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THE DYNAMIC ANALYSIS OF A BEAM–MASS SYSTEM DUE TO THE OCCURRENCE OF TWO-COMPONENT PARAMETRIC RESONANCE

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The objective of this paper is an analytical and numerical study of the dynamics of a beam–mass system. Special attention is given to the phenomena arising due to the motion of the attached mass and modal interactions produced by the existence of multi-component, specifically two-component, parametric resonance under primary resonance. The two-component parametric resonance occurs when the sums or the differences among internal frequencies are the same, or close, as the dimensionless speed parameter of the moving mass. The effects of two-component parametric resonance post on dynamic condition are investigated. Resonance generated by more than two-component modes are neglected due to its remote probability of occurrence in nature.

The mechanics of the problem is Newtonian. Based on the assumption that when the moving mass is set in motion the mass is assumed to be rolling on the beam, the mechanics, including the effects due to friction and convective accelerations, of the interface between the moving mass and the beam are determined.

Based on the Bernoulli–Euler beam theory, the coupled non-linear equations of motion of an inextensible beam with an attached moving mass are derived. By employing Galerkin procedure, the partial differential equations which describe the motion of a beam–mass system are reduced to an initial-value problem with finite dimensions. The method of multiple time scales is applied to consider the solutions and the occurrence of internal resonance of the resulting multi-degree-of-freedom beam–mass system with time dependent coefficients.

1. INTRODUCTION

Vibrations of flexible structures with attached moving masses have been the subject of many studies [1–14]. In references [1, 2], the response of a simply supported finite beam with and without elastic foundation under a moving load was studied. It shows the existence of a truly critical speed and the impossibility of the occurrence of a steady state when the load speed is equal to either the shear or the bar velocity.

The problem regarding the interaction between the moving mass and the supporting structure was first considered by Ting *et al.* [3]. The result indicates that the convective acceleration terms should be included if "correct" formulation is desired.

Recently, sophisticated effects, such as longitudinal deflections, inertia, and non-linearities of the beam and the variation of moving masses, to the response due to the motion of moving loads have been taken into account in references [4, 5]. The result of reference [4] shows that the largest amplitude of response occurs in the linear model.

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The phenomenon of negative deflection of a flexible structure due to the motion of attached masses has been found in many studies, such as references [3, 8, 10]. However, this event was generally not discussed in detail. Investigation of the occurrence and relative conditions of the separation between flexible structures and riding masses has been done in reference [6]. It discloses that in certain conditions, the effects of separation are of critical importance.

In reference [7], the author discussed critical speeds and the response of a tensioned beam on an elastic foundation to repetitive moving loads. The possibilities of the existence and the occurrence of critical speeds and the effects of damping, beam tension, and elastic foundation stiffness on the response of the system are studied.

A new model that includes the effect of rolling friction between the rollers of the moving mass and the beam was established by Wang [8]. An important feature that is carried out in the analysis is the ability to bring the mass to a halt at desired points along the beam. The transient vibrations of an inextensible beam with a riding accelerating mass are studied in detail.

In this study, the vibrations of a finite inextensible beam with a riding mass are investigated. The beam rests on a uniform elastic foundation. The gravity of the beam then is assumed to be taken by the foundation preload.

In the modelling, effects due to friction and convective accelerations of the interface between the moving mass and the beam are considered. This results in variable velocity and acceleration, and unknown location of the mass along the beam. The mass is able to be accelerated by a forward force or reduces speed and brakes to a halt at desired position on the beam by applying a reverse force to the mass and/or increasing the friction between the mass and the beam. In the analysis, the moving mass with constant velocity is assumed and the method of multiple time scales is employed to evaluate the dynamics of a moving mass and obtain solutions of the analysis.

Results of the present study show that for the case of two-component resonance, new regions of the growth of small amplitude of vibrations into large motion regime are found for the first mode even the excitation due to the motion of the attached mass is not close to the fundamental mode. This is due to modal interactions caused by the existence of two-component parametric resonance.

2. BASIC FORMULAS

In this study, a finite inextensible beam rested on a uniform elastic foundation and having a length of l is considered. The static state of the beam is obtained by assuming that the gravity of the beam and the foundation preload are in the state of equilibrium.

