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STABILITY ANALYSIS OF PARAMETRICALLY EXCITED SYSTEMS USING SPECTRAL COLLOCATION

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The spectral collocation method is used to determine the stability of parametrically excited systems and compared with the traditional transition matrix approach. Results from a series of test problems demonstrate that spectral collocation converges rapidly. In addition, the spectral collocation method preserves the sparsity of the underlying system matrices, a property not shared by the transition matrix approach. As a result, spectral collocation can be used for very large systems and can utilize sparse eigensolvers to reduce computational memory and time. For the large-scale system studied (up to 40 degrees of freedom), the spectral collocation method was on average an order of magnitude faster than the transition matrix approach using Matlab. This computational advantage is implementation specific; in a C implementation of the algorithm, the transition matrix method is faster than the spectral collocation. Overall, the method proves to be simple, efficient, reliable, and generally competitive with the transition matrix method.

1. INTRODUCTION

This paper evaluates a new numerical algorithm for determining the stability of parametrically excited systems, i.e., systems of homogeneous ordinary differential equations with periodic coefficients. These mathematical models have been used to describe numerous mechanical systems including reciprocating machinery [1], belt drives and moving bands [2–6], asymmetric circular saws [7], and disk drives [8, 9]. The design of these devices depends in part on their dynamic stability and, to that end, efficient algorithms for determining stability are helpful.

Numerous algorithms are available for evaluating the stability of parametrically excited systems. Analytic procedures including perturbation and Lyapunov methods can be employed if the system under evaluation has either a small perturbation parameter or a known Lyapunov function [10]. When applicable, analytic methods can be efficient and insightful. However, the class of applicable systems is limited, and the methods are normally only applicable to systems with relatively low degrees of freedom due, in part, to algebraic complexity. Wu *et al.* [11] present a modified perturbation algorithm for large-scale systems, but their principle example of a large-scale system possesses only 11 degrees of freedom.

The most important and widely used numerical method for determining the stability of parametrically excited systems involves calculating the eigenvalues of the transition matrix [10]. Although robust, the transition matrix approach is computationally expensive. For an m degree of freedom, second order system, computation of the transition matrix

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requires $O(m^3)$ operations: integration of $2m$ initial conditions with evaluation of the system derivatives requiring $O(m^2)$ numerical operations at each time step, assuming that the leading order matrices have been previously inverted or decomposed. Previous researchers have proposed alternative integration procedures in order to reduce the computational expense of computing the transition matrix [12, 13]. However, subsequent to computing the transition matrix, its eigenvalues must be computed, which requires $O(m^3)$ operations as well [14]. Hence, regardless of any efficiencies in calculating the elements of the transition matrix, stability evaluation using the numerical transition matrix approach requires $O(m^3)$ operations.

More important than computational expense, perhaps, is the fact that all elements of the transition matrix must be computed and stored. This means that the transition matrix will be full and the algorithm will have no advantages in storage or efficiency when the underlying system is sparse [14].

This paper evaluates the use of spectral collocation to determine the stability of parametrically excited systems. For an m-degree-of-freedom system with N collocation points per degree of freedom, the method involves solving an mN-degree-of-freedom quadratic eigenvalue problem. Any sparsity in the system matrices is retained by the stability eigenvalue problem. Consequently, the method can take full advantage of sparse computer storage and sparse eigenvalue solvers to efficiently evaluate the stability of large-scale systems. Although this new analysis procedure uses well-established techniques [15], the authors are unaware of any previous investigation.

Numerical experiments demonstrate the accuracy of the spectral collocation method on a number of test problems. Convergence criteria are set forth that ensure the accuracy of the Floquet eigenvalues and eliminate spurious, unstable eigenvalues. Comparisons of computational effort prove ambiguous: in a Matlab implementation of the algorithm, spectral collocation was roughly an order of magnitude faster than the transition matrix method; in a C implementation, the time required for spectral collocation method was approximately the same as with Matlab, but the transition matrix method was an order of magnitude faster than spectral collocation. Given these conflicting results, we conclude only that the spectral collocation method is competitive with the transition matrix approach. The method is simple and reliable and offers an attractive alternative to other numerical stability algorithms for parametrically excited systems.

2. STABILITY ANALYSIS

Consider the second order, m degrees of freedom, linear parametrically excited system

$$\mathbf{M}\mathbf{u}_{,tt} + \mathbf{C}\mathbf{u}_{,t} + \mathbf{K}\mathbf{u} + \varepsilon\, \mathbf{P}(t)\mathbf{u} = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are constant $m \times m$ matrices, $\mathbf{u}$ is an m dimensional column vector, t is time, a comma indicates differentiation, ε is a constant measuring excitation magnitude, and $\mathbf{P}$ is a periodic, $m \times m$ matrix of period $2\pi/\Omega$, i.e.,

$$\mathbf{P}(t + 2\pi/\Omega) = \mathbf{P}(t). \tag{2}$$

We normalize t so that the period is 2π:

$$\tau = \Omega t. \tag{3}$$

According to the Floquet theory, the $2m$ fundamental solutions of equation (1) can be written as

$$\mathbf{u}(t) = \mathrm{e}^{\alpha\tau}\mathbf{v}(\tau), \tag{4}$$

where α is a complex constant known as a Floquet eigenvalue and $\mathbf{v}(\tau + 2\pi) = \mathbf{v}(\tau)$. The solution from (4) is unique only up to changes in α of an integer multiple of $\mathrm{i} = \sqrt{-1}$. The system stability is determined by α: if all $\mathrm{Re}[\alpha] < 0$ or if all $\mathrm{Re}[\alpha] \leqslant 0$ and all purely imaginary α are semi-simple (i.e., each possesses a distinct, linearly independent eigenvector), the system is stable; otherwise, it is unstable. Substituting equations (3) and (4) into equation (1) gives the eigenvalue equation

