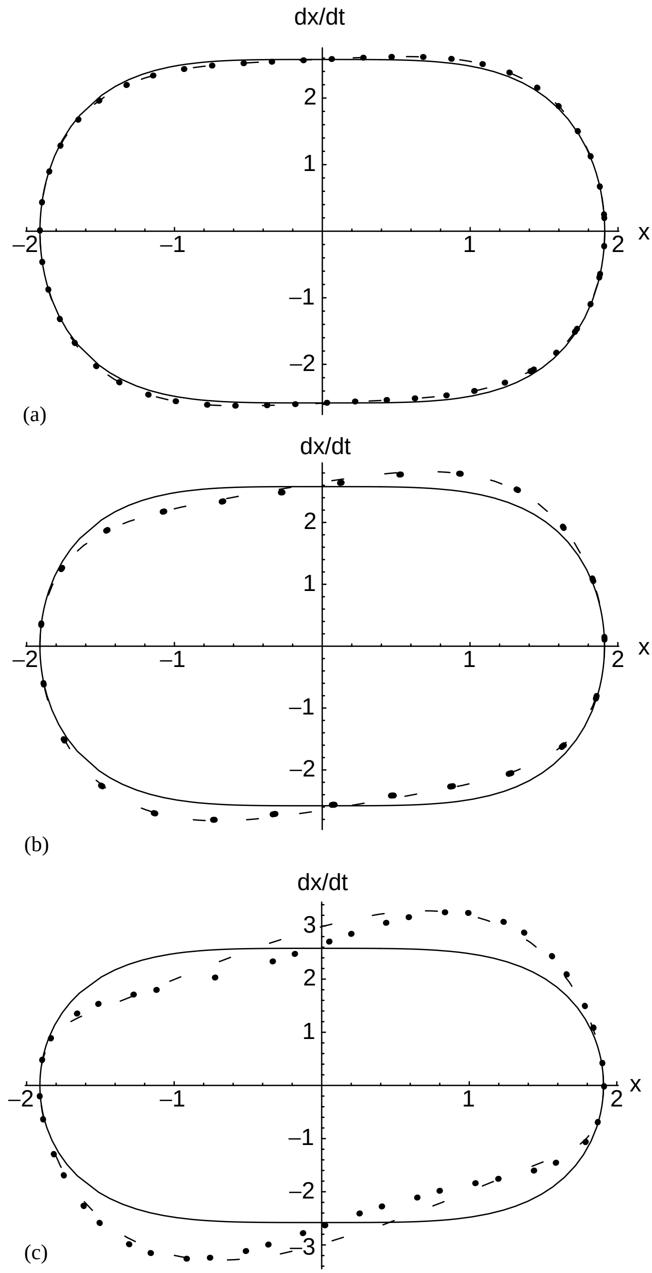

Figure 2. Limit cycle of equation (71) for (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}4$; (c) $\varepsilon = 1$. (——) First order elliptic perturbation method, (- - -) present method, (···) R–K method.

Consequently,

$$x_0' = -2A_1(\mathrm{sn})(\mathrm{cn})(\mathrm{dn}), \tag{78}$$

$$x_0'' = 2A_1[k'^2 + 2(k^2 - k'^2)(\mathrm{cn})^2 - 3k^2(\mathrm{cn})^4]. \tag{79}$$