From Figures 1 and 2, the equations governing the motion of the system can be derived from the dynamic equilibrium of forces and momenta and are given as [8]

$$[(T\cos\theta - V\sin\theta)\mathbf{i} + (T\sin\theta + V\cos\theta)\mathbf{j}]_{,s} + \mathbf{f} = m\mathbf{r}_{,tt}, \quad 0<s<l, t>0, \tag{1}$$

$$EIv_{,sss} + V = 0 \tag{2}$$

with the inextensibility constraint $\mathbf{r}_{,s} \cdot \mathbf{r}_{,s} = 1$. In the above equations, T is the axial force in the beam; V is the transverse force in the beam; θ is the angle between the neutral axis of the beam and the x-axis. The subscript s and t denote the s and t differentiation and where m indicates the mass per unit length of the beam. E and I are Young's modulus and the area moment of inertia of the beam. $\mathbf{r}(s, t)$ is the Cartesian position vector of point s along

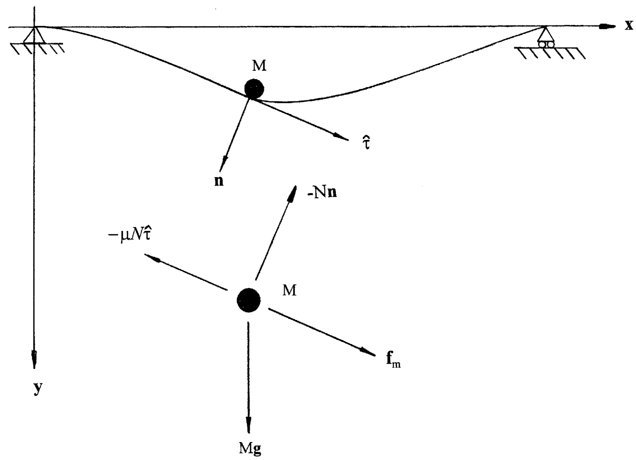

Figure 1. System configuration.

the beam at time t and has the form

$$\mathbf{r}(s,t) = (x(s) + u(s,t))\mathbf{i} + v(s,t)\mathbf{j}, \tag{3}$$

where $u(s,t)$ and $v(s,t)$ are the axial displacement and the transverse displacement of the beam from the undeformed state respectively. The force $\mathbf{f}$ denotes the external forces including the weight of the moving mass and the moving reaction of the mass upon the beam and can be stated by

$$\mathbf{f} = -kv\mathbf{j} + (N\mathbf{n} + \mu N\hat{\boldsymbol{\tau}})\bar{\delta}(s - \bar{s}), \tag{4}$$

where k, N, μ, and $\bar{\delta}(s - \bar{s})$ denote the foundation stiffness per unit length, the reaction of beam on the mass, the coefficient of friction, and the Dirac delta function respectively.

The equation of motion of the moving mass obeys (Figure 1)

$$M\mathbf{a}_M = M\mathbf{g} + \mathbf{f}_m - \mu N\hat{\boldsymbol{\tau}} - N\mathbf{n}, \tag{5}$$

where M represents the mass of the moving mass and $\mathbf{f}_m = Mf\hat{\boldsymbol{\tau}} = Mf[(1 + u_{,s})\mathbf{i} + v_{,s}\mathbf{j}]$, $\mathbf{g} = g\mathbf{j}$, $\tau = (1 + u_{,s})\mathbf{i} + v_{,s}\mathbf{j} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, and $\mathbf{n} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. Note that here it is assumed that whenever a mass is being propelled by a force along a beam, the force on the mass will be along the tangent to the vibrating beam. Hence, $\mathbf{f}_m = Mf\hat{\boldsymbol{\tau}}$ and f is a prescribed function of time. For example, f may be a positive constant to increase speed and a negative constant to reduce speed and to come to a halt at a desired position on the beam.

The acceleration of the moving mass $\mathbf{a}_M$ is obtained from

$$\mathbf{a}_M = \frac{\mathrm{d}^2}{\mathrm{d}t^2}[\mathbf{r}(\bar{s}(t), t)] = \mathbf{r}_{,ss}(\bar{s}_{,t})^2 + 2\mathbf{r}_{,st}\bar{s}_{,t} + \mathbf{r}_{,s}\bar{s}_{,tt} + \mathbf{r}_{,tt} \tag{6}$$

in which $\bar{s}(t)$ is the distance along the arc of the beam which represents the position of the moving mass.