$$\Omega^2\mathbf{M}(\mathbf{v}_{,\tau\tau} + 2\alpha\mathbf{v}_{,\tau} + \alpha^2\mathbf{v}) + \Omega\mathbf{C}(\mathbf{v}_{,\tau} + \alpha\mathbf{v}) + \mathbf{K}\mathbf{v} + \varepsilon\,\mathbf{F}(\tau)\mathbf{v} = \mathbf{0}, \tag{5}$$

where $\mathbf{F}(\tau + 2\pi) = \mathbf{F}(\tau) \equiv \mathbf{P}(t)$.

The classical method for determining the Floquet eigenvalues α of equation (1) and, hence, system stability, is to compute the transition matrix [10]. Each column of the transition matrix is given by numerically integrating equation (1), rewritten as a $2m$ degrees of freedom, first order system, from $t = 0$ to $2\pi/\Omega$ using the corresponding column of the identity matrix as initial conditions. The eigenvalues of the transition matrix, denoted by λ, are known as Floquet multipliers, and are related to α by

$$\lambda = \mathrm{e}^{2\pi\alpha}. \tag{6}$$

In this paper, we investigate the α using spectral collocation. The spectral collocation method employed here utilizes the discrete Fourier transform to obtain the differentiation matrices; a detailed derivation can be found in reference [15]. The interval $0 < \tau \leqslant 2\pi$ is divided into N evenly spaced grid points,

$$\tau_j = 2\pi j/N, \quad j = 1, 2, \ldots, N, \tag{7}$$

where, for simplicity, we assume N to be even. (We could also use odd N, but would have to use a different set of differentiation matrices $D_N^{(1)}$ and $D_N^{(2)}$ in the following; for simplicity, we restrict ourselves to even N.) At each τ_j, the ith component $\mathrm{v}_i(\tau_j)$ of the 2π-periodic vector function $\mathbf{v}(\tau)$ is approximated by the discrete value v_{ij}. The derivatives at each of the N grid points are given by

$$v_{i,\tau}(\tau_j) = \sum_{k=1}^{N} \left[D_N^{(1)}\right]_{jk} v_{ik}, \quad v_{i,\tau\tau}(\tau_j) = \sum_{k=1}^{N} \left[D_N^{(2)}\right]_{jk} v_{ik}, \tag{8}$$

where $D_N^{(1)}$ and $D_N^{(2)}$ are the $N \times N$ differentiation matrices

$$\left[D_N^{(1)}\right]_{jk} = \begin{cases} 0, & j = k, \\ (-1)^{j-k}\cot((j-k)h/2)/2, & j \neq k, \end{cases} \tag{9}$$

$$\left[D_N^{(2)}\right]_{jk} = \begin{cases} -1/6 - \pi^2/3h^2, & j = k, \\ (-1)^{j-k}/2\sin^2((j-k)h/2)/2, & j \neq k, \end{cases} \tag{10}$$

where $h = 2\pi/N$. Satisfaction of equation (5) at each of N grid points yields a quadratic eigenvalue problem with mN degrees of freedom:

$$\left(\alpha^2\mathbf{L}^{(2)} + \alpha\mathbf{L}^{(1)} + \mathbf{L}^{(0)}\right)\mathbf{w} = 0, \tag{11}$$

where

$$\begin{aligned} \mathbf{L}^{(2)} &= \Omega^2 kron(\mathbf{M}, \mathbf{I}_N),\ \mathbf{L}^{(1)} = 2\Omega^2 kron\left(\mathbf{M}, \mathbf{D}_N^{(1)}\right) + \Omega kron(\mathbf{C}, \mathbf{I}_N), \\ \mathbf{L}^{(0)} &= \Omega^2 kron\left(\mathbf{M}, \mathbf{D}_N^{(2)}\right) + \Omega kron\left(\mathbf{C}, \mathbf{D}_N^{(1)}\right) + kron(\mathbf{K}, \mathbf{I}_N) \\ &\quad + \varepsilon\, kron(\mathbf{F}, \mathrm{diag}(\delta(\tau - \tau_1), \delta(\tau - \tau_2), \ldots, \delta(\tau - \tau_N))) \end{aligned} \tag{12}$$

and $\mathbf{I}_N$ is the $N \times N$ identity matrix, *kron*(,) is the Kronecker tensor product, and we use $\delta(\tau - \tau_j)$ to indicate evaluation of the appropriate component of $\mathbf{F}(\tau)$ at $\tau = \tau_j$.

3. CONVERGENCE

We compare convergence performance of the transition matrix and spectral collocation methods for the damped Mathieu equation

$$m = 1, \quad \mathbf{M} = 1, \quad \mathbf{C} = \mu, \quad \mathbf{K} = 1, \quad \mathbf{F} = \sin(\tau). \tag{13}$$

The stability properties of this system are well known [10]. We wish to determine the α to an absolute accuracy of $\beta > 0$ so that stability requires $\mathrm{Re}[\alpha] \leqslant \beta$ for all α.