Substituting equation (77) and its derivatives into equation (75) yields

$$Z_1 + Z_2(\mathrm{cn})^2 + Z_3(\mathrm{cn})^4 = 0, \tag{80}$$

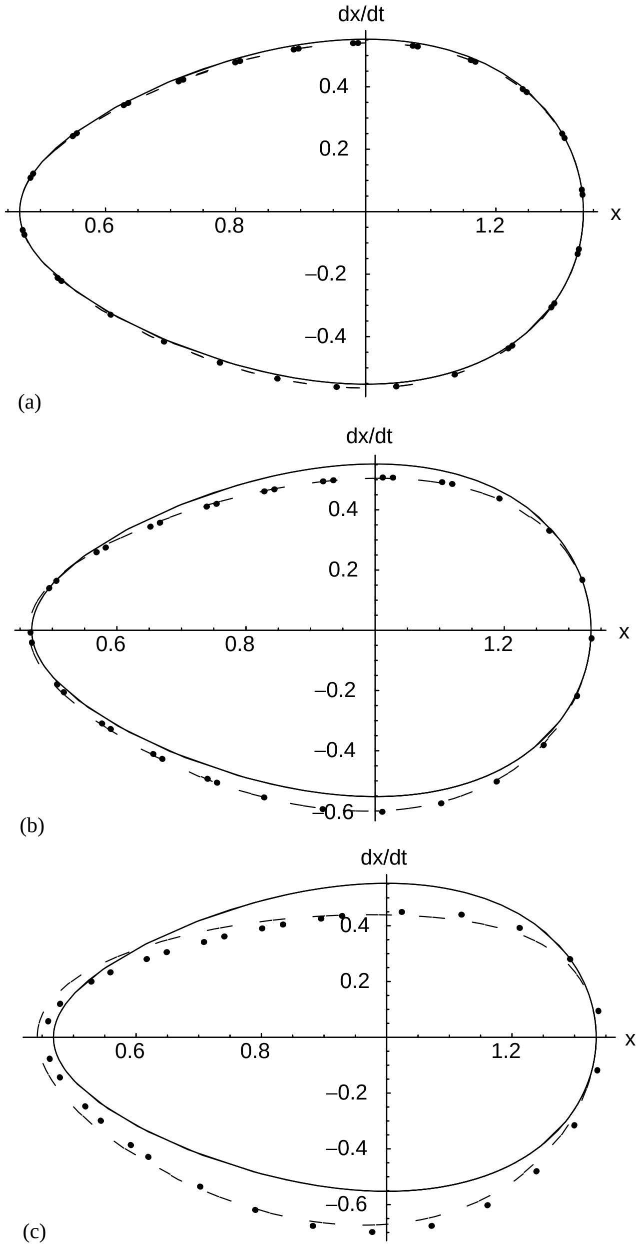

Figure 3. Inner limit cycle of equation (72) for (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}4$; (c) $\varepsilon = 1$. (——) First order elliptic perturbation method, (- - -) present method, (···) R–K method.

where

$$Z_1 = c_1 A_2 + 2k'^2\omega^2 A_1 + c_2 A_2^2, \tag{81}$$

$$Z_2 = A_1[c_1 + 2c_2 A_2 + 4k^2\omega^2 - 4k'^2\omega^2], \tag{82}$$

$$Z_3 = A_1[c_2 A_1 - 6k^2\omega^2]. \tag{83}$$

Requiring Z_1, Z_2 and Z_3 to vanish gives three non-linear algebraic equations relating the four unknowns A_1, A_2, k and ω (c_1, c_2 and μ are assumed to be known). These relations