In this study, the system under consideration is a finite, simply supported, Bernoulli–Euler beam on a uniform elastic foundation with an attached mass. Hence, the boundary conditions are

$$u(0,t) = v(0,t) = v(l,t) = \frac{\partial^2 v(0,t)}{\partial s^2} = \frac{\partial^2 v(l,t)}{\partial s^2} = 0, \tag{7}$$

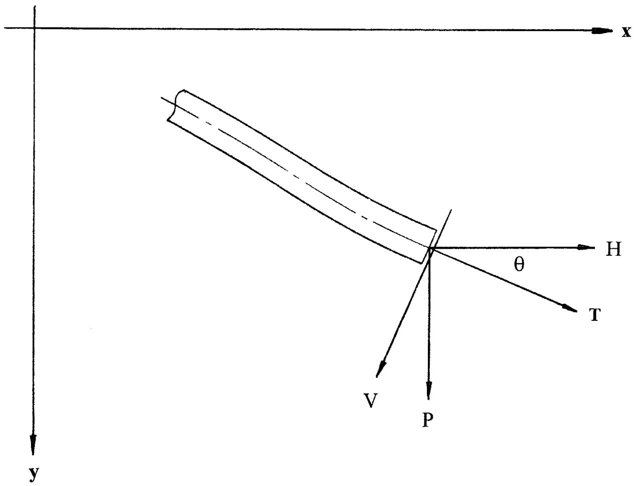

Figure 2. Force equilibrium diagram.

$$H(l,t) = T(l,t)(1 + u_{,s}) + EIv_{,sss}\, v_{,s} = 0, \quad \text{at } s = l. \tag{8}$$

Introducing the following dimensionless quantities:

$$\tau = \sqrt{\frac{EI}{ml^4}}\, t, \qquad \hat{M} = \frac{M}{ml}, \qquad \hat{T} = \frac{l^2}{EI} T, \qquad \hat{f} = \frac{ml^3}{EI} f,$$
$$\hat{g} = \frac{ml^3}{EI} g, \qquad \hat{k} = \frac{kl^3}{EI}, \qquad \eta = \frac{s}{l}, \qquad \xi = \frac{\bar{s}}{l}, \qquad \hat{v} = \frac{v}{l}, \qquad \hat{u} = \frac{u}{l} \tag{9}$$

and substituting equations (3)–(6) and (9) with equation (2) into equation (1), the equations of motion of the combined system in directions **i** and **j** yields, in dimensionless form

$$[\hat{T}(1 + \hat{u}') + \hat{v}'''\hat{v}']' + \hat{M}\hat{f}(1 + \hat{u}')$$
$$= \ddot{\hat{u}} + \hat{M}[\hat{u}''(\dot{\xi})^2 + 2\dot{\hat{u}}'\dot{\xi} + (1 + \hat{u}')\ddot{\xi} + \ddot{\hat{u}}]\delta(\eta - \xi), \quad 0<\eta<1, \ \tau > 0, \tag{10}$$

$$[\hat{T}\hat{v}' - \hat{v}'''(1 + \hat{u}')]' + \hat{M}(\hat{f}\hat{v}' + \hat{g})\delta(\eta - \xi)$$
$$= \ddot{\hat{v}} + \hat{k}\hat{v} + \hat{M}[\hat{v}''(\dot{\xi})^2 + 2\dot{\hat{v}}'\dot{\xi} + \hat{v}'\ddot{\xi} + \ddot{\hat{v}}]\delta(\eta - \xi), \quad 0<\eta<1, \ \tau > 0. \tag{11}$$

Similarly, considering the equation of motion of the moving mass, equation (5), one has

$$(1 + \hat{u}'^2)\ddot{\xi} - [\mu\hat{v}'' - \hat{v}'\hat{v}'' - \hat{u}'' + \mu(\hat{u}'\hat{v}'' - \hat{u}''\hat{v}')](\dot{\xi})^2$$
$$- 2[\mu\dot{\hat{v}}' - \dot{\hat{v}}'\hat{v}' - \dot{\hat{u}}' + \mu(\hat{u}'\dot{\hat{v}}' - \dot{\hat{u}}'\hat{v}')]\dot{\xi}$$
$$= \hat{g}[\hat{v}' - \mu(1 + \hat{u}')] + \hat{f}\{(1 + \hat{u}')[\mu\hat{v}' + (1 + \hat{u}')] + \hat{v}'[\hat{v}' - \mu(1 + \hat{u}')]\}$$
$$- \ddot{\hat{v}}[\hat{v}' - \mu(1 + \hat{u}')] - \ddot{\hat{u}}[\mu\hat{v}' + (1 + \hat{u}')], \quad \eta = \xi, \ \tau > 0, \tag{12}$$

where superposed prime and dot denote the η and τ differentiation. It is mentioned here that equation (12) is obtained by eliminating the normal reaction force of the beam on the mass, N, between the two equations in directions **i** and **j** of equation (5), respectively, and using the inextensibility constraint. Therefore, equations (10)–(12) with the inextensibility constraint account for $\hat{u}(\eta,\tau), \hat{v}(\eta,\tau)$, $\hat{T}$ and ξ when $\hat{M}$, μ, $\hat{g}$ and the boundary conditions are specified, equations (7) and (8).