For the transition matrix method, errors in α are determined principally by the relative accuracy of numerical integration. Tables 1–4 list the α determined by the transition matrix method for different integration accuracies for $\varepsilon = 0{\cdot}5$ Table 1, $\Omega = 0{\cdot}5$, $\mu = 0$ (stable); Table 2, $\Omega = 1{\cdot}0$, $\mu = 0$ (unstable); Table 3, $\Omega = 0{\cdot}5$, $\mu = 0{\cdot}1$ (stable); Table 4, $\Omega = 1{\cdot}0$, $\mu = 0{\cdot}1$ (unstable). For each α, the digits thought to have converged are shadowed. From these results, it is evident that the integration accuracy used must be approximately an order of magnitude less than β in order to correctly predict stability. In the sequel, we use a relative numerical integration accuracy of $\beta/5$.

For spectral collocation, better accuracy is achieved by increasing N. The results of spectral collocation for $N = 4$, 8, and 16 are also given in Tables 1–4. These results exhibit two important properties found in all examples tested. First, there is one spurious, incorrect, unstable and one spurious, incorrect, stable α in every case. (For higher dimensional problems, we obtain one pair of spurious α for each degree of freedom.) These spurious pairs of eigenvalues have large real parts of differing signs. The magnitudes of the real parts increase with increasing N and do not converge. The eigensolutions corresponding to these eigenvalues possess rapid oscillations unlike those of the other eigensolutions. Second, apart from the pair of spurious eigenvalues, the other α converges smoothly with increasing N, with those with smallest magnitude converging most rapidly. The transition matrix method predicts only 2 eigenvalues, while the spectral collocation method gives $2N$ eigenvalues in each case. Apart from the spurious eigenvalues, these additional eigenvalue estimates converge to values that differ by integer multiples of i.

In order to predict stability with spectral collocation, an algorithm must be used that filters out the spurious eigenvalues and estimates the accuracy of those that remain. The algorithm adopted here was as follows: eigenvalues were calculated for N and for $N+2$ collocations points. If $2m$ or more eigenvalue of the first set differed in absolute (complex) magnitude from those of the second set by less than β, the $2m$ with the smallest magnitude were taken to have converged and used to determine stability. If there are fewer than $2m$ converged eigenvalues, but one of the converged eigenvalues satisfied $\mathrm{Re}[\alpha] > \beta$, the system was deemed unstable; otherwise, the comparison was repeated using $N+2$ and $N+4$ collocation points, and so on.

Figure 1 shows the stability regions of the Mathieu equation in the $\Omega - \varepsilon$ plane for $\mu = 0$ and $\mu = 0{\cdot}1$ created using spectral collocation. This plot was created using a contour plotting routine which exhibits two artifacts: first, the instability regions for $\mu = 0$ do not fully extend to $\varepsilon = 0$; and second, the contours are not completely smooth. These two artifacts are present if the plot is produced using the transition matrix algorithm. Hence, they represent contour plotting errors rather than errors in the spectral collocation method.

In Figure 2, the number of collocation points N required for $\beta = 10^{-8}$ is superimposed on the stability chart for $\mu = 0$. Over the entire region plotted, no more than 14 collocation points were required; for smaller ε, even fewer collocation points were required.

TABLE 1

α obtained by transition matrix and spectral collection methods for the Mathieu equation with $\varepsilon = 0{\cdot}5$, $\mu = 0$, $\Omega = 0{\cdot}5$ *(stable)*

Transition matrix method Integration accuracy	Re[α]	Im[α]
5E−05	−0·000004285689833	± 0·035676839133928
5E−07	−0·000000048694277	± 0·035680318444710
5E−09	−0·000000000498641	± 0·035680332256699
5E−11	−0·000000000005007	± 0·035680332312127
5E−13	−0·000000000000050	± 0·035680332312350
5E−15	−0·000000000000002	± 0·035680332312349
Spectral collocation method Number of collocation points	Re[α]	Im[α]
4	± 0·734394000180320	0·000000000000001
	0·000000000000000	± 1·101577976626948
	0·000000000000000	± 1·970731568861290
	0·000000000000001	± 3·072796379912738
8	± 3·468273044996834	0·000000000000000
	0·000000000000000	± 0·035843200675851
	0·000000000000000	± 0·964070070604511
	0·000000000000000	± 1·068949418894991
	0·000000000000000	± 1·963563673513298
	0·000000000000000	± 2·964299955181088
	0·000000000000003	± 3·971628065420576
	0·000000000000000	± 5·053618822602433
16	± 7·746128226715440	0·000000000000024
	0·000000000000000	± 0·035680332312347
	0·000000000000002	± 0·964319667687646
	−0·000000000000001	± 1·035680332313441
	0·000000000000003	± 1·964319667667444
	0·000000000000001	± 2·035680335175481
	0·000000000000008	± 2·964319648155042
	0·000000000000006	± 3·035682900238734
	0·000000000000006	± 3·964313585971642
	0·000000000000007	± 4·036418049185600
	0·000000000000009	± 4·964041397211546
	0·000000000000012	± 5·084949863231436
	−0·000000000000004	± 5·963551586883235
	−0·000000000000014	± 6·964231974299665
	−0·000000000000005	± 7·970960274607946
	−0·000000000000019	± 9·049089358281088

Note: Digits thought to have converged are shadowed.