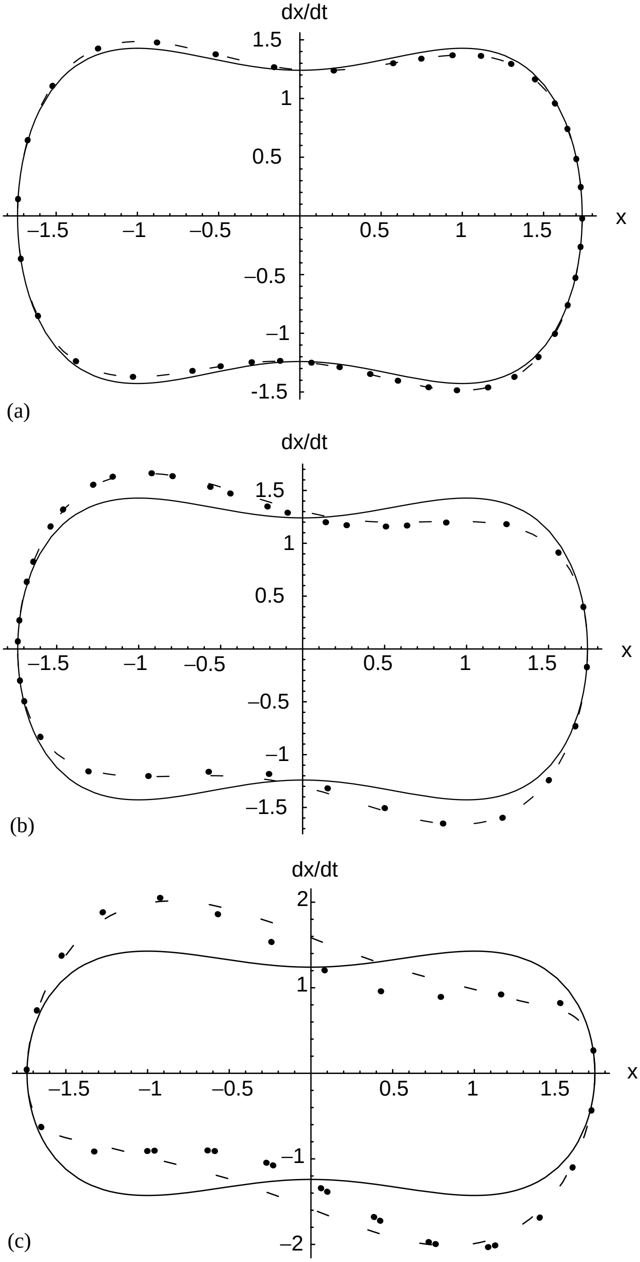

Figure 4. Outer limit cycle of equation (72) for (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}4$; (c) $\varepsilon = 1$. (——) First order elliptic perturbation method, (- - -) present method, (···) R–K method.

can be deduced directly form Table 1 and equation (27). Solving for A_1, A_2 and ω in terms of k, c_1 and c_2 provides

$$\omega = \left[\frac{c_1^2}{16\lambda}\right]^{1/4}, \qquad A_1 = \frac{6\omega^2 k^2}{c_2}, \tag{84, 85}$$

$$A_2 = -[4\omega^2(k^2 - k'^2) + c_1]/(2c_2), \tag{86}$$

where $\lambda = k'^2 + k^4$.

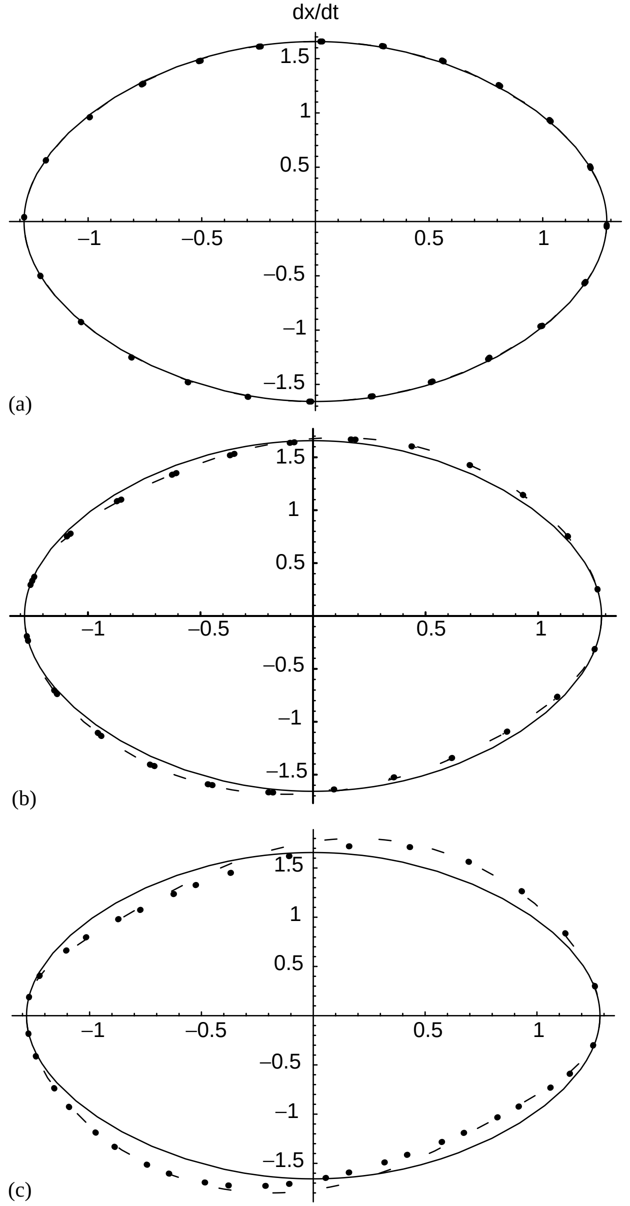

Figure 5. Limit cycle of equation (73) for (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}4$; (c) $\varepsilon = 1$. (——) First order elliptic perturbation method, (- - -) present method, (···) R–K method.

A fourth equation is given by the vanishing of amplitude modulation given by

$$(D_1 A_1) = \frac{\int_0^{2K} g(\mu, x_0, \omega x_0') x_0' \, \mathrm{d}u}{8\omega A_1 \int_0^{2K} (\mathrm{sn})^2 (\mathrm{cn})^2 (\mathrm{dn})^2 \, \mathrm{d}u} = 0. \tag{87}$$

The drift A_2 is assumed constant and independent of all scales of time. The modulations of the phase and the solution x_1 will be computed following the discussions given in section 2.