Table 5 presents a comparison of the real time execution times of the transition matrix and spectral algorithms for the undamped ($\mu = 0$) Mathieu equation (13). In each case, stability was computed over a 20×20 grid of equally spaced points in the $\log(\Omega) - \varepsilon$ plane ranging from $0{\cdot}3 \leqslant \Omega \leqslant 4$ to $0 \leqslant \varepsilon \leqslant 1$. This is the same range shown in Figure 1. The computations were performed on a Pentium III, 600 MHz Dell computer using either Matlab or a set of C programs. The time shown in each case is the real time required to compute the stability of the 400 parameter sets. It is evident from these results that computational efficiency is implementation dependent. In both implementations, spectral collocation takes approximately the same time to execute. In C, the transition matrix

TABLE 2

α obtained by transition matrix and spectral collocation methods for the Mathieu equation with ε = 0·5, μ = 0, Ω = 1 (unstable)

Integration accuracy	Re[α]	Im[α]
Transition matrix method		
5E−05	± 0·0232148834271 42	0.000000000000000
5E−07	± 0·0232151436822 77	0·000000000000000
5E−09	± 0·0232151616155 50	0·000000000000000
5E−11	± 0·0232151618424 91	0·000000000000000
5E−13	± 0·0232151618449 36	0·000000000000000
5E−15	± 0·0232151618449 63	0·000000000000000
Spectral collocation method		
Number of collocation points	Re[α]	Im[α]
4	± 1·7372077972856 18	0·000000000000000
	0·0000000018259 43	± 0·0000000000027 22
	0·0000000000000 00	± 0·9794454096603 76
	0·0000000000000 01	± 2·0145911794816 26
8	± 3·8731750613307 74	0·000000000000005
	± 0·0232151430966 03	0·000000000000000
	± 0·0232098925429 12	± 1·0000074062098 06
	± 0·0185172462234 32	± 2·0046560586324 18
	−0·0000000000000 02	± 2·9793443270669 84
	0·0000000000000 00	± 4·0112308712693 56
16	± 7·9372633272983 22	0·000000000000010
	± 0·0232151618449 63	0·000000000000001
	± 0·0232151618449 63	± 1·0000000000000 04
	± 0·0232151618449 57	± 2·0000000000000 07
	± 0·0232151618443 49	± 3·0000000000009 30
	± 0·0232151574856 47	± 4·0000000063130 18
	± 0·0232053654404 49	± 5·0000132685478 68
	± 0·0175575096183 61	± 6·0053367639212 83
	−0·0000000000000 21	± 6·9792796777881 68
	−0·0000000000000 40	± 8·0104895943417 77

Note: Digits thought to have converged are shadowed.

method is about an order of magnitude faster than spectral collocation. In Matlab, the transition matrix method is about an order of magnitude slower than spectral collocation. This slowness may be attributable to the speed at which external functions are called by the numerical integrator in Matlab.

4. EXAMPLES

In this section, we summarize the results of three additional test problems.

4.1. COMBINATION RESONANCES

The two-degree-of-freedom system

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}, \quad \mathbf{F} = \sin(\tau) \begin{bmatrix} 0 & 1 \\ \rho & 0 \end{bmatrix} \tag{14}$$

TABLE 3

α obtained by transition matrix and spectral collocation methods for the Mathieu equation with $\varepsilon = 0{\cdot}5$, $\mu = 0{\cdot}1$, $\Omega = 0{\cdot}5$ (stable)

Integration accuracy	Re[α]	Im[α]
Transition matrix method		
5E−05	−0·100007214002479	±0·038382591871465
5E−07	−0·100000071913524	±0·038385834463077
5E−09	−0·100000000689207	±0·038385833180511
5E−11	−0·100000000006751	±0·038385833042948
5E−13	−0·100000000000067	±0·038385833041096
5E−15	−0·100000000000003	±0·038385833041072
Spectral collocation method		
Number of collocation points	Re[α]	Im[α]
4	±0·640318142303661	0·000000000000000
	−0·100000000000000	±1·098631272776534
	−0·100000000000000	±1·968122770218738
	−0·100000000000000	±3·070435317613426
8	±3·369711235034758	0·000000000000003
	−0·100000000000000	±0·038540810843795
	−0·100000000000000	±0·961387640121090
	−0·100000000000000	±1·071761906165758
	−0·100000000000000	±1·960895883350228
	−0·100000000000000	±2·961643615510970
	−0·100000000000000	±3·969022099000350
	−0·100000000000000	±5·051226508118750
16	±7·846773657183812	−0·000000000000014
	−0·100000000000000	±0·038385833041072
	−0·100000000000002	±0·961614166958922
	−0·100000000000002	±1·038385833042177
	−0·099999999999998	±1·961614166941442
	−0·100000000000002	±2·038385835926445
	−0·099999999999994	±2·961614149998386
	−0·099999999999999	±3·038388415687972
	−0·099999999999989	±3·961608857426128
	−0·099999999999996	±4·039126250354346
	−0·099999999999993	±4·961362476564584
	−0·099999999999987	±5·087758363855455
	−0·100000000000001	±5·960883518611525
	−0·100000000000005	±6·961574930237023
	−0·100000000000016	±7·968351162942833
	−0·100000000000022	±9·046690317396536

Note: Digits thought to have converged are shadowed.

possesses a combination resonance of the sum type at $\Omega = \omega_1 + \omega_2$ when $\rho = 1$ and of the difference type at $\Omega = \omega_2 - \omega_1$ when $\rho = -1$. Figure 3 shows the stability regions of the sum and differences cases for $\omega_1 = 1$ and $\omega_2 = 1{\cdot}75$ for both $\mu = 0$ and $0{\cdot}1$. Identical results are obtained using the transition matrix approach.

The real execution times in C and Matlab for this example with $\rho = 1$ over the same 20×20 grid used in the previous section are given in Table 5. Once again, the execution times are approximately the same for spectral collocation in both implementations. In C,

TABLE 4

α obtained by transition matrix and spectral collocation methods for the Mathieu equation with ε = 0·5, μ = 0·1, Ω = 1 (unstable)

Integration accuracy	Re[α]	Im[α]
Transition matrix method		
5E−05	−0·027620056421262	0·000000000000000
	−0·072086419315790	0·000000000000000
5E−07	−0·027919074952898	0·000000000000000
	−0·072080985620024	0·000000000000000
5E−09	−0·027919042675397	0·000000000000000
	−0·072080957898537	0·000000000000000
5E−11	−0·027919042307970	0·000000000000000
	−0·072080957697710	0·000000000000000
5E−13	−0·027919042304099	0·000000000000000
	−0·072080957695958	0·000000000000000
5E−15	−0·027919042304059	0·000000000000000
	−0·072080957695944	0·000000000000000
Spectral collocation method		
Number of collocation points	Re[α]	Im[α]
4	± 1·687924760093903	0·000000000000000
	−0·050000000000000	± 0·007422287901469
	−0·050000000000000	± 0·978098465685675
	−0·050000000000000	± 2·013368017232088
8	± 3·823497751243955	−0·000000000000002
	−0·027919062065250	0·000000000000000
	−0·072080937934749	0·000000000000000
	−0·027925065130059	± 1·000007443624584
	−0·072074934869941	± 1·000007443624585
	−0·066685311072651	± 2·004668607863050
	−0·033314688927348	± 2·004668607863057
	−0·049999999999999	± 2·977996308988569
	−0·050000000000003	± 4·010002438373366
16	± 7·987420810380660	0·000000000000038
	−0·027919042304055	0·000000000000000
	−0·072080957695945	0·000000000000000
	−0·027919042304057	± 1·000000000000000
	−0·072080957695941	± 1·000000000000004
	−0·027919042304057	± 1·999999999999999
	−0·072080957695939	± 2·000000000000010
	−0·027919042304755	± 3·000000000000929
	−0·072080957695233	± 3·000000000000944
	−0·027919047301167	± 4·000000006347660
	−0·072080952698814	± 4·000000006347679
	−0·072069802976370	± 5·000013323465184
	−0·027930197023615	± 5·000013323465184
	−0·065552809312609	± 6·005350009060750
	−0·034447190687382	± 6·005350009060836
	−0·050000000000020	± 6·977931218445101
	−0·050000000000033	± 8·009260081167453

Note: Digits thought to have converged are shadowed.

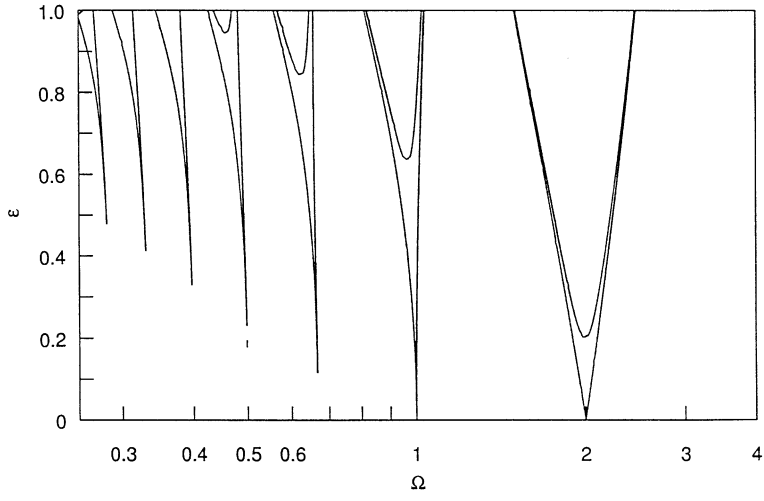

Figure 1. Stability regions of the Mathieu equation for $\mu = 0$ and 0·1.

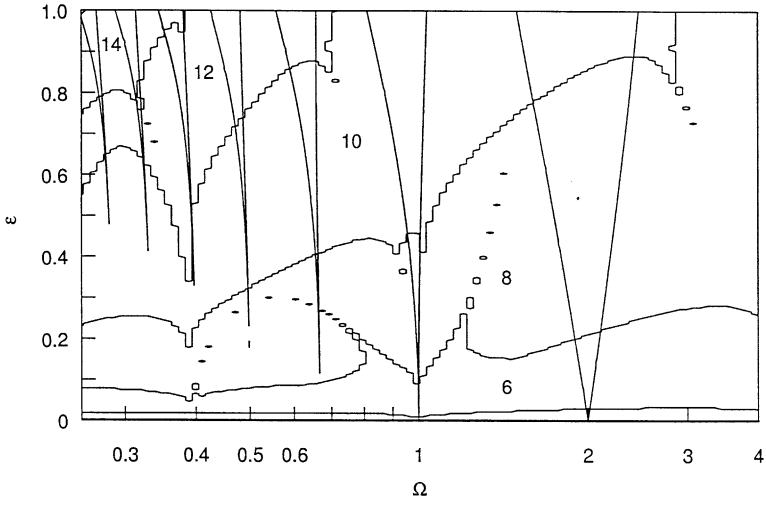

Figure 2. Number of collocation points required to determine the stability of the undamped Mathieu equation to an accuracy of 10^{-8}.