3.2.1. *Example*

Let us consider the following oscillator:

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon[(\mu_1 + \mu_2 x^2)\dot{x} + \mu_3 x]. \tag{88}$$

Here $g(\mu, x, \dot{x}) = (\mu_1 + \mu_2 x^2)\dot{x} + \mu_3 x$. Through the discussions in section 2, the secular terms caused by the two first terms of g (since they are of the form $\dot{x}^{2p+1}$, with $p = 0$ in our case) are killed by the vanishing of the amplitude modulation equation. Those related to the third term of g (since it is of the form $\dot{x}^{2p}$, with $p = 0$ in our case) will be taken into account by the modulation of the phase.

From equation (87), the modulation of the amplitude A_1 is given by

$$(D_1 A_1) = 4\omega A_1^2 \frac{N_1}{N_2}, \tag{89}$$

where

$$N_1 = C_a II_{11}^{2K} + C_b II_{12}^{2K} + C_c II_{13}^{2K}, \tag{90}$$

$$N_2 = 8\omega A_1 II_{11}^{2K}. \tag{91}$$

Here,

$$C_a = \mu_1 + \mu_2 A_2^2, \qquad C_b = 2\mu_2 A_1 A_2, \tag{92, 93}$$

$$C_c = \mu_2 A_1^2, \tag{94}$$

$$II_{11}^{2K} = \int_0^{2K} \mathrm{sn}^2(\sigma)\mathrm{cn}^2(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma, \tag{95}$$

$$II_{12}^{2K} = \int_0^{2K} \mathrm{sn}^2(\sigma)\mathrm{cn}^4(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma, \tag{96}$$

$$II_{13}^{2K} = \int_0^{2K} \mathrm{sn}^2(\sigma)\mathrm{cn}^6(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma. \tag{97}$$

The three latter integrals are deduced from the following integrals:

$$\begin{aligned} II_{11}(u) &= \int_0^{u} \mathrm{sn}^2(\sigma)\mathrm{cn}^2(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma \\ &= \left[\left[k'^2(k^2-1) + 2(k^4+k'^2)\frac{E}{K}\right]u \right. \\ &\quad + 2(k^4+k'^2)Z(u) - 3k^4\,\mathrm{sn}(u)\mathrm{cn}^3(u)\mathrm{dn}(u) \\ &\quad \left. + k^2(2k^2-1)\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u)\right]/(15k^4), \end{aligned} \tag{98}$$

$$\begin{aligned} II_{12}(u) &= \int_0^{u} \mathrm{sn}^2(\sigma)\mathrm{cn}^2(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma \\ &= \left[\left[k'^2(3k^4-15k^2+8) + (2k^2-1)(3k^4-3k^2+8)\frac{E}{K}\right]u \right. \\ &\quad + (2k^2-1)(3k^4-3k^2+8)Z(u) + k^2[4(2k^2-1)^2 \\ &\quad + 10k^2k'^2]\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u) \\ &\quad + 3k^4(2k^2-1)\mathrm{sn}(u)\mathrm{cn}^3(u)\mathrm{dn}(u) \\ &\quad \left. - 15k^6\,\mathrm{sn}(u)\mathrm{cn}^5(u)\mathrm{dn}(u)\right]/(105k^6), \end{aligned} \tag{99}$$

$$
\begin{aligned}
I_{13}(u) &= \int_0^u \mathrm{sn}^2(\sigma)\mathrm{cn}^6(\sigma)\mathrm{dn}^2(\sigma)\,\mathrm{d}\sigma \\
&= \Big[\Big[k'^2(5k^6 - 45k^4 + 48k^2 - 16) + (10k^8 - 20k^6 \\
&\quad + 66k^4 - 56k^2 + 16)\frac{E}{K}\Big]u \\
&\quad + (10k^8 - 20k^6 + 66k^4 - 56k^2 + 16)Z(u) \\
&\quad + k^2(2k^2 - 1)[8(2k^2 - 1)^2 + 27k^2k'^2]\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{dn}(u) \\
&\quad + k^4[6(2k^2 - 1)^2 + 14k^2k'^2]\mathrm{sn}(u)\mathrm{cn}^3(u)\mathrm{dn}(u) \\
&\quad + 5k^6(2k^2 - 1)\mathrm{sn}(u)\mathrm{cn}^5(u)\mathrm{dn}(u) \\
&\quad - 35k^8\,\mathrm{sn}(u)\mathrm{cn}^7(u)\mathrm{dn}(u)\Big]\Big/(315k^8).
\end{aligned} \tag{100}
$$