TABLE 5

Real execution times computing stability on a grid of 400 *different parameter sets*

		C (s)	Matlab (s)
The undamped Mathieu equation (13) with $\mu = 0$	Transition matrix	1·57	597·9
	Spectral collocation	5·80	10·5
Combination resonances (14) with $\rho = 1$	Transition matrix	7·56	2432·43
	Spectral collocation	55·29	49·35

the transition matrix method is approximately an order of magnitude faster than spectral collocation; in Matlab, the transition matrix method is approximately an order of magnitude slower than spectral collocation.

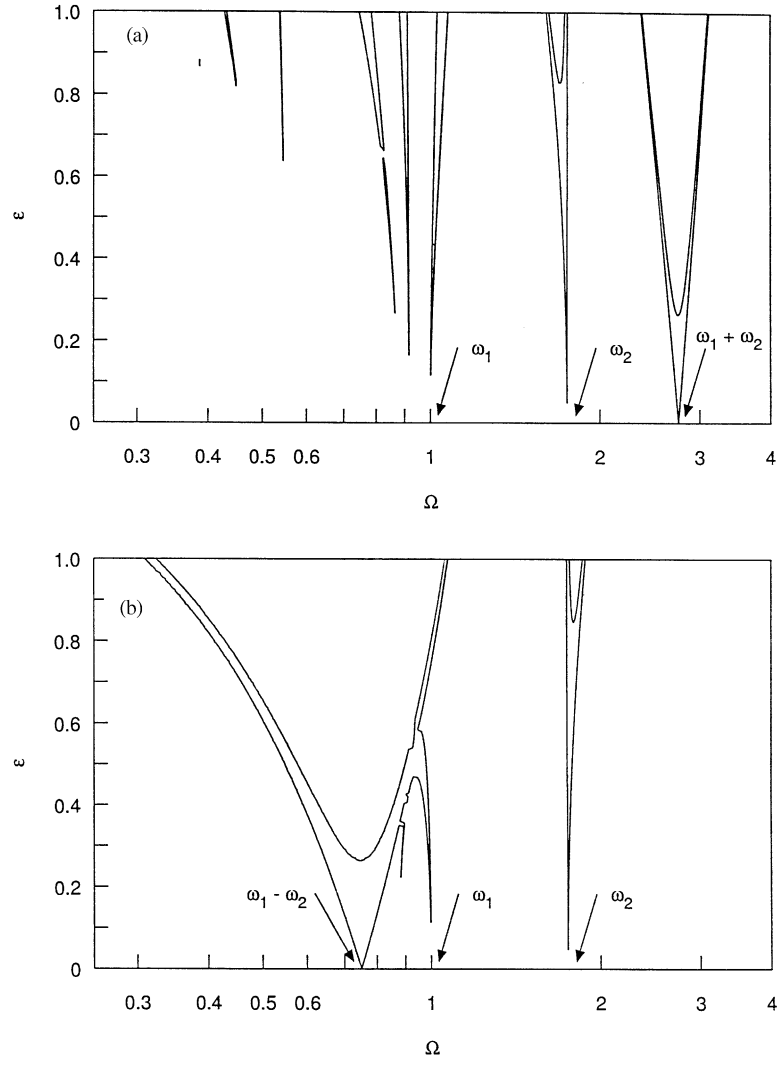

Figure 3. Stability regions of the combination resonances of the (a) sum and (b) differences type for $\mu = 0$ and 0·1. $\omega_1 = 1{\cdot}0$ and $\omega_2 = 1{\cdot}75$.

4.2. A THREE-DEGREE-OF-FREEDOM EXAMPLE

Tyc *et al.* [16] use a three-degree-of-freedom system to describe a satellite in a circular orbit including a nutation damper:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0{\cdot}01 \\ 0 & 0{\cdot}01 & 0{\cdot}0099 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & w \\ 0 & w & k_{dv} \end{bmatrix}, \quad \mathbf{F} = 2\varepsilon \cos(\tau) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\varepsilon \sin(\tau) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (15)$$

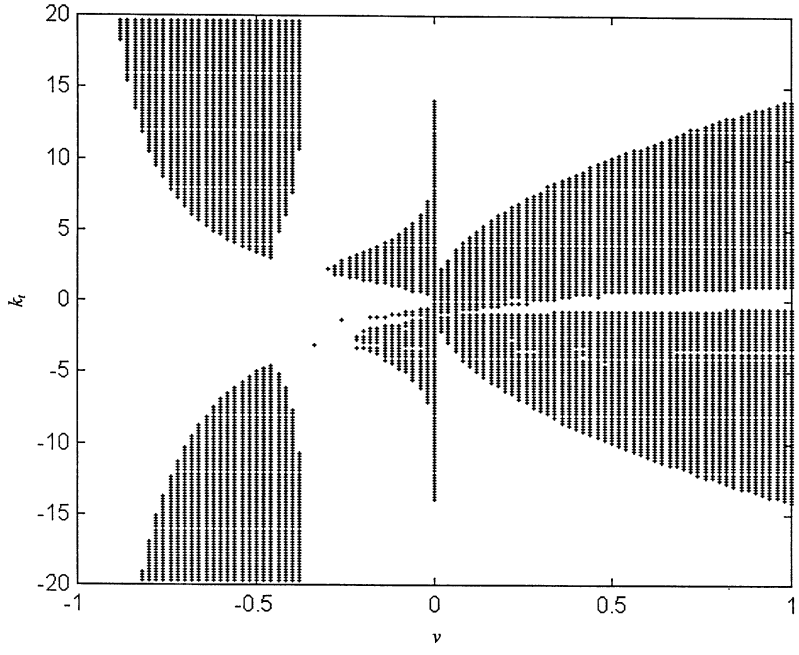

Figure 4. Stability regions of a satellite with a nutation damping in a circular orbit [16]. A mark is plotted on each point of the grid at which the system is stable.