From the amplitude modulation equation (89), the stationary amplitude is obtained by solving the algebraic equation $N_1 = 0$. Thus, the stationary amplitude A_1, which must be positive, is given by

$$
A_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{101}
$$

where

$$
a = \mu_2 II_{13}^{2K}, \qquad b = 2\mu_2 A_2 II_{12}^{2K}, \qquad c = C_a II_{12}^{2K}. \tag{102−104}
$$

Following equation (37) the solution x_1 is given by

$$
x_1(u) = x_0'(u)\int_0^u \frac{I_1(\sigma)}{\omega x_0'^2}\,\mathrm{d}\sigma. \tag{105}
$$

Here we have

$$
\begin{aligned}
I_1(\sigma) &= \int_0^\sigma g(\mu, x_0, \omega x_0')x_0'\,\mathrm{d}\sigma_1 \\
&= 4\omega A_1^2[\alpha_0\sigma + \alpha_1 Z(\sigma) + \alpha_2(\mathrm{sn})(\mathrm{cn})(\mathrm{dn}) \\
&\quad + \alpha_3(\mathrm{sn})(\mathrm{cn})^3(\mathrm{dn}) + \alpha_4(\mathrm{sn})(\mathrm{cn})^5(\mathrm{dn}) \\
&\quad + \alpha_5(\mathrm{sn})(\mathrm{cn})^7(\mathrm{dn})] + \frac{\mu_3}{2}[A_1^2(\mathrm{cn})^4 + 2A_1A_2(\mathrm{cn})^2 - 2A_1A_2 - A_1^2],
\end{aligned} \tag{106}
$$

$$
\alpha_0 = N_1, \tag{107}
$$

$$
\begin{aligned}
\alpha_1 =& \frac{2C_a}{15k^4}(k^4 + k'^2) + \frac{C_b}{105k^6}(2k^2 - 1)(3k^4 - 3k^2 + 8) \\
&+ \frac{C_c}{315k^8}(10k^8 - 20k^6 + 66k^4 - 56k^2 + 16),
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\alpha_2 =& \frac{C_a}{15k^2}(2k^2 - 1) + \frac{C_b}{105k^4}[4(2k^2 - 1)^2 + 10k^2k'^2] \\
&+ \frac{C_c}{315k^6}[(2k^2 - 1)(8(2k^2 - 1)^2 + 27k^2k'^2)],
\end{aligned} \tag{109}
$$

$$
\alpha_3 = -\frac{C_a}{5} + \frac{C_c}{35k^2}(2k^2 - 1) + \frac{C_c}{315k^4}[6(2k^2 - 1)^2 + 14k^2k'^2], \tag{110}
$$

$$
\alpha_4 = -\frac{C_b}{7} + \frac{C_c}{63k^2}(2k^2 - 1), \qquad \alpha_5 = -\frac{C_c}{9}. \tag{111, 112}
$$

Taking into account the stationarity of the amplitude A_1 means $\alpha_0 = 0$. Set

$$
\begin{aligned}
D_{11} &= \int \frac{Z(\sigma)}{\mathrm{sn}^2(\sigma)\mathrm{cn}^2(\sigma)\mathrm{dn}^2(\sigma)}\,\mathrm{d}\sigma \\
&= \frac{Z(u)}{k'^4}\left[\frac{(\mathrm{sn})(\mathrm{dn})}{(\mathrm{cn})} - k'^4\frac{(\mathrm{cn})(\mathrm{dn})}{(\mathrm{sn})} + k^6\frac{(\mathrm{sn})(\mathrm{cn})}{(\mathrm{dn})}\right] - \frac{(1+k^4+k'^4)}{2k'^4}Z^2(\sigma) \\
&\quad + \frac{E}{k'^4K}[-(1+k^4+k'^4)\ln(\Theta(u)) - \ln(\mathrm{cn}) - k'^4\ln(\mathrm{sn}) - k^4\ln(\mathrm{dn})] \\
&\quad - \frac{1}{k'^4}\Big[-k'^2(1+k'^2)\ln(\Theta) - k'^2\ln(\mathrm{cn}) \\
&\quad + \frac{k^2}{2}(\mathrm{sn})^2 - k'^4\ln(\mathrm{sn}) + \frac{k^2k'^4}{2}(\mathrm{sn})^2 + \frac{k^6}{2}(\mathrm{sn})^2\Big],
\end{aligned} \tag{113}
$$

$$
\begin{aligned}
D_{12} &= \int \frac{1}{\mathrm{sn}(\sigma)\mathrm{cn}(\sigma)\mathrm{dn}(\sigma)}\,\mathrm{d}\sigma \\
&= \ln(\mathrm{sn}(\sigma)) - \frac{ln(\mathrm{cn}(\sigma))}{k'^2} + \frac{k^2}{k'^2}\ln(\mathrm{dn}(\sigma)),
\end{aligned} \tag{114}
$$

$$D_{13} = \int \frac{\mathrm{cn}(\sigma)}{\mathrm{dn}(\sigma)\mathrm{sn}(\sigma)}\,\mathrm{d}\sigma = \ln\left(\frac{\mathrm{sn}(\sigma)}{\mathrm{dn}(\sigma)}\right), \tag{115}$$

$$D_{14} = \int \frac{\mathrm{cn}^3(\sigma)}{\mathrm{dn}(\sigma)\mathrm{sn}(\sigma)}\,\mathrm{d}\sigma = \ln(\mathrm{sn}(\sigma)) + \frac{k'^2}{k^2}\ln(\mathrm{dn}(\sigma)), \tag{116}$$

$$D_{15} = \int \frac{\mathrm{cn}^5(\sigma)}{\mathrm{dn}(\sigma)\mathrm{sn}(\sigma)}\,\mathrm{d}\sigma = \ln(\mathrm{sn}(\sigma)) + \frac{\mathrm{cn}^2(\sigma)}{2k^2} - \left(\frac{k'^4}{k^4}\right)\ln(\mathrm{dn}(\sigma)). \tag{117}$$

The solution x_1 given by equation (105) is then expressed as follows:

$$x_1 = \frac{x_0'}{\omega}\left[\sum_{j=1}^{6}\alpha_j D_{1j} - \lim_{u\to 0}\sum_{j=1}^{6}\alpha_j D_{1j}\right], \tag{118}$$

where

$$\alpha_6 = \frac{\mu_3}{8\omega k'^4 A_1^2}, \tag{119}$$

$$D_{16} = \left[\beta_1 Z(u) + \beta_2\frac{(\mathrm{sn})(\mathrm{cn})}{(\mathrm{dn})} + \beta_3\frac{(\mathrm{sn})(\mathrm{dn})}{(\mathrm{cn})}\right], \tag{120}$$

$$\beta_1 = (1+k^2)(A_1^2 + 2A_1A_2) - k'^2A_1^2, \tag{121}$$

$$\beta_2 = -k^4(A_1^2 + 2A_1A_2) + k'^2k^2A_1^2, \tag{122}$$

$$\beta_3 = -A_1^2 - 2A_1A_2. \tag{123}$$

The elimination of secular terms, caused by the linear perturbation related to μ_3, leads to the definition of the modulation of the phase

$$(D_1\Phi) = \frac{\mu_3}{8\omega k'^4 K A_1^2}[A_1^2(-k'^2K + 2k^2E) + 2A_1A_2(-k'^2K + (1+k^2)E)]. \tag{124}$$