where $a = k_t(v^2 + 3/2)/4(1+v)^2$, $c = (1-k_t)v/2(1+v)$, $w = 0.01v^2/4(1+v)^2$, $k_{dv} = 0.005/(1+v)^2$, and $\varepsilon = 3k_t/16(1+v)^2$. The stability of this system in the k_t–v plane is shown in Figure 4. For this plot, rather than trace a contour plot, we calculate stability over a grid of points and plot a mark on each point of the grid at which the system is stable. This chart is identical to the results given in reference [16] using both the transition matrix and infinite eigenvalue methods.

4.3. AXIALLY MOVING STRING SUBJECT TO OSCILLATING TENSION

The spectral collocation method can be particularly advantageous for systems with large degrees of freedom. We therefore consider the following continuous system exhibiting parametric resonance. Consider an axially moving string subjected to a parametrically excited relative end displacement [4, 17]. The transverse displacement of the string is $w(x, t)$, x is axial position, t is time, and the axial speed is v. The equation of motion is

$$w_{,tt} + 2vw_{,tx} + (v^2 - 1 + \varepsilon\cos(\Omega t))w_{,xx} = 0, \tag{16}$$

where ε is the excitation magnitude and, for physical reasons, $0 \leqslant \varepsilon < 1$. The boundary conditions are $w(0,t) = w(1,t) = 0$. Stability boundaries in the ε–Ω plane have been computed for the primary parametric instabilities and combination resonances [4]. Here we verify these stability predictions numerically.

Two different discretization are used and compared. First, Galerkin's method is used with the axially moving string eigenfunctions as trial functions. Since these trial functions are derived from the eigenvalue problem associated with the unexcited system (16), they are efficient for analytic analysis: reference [4] deduce stability boundaries using two and four term expansions of these trial functions. However, for the general Galerkin

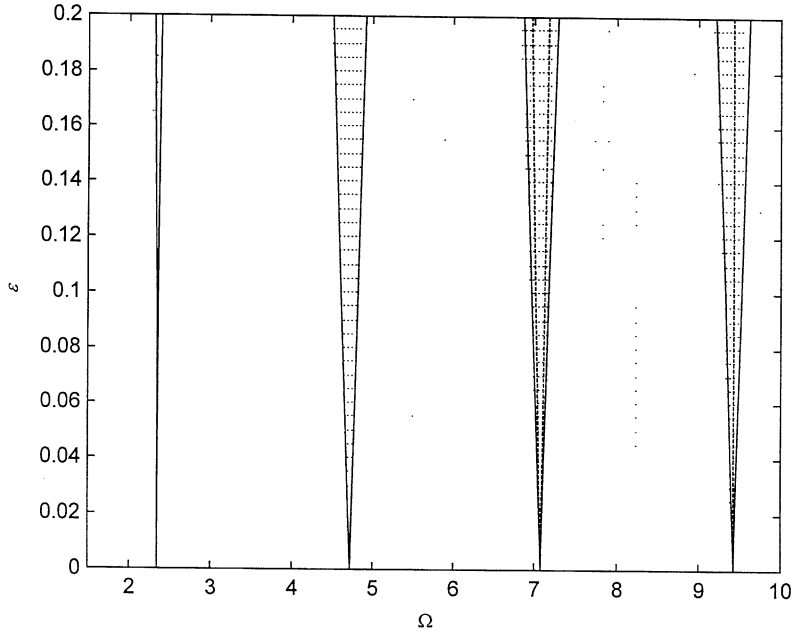

Figure 5. Stability boundary of the axially moving string subject to oscillating tension. ·, the unstable points on the grid predicted by the spectral collocation; ——, stability boundaries predicted by the method of multiple scales; ------, stability boundaries predicted by reference [4].

procedure, the system matrices resulting from this method are full and there is no sparsity to exploit. Second, the finite element method is used using cubic interpolation functions. The system matrices for this discretization have a bandwidth of seven, and, hence, are sparse.

We use Matlab to determine stability in both cases. For the finite element discretization, we use the sparse eigenvalue solvers to expedite the solution. Figure 5 shows the stability boundary in the $\Omega-\varepsilon$ plane from $1{\cdot}5 \leqslant \Omega \leqslant 10$ to $0 \leqslant \varepsilon \leqslant 0{\cdot}2$, where unstable points on the grid are marked (the opposite of Example 4·2). $v = 0{\cdot}5$. The first natural frequencies of the unexcited system are 2·36, 4·71 and 7·07. The results show primary instabilities emanating from $\Omega = 4{\cdot}71$, 7·07, and 9·42, i.e., $2\omega_1$, $\omega_1 + \omega_2 = \omega_3$, $\omega_1 + \omega_3$, etc. A secondary low-frequency instability is evident at $\Omega = 2{\cdot}36$. A few other, very narrow unstable regions are also evident.

Figure 6 shows the execution time of the stability calculation using Matlab for the parameters $\Omega = 4{\cdot}7$ and $\varepsilon = 0{\cdot}1$ as a function of m, the degree of freedom of the resulting systems. For each set of data, the number of degrees of freedom used was increased until the memory required was greater than the computer storage available. Only Matlab results are shown since a sparse eigenvalue solver in C was unavailable. This is unfortunate since the comparisons of computational efficiency would undoubtedly be different in C. Using the finite element discretization, spectral collocation was on average an order of magnitude faster than the transition matrix approach up to 40 degrees of freedom. Similar results occur for the Galerkin discretization, although memory requirements prohibit $m > 20$.