As an example, we consider the equation

$$\ddot{x} + x + x^2 = \varepsilon(0{\cdot}1 - x^2)\dot{x}, \tag{125}$$

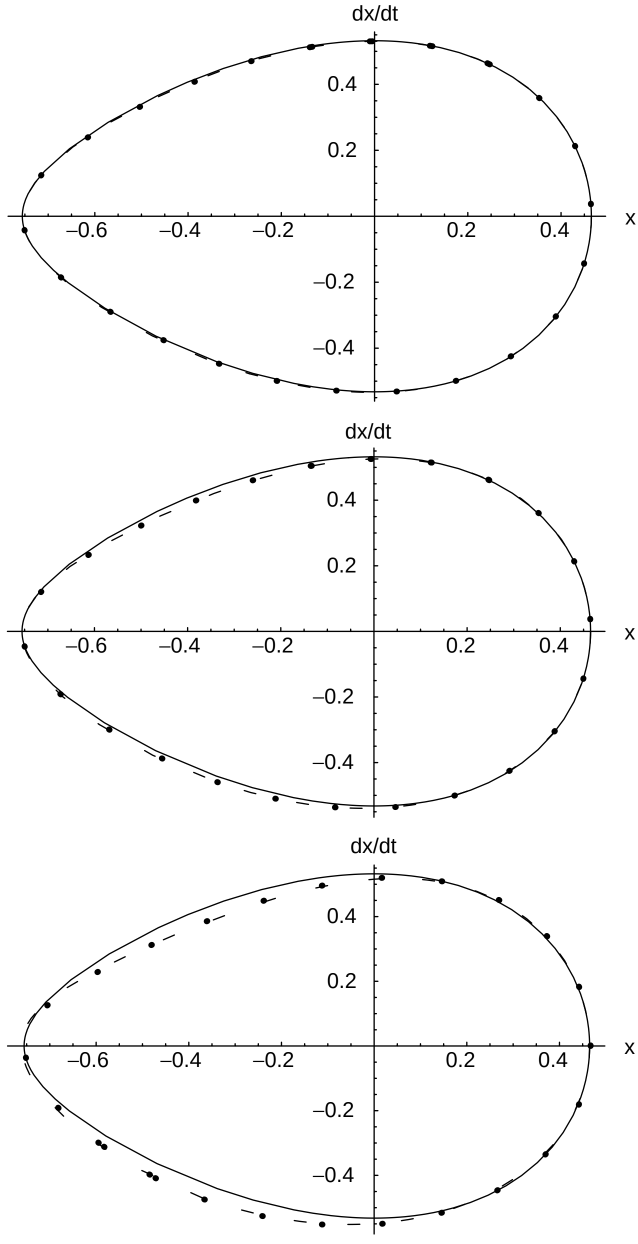

Figure 6. Limit cycle of equation (125) for (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}4$; (c) $\varepsilon = 1$. (——) First order elliptic perturbation method, (- - -) present method, (···) R–K method

where $c_1 = 1$, $c_2 = 1$, $\mu_1 = 0{\cdot}1$, $\mu_2 = -1$ and $\mu_3 = 0$. From equations (84)–(86) and (101), we have $\omega = 0{\cdot}528297$, $A_1 = 1{\cdot}22047$, $A_2 = -0{\cdot}755452$ and $k^2 = 0{\cdot}72882$.

In Figures 6, the limit cycle portraits for $\varepsilon = 0{\cdot}1$, $0{\cdot}4$ and 1 are shown.

4. CONCLUSION

We have extended the multiple scales method (MSM), initially developed for weakly non-linear systems, to strongly non-linear self-excited systems. Two types of non-linearities are considered for the unperturbed system: quadratic and cubic. The solutions are expressed in terms of Jacobian elliptic functions. Higher order approximations, of periodic solutions as well as modulations of amplitude and phase, are derived. Moreover, applications of the MSM have shown good agreement with numerical integration even for moderately large values of the perturbation parameter ε.

However, more applications are required in order to strengthen the proposed method. The future challenge is to apply the proposed MSM to non-autonomous oscillators and consequently to deal with resonances and complicated dynamics.

ACKNOWLEDGMENTS

The work was partially supported by DFG priority program "Analysis, Modelling and Simulation of Multiscale Problems".

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APPENDIX A: ELIPTIC FUNCTIONS

For the convenience of readers, we collect some facts on Jacobian elliptic functions. See reference [29] for details. Jacobian elliptic functions satisfy algebraic relations which are analogous to those for trigonometric functions. The fundamental three elliptic functions are $\mathrm{cn}(u,k)$, $\mathrm{sn}(u,k)$ and $\mathrm{dn}(u,k)$. Each of the elliptic functions depends on the modulus k as well as the argument u. Note that the elliptic functions sn and cn may be thought of as generalizations of sin and cos where their period depends on the modulus k.

The elliptic functions satisfy the following identities, which are analogous to $\sin^2 + \cos^2 = 1$:

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1, \qquad k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1, \quad 1 - k^2 + k^2\,\mathrm{cn}^2 = \mathrm{dn}^2. \tag{A.1}$$

Only two of these three relations are algebraically independent. In Table A1, additional properties of Jacobi elliptic functions are summarized.

Here $K(k)$ is the complete elliptic integral of the first kind,

$$K(0) = \pi/2, \quad K(1) = +\infty.$$

In addition to the sn, cn and dn functions, there are three other frequently encountered elliptic functions. First, there is the amplitude $\mathrm{am}(u,k) = \theta$ which is the inverse of the

TABLE A1

Properties of Jacobi elliptic functions

Property	$\mathrm{sn}(\cdot,k)$	$\sin(\cdot)$	$\mathrm{cn}(\cdot,k)$	$\cos(\cdot)$	$\mathrm{dn}(\cdot,k)$
Max. value	1	1	1	1	1
Min. value	-1	-1	-1	-1	$\sqrt{1-k^2}$
Period	$4K(k)$	2π	$4K(k)$	2π	$2K(k)$
Parity	Odd	Odd	Even	Even	Even
$\mathrm{d}f/\mathrm{d}u$	cn dn	cos	$-$sn dn	$-$sin	$-k^2$ sn cn
$f_{k=0}$	sin	sin	cos	cos	1
$\int(.)\,\mathrm{d}u$	$(1/k)\ln(\mathrm{dn} - k\,\mathrm{cn})$	$-$cos	$(1/k)\arcsin(k\,\mathrm{sn})$	sin	am