Apart from rendering the stability of large systems practical, the results shown in Figure 5 also can be used to verify low order, analytical results. Using the method of multiple scales, the authors obtain the stability boundary of the primary instability of the

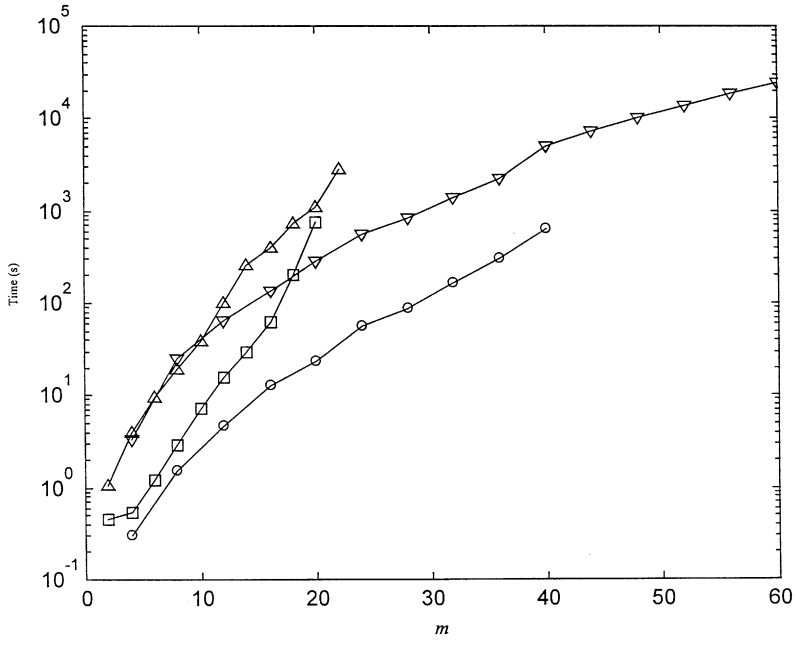

Figure 6. Execution time of the stability calculation for the parameters $\Omega = 4{\cdot}7$ and $\varepsilon = 0{\cdot}1$ for each m. □, the spectral collocation with the Galerkin discretization; △, the transition matrix with the Galerkin discretization; ○, the spectral collocation with the finite element discretization; ▽, the transition matrix with the finite element discretization.

nth mode as

$$\Omega = 2n\pi\left(1 - v^2\right) \pm \frac{\varepsilon \sin(n\pi v)}{2v}, \tag{17}$$

$n = 1, 2, 3, \ldots$; for the combination of the first and second modes as

$$\Omega = 3\pi\left(1 - v^2\right) \pm \frac{8\sqrt{2}\,\varepsilon v \cos(3\pi v/2)}{3(9v^2 - 1)}; \tag{18}$$

for the first and third modes as

$$\Omega = 4\pi\left(1 - v^2\right) \pm \frac{3\sqrt{3}\,\varepsilon v \sin(2\pi v)}{4(4v^2 - 1)}; \tag{19}$$

and for the secondary instability of the first mode as

$$\Omega = \pi\left(1 - v^2\right) + \varepsilon^2(-\Gamma_1 \pm \Gamma_2), \tag{20}$$

where

$$\Gamma_1 = \frac{-\pi}{32v^2\left(-(\Omega + \omega_1)^2 + \omega_1^2\right)}\left(-\Omega + \Omega\cos(2\pi v) + (\Omega + 2\omega_1)2\pi^2 v^2\left(1 + v^2\right)^2\right),$$

$$\Gamma_2 = \frac{\pi^3\left(1 - v^4\right)\sin(\pi v)}{8v^3\left(-(\Omega - \omega_1)^2 + \omega_1^2\right)}. \tag{21}$$

These boundaries are plotted in Figure 5 using the solid lines and agree well with the numerical predictions. Also shown in Figure 5 using a dotted line are the stability

boundaries predicted by Mockensturm *et al.* [4]. Because they did not examine the combination resonance of the first and third modes, they cannot accurately predict stability near $\Omega = 9{\cdot}42$ which is dominated by that combination resonance. In addition, their formula for the combination resonance of the first and second modes at $\Omega = 7{\cdot}07$ appears to be off by a factor of 2. Their prediction near $\Omega = 4{\cdot}71$ is identical to equation (17).

5. DISCUSSION AND CONCLUSION

The spectral collocation method is used to determine the stability of parametrically excited systems and compared to the traditional transition matrix approach. The hallmark of spectral methods is very rapid convergence for smooth functions. This is well suited to the parametric excitation since the periodic, fundamental solutions are smooth. Results on a series of test problems confirm these predictions. The simplicity of the method makes it attractive.

The spectral collocation method preserves the sparsity of the underlying system matrices, a property not shared by the transition matrix approach. In fact, if the eigensolver only requires vector multiplication, as some large solvers do, no additional storage is required. As a result, spectral collocation can potentially be used for very large systems. Unfortunately, this saving in storage does not necessarily lead to a savings in computational effort. The computational effort appears to be implementation dependent. Specifically, the spectral method is roughly an order of magnitude faster than the transition matrix method in Matlab. For other implementations, we conclude only that spectral collocation is competitive with the numerical transition matrix approach.

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