TABLE A2

Evaluation of elliptic quantities for various values of k^2

IMT $k^2 \leqslant 0$	$k^2 = 0$	Normal range $0 < k^2 < 1$	$k^2 = 1$	RMT $1 < k^2$
$\hat{k}^2 = -k^2/k'^2$	0	k^2	1	$\hat{k}^2 = 1/k^2$
$\hat{k}' K(\hat{k})$	$\pi/2$	$K(k)$	∞	
$E(\hat{k})/\hat{k}'$	$\pi/2$	$E(k)$	1	
$4\hat{k}' K(\hat{k})$	2π	$T(k) = 4K(k)$	∞	$4\hat{k}K(\hat{k})$
$\hat{u} = k'u$		u	u	$\hat{u} = ku$
$\mathrm{cn}(\hat{u}, \hat{k})/\mathrm{dn}(\hat{u}, \hat{k})$	$\cos u$	$\mathrm{cn}(u, k)$	$\mathrm{sech}\, u$	$\mathrm{dn}(\hat{u}, \hat{k})$
$1/\mathrm{dn}(\hat{u}, \hat{k})$	1	$\mathrm{dn}(u, k)$	$\mathrm{sech}\, u$	$\mathrm{cn}(\hat{u}, \hat{k})$
$\hat{k}' \mathrm{sn}(\hat{u}, \hat{k})/\mathrm{dn}(\hat{u}, \hat{k})$	$\sin u$	$\mathrm{sn}(u, k)$	$\tanh u$	$\hat{k}\mathrm{sn}(\hat{u}, \hat{k})$
$\dfrac{Z(\hat{u}, \hat{k})}{\hat{k}'} - \dfrac{(\hat{k}^2 \mathrm{sn}(\hat{u}, \hat{k})\mathrm{cn}(\hat{u}, \hat{k})}{(\hat{k}' \mathrm{dn}(\hat{u}, \hat{k})))}$	0	$Z(u, k)$	$\tanh u$	—

incomplete elliptic integral of the first kind $F(\theta, k)$. This function maps the elliptic argument u onto trigonometric argument θ so that the period $4K(k)$ in u equals the period 2π in θ.

Second, there is $E(\theta, k)$, the incomplete elliptic integral of the second kind. It is often written in abbreviated notation as $E(u)$ since θ depends on u (via am function) and the dependence on k is understood. Both $E(u)$ and $\mathrm{am}(u)$ are not periodic in u.

Finally, there is the Jacobi zeta function $Z(\theta, k)$, a linear combination of $E(u)$ and u

$$Z(\theta, k) = E(\theta, k) - \frac{E(k)}{K(k)} F(\theta, k), \tag{A.2}$$

$$Z(u) = \frac{\Theta(u)'}{\Theta(u)}, \tag{A.3}$$

where $\Theta(u)$ is a theta function defined by

$$\Theta(u) = 1 + 2 \sum_{1}^{\infty} (-1)^m q^{m^2} \cos(2mv), \tag{A.4}$$

$$q = e^{-(\pi K'/K)}, \quad v = \frac{\pi u}{2K}, \tag{A.5}$$

where $K' = K(k')$ with $k'^2 = 1 - k^2$.

We have used in equation (66) the expression

$$\lim_{u \to 0} \frac{Z(u, k)}{\mathrm{sn}(u, k)} = 1 - \frac{E(k)}{K(k)}. \tag{A.6}$$

The argument u is identified as the incomplete elliptic integral of the first kind which is denoted $F(\theta, k)$. This identification shows that u also depends on k. The value of k normally ranges from 0 to 1.

Outside the normal range of k, elliptic functions and integrals can be computed using modulus transformations. Table A2 lists the RMT and the IMT that are used to evaluate elliptic functions and integrals outside the range $0 \leqslant k^2 < 1$. $T(k)$ denotes the *real* period of $\mathrm{cn}(u, k)$ in u. For the IMT, the real period is simply the transformation of $4K(k)$, but for the RMT the real period is $\mathrm{Re}(4K(k))$ with $K(k)$ being complex for $k^2 > 1$.