The axial force $\hat{T}$ can be determined by the assumption that the variation of axial force is assumed to remain continuous at $\eta = \xi(\tau)$. Integrating equation (10) and using the boundary condition, equation (8), yields

$$\hat{T}(\eta,\tau) = \frac{-1}{1+\hat{u}'}\left[\hat{v}'''\hat{v}' + \int_{\eta}^{1} \ddot{\hat{u}}\,\mathrm{d}\eta\right], \quad 0<\eta<1,\ \tau>0. \tag{13}$$

Now, the condition of small deformations is assumed. For this, one neglects the non-linear terms when compares these terms with the linear term of $\hat{v}(\eta,\tau)$ and unity. The equation of motion of the beam, equation (11), then can be simplified as

$$\ddot{\hat{v}} + \hat{v}'''' + \hat{k}\hat{v} + \hat{M}[\hat{v}''(\dot{\xi})^2 + 2\dot{\hat{v}}'\dot{\xi} + \hat{v}'\ddot{\xi} + \ddot{\hat{v}} - \hat{f}\hat{v}']\delta(\eta-\xi) = \hat{M}\hat{g}\delta(\eta-\xi), \quad 0<\eta<1,\ \tau>0, \tag{14}$$

while the equation of motion of the moving mass, equation (12), becomes

$$\ddot{\xi} - \mu\hat{v}''(\dot{\xi})^2 - 2\mu\dot{\hat{v}}'\dot{\xi} = \hat{f} - \mu\hat{g} + \hat{g}\hat{v}' + \mu\ddot{\hat{v}}, \quad \eta=\xi,\ \tau>0. \tag{15}$$

In the following, the condition that the attached mass is assumed to move along the beam with constant velocity is considered. Therefore, substitution of equation (15) into equation (14) and neglect of non-linear terms yields

$$\ddot{\hat{v}} + \hat{v}'''' + \hat{k}\hat{v} = \hat{M}[(1+\mu\hat{v}')\hat{g} - (\hat{v}''(\dot{\xi})^2 + 2\dot{\hat{v}}'\dot{\xi} + \ddot{\hat{v}})]\delta(\eta-\xi), \quad 0<\eta<1,\ \tau>0, \tag{16}$$

Examination of the dynamics governed by equation (16) is the main purpose in this study.

Representing $\hat{v}$ as a continuous function and letting

$$\hat{v} = \sum_{n=1}^{\infty} A_n(\tau)\sin n\pi\eta, \quad 0<\eta<1,\ \tau>0, \tag{17}$$

hence, the boundary condition, equation (7), is satisfied. The approximate solution of the beam–mass system is to be obtained by employing Galerkin‘s method. Using Galerkin’s procedure for minimizing error, one obtains

$$\begin{aligned}\ddot{A}_j(\tau) + \omega_j^2 A_j(\tau) = 2\hat{M}\Bigg\{ &\hat{g}\hat{S}_j(\xi) + \mu\hat{g}\sum_{n=1}^{\infty} R_{jn}(\xi)A_n(\tau) \\ &- \sum_{n=1}^{\infty}[\hat{S}_{jn}(\xi)\ddot{A}_n(\tau) + 2\dot{\xi}R_{jn}(\xi)\dot{A}_n(\tau) - (\dot{\xi})^2 S_{jn}(\xi)A_n(\tau)]\Bigg\}, \\ &0<\eta<1,\ \tau>0,\end{aligned} \tag{18}$$

where $\omega_j^2 = ((j\pi)^4 + \hat{k})$, $R_{jn}(\xi) = (n\pi)\cos n\pi\xi \sin j\pi\xi$, $\hat{S}_{jn}(\xi) = \sin n\pi\xi \sin j\pi\xi$, $S_{jn}(\xi) = (n\pi)^2\hat{S}_{jn}(\xi)$ and $\hat{S}_n(\xi) = \sin n\pi\xi$.

To analyze the system governed by equation (18), one allows the response of the system to be small but finite. Thus, the method of multiple time scales can be used to predict the responses of the system. According to this method, it is assumed that the amplitude,

$A_j(\tau)$, has the expansion [14]

$$
\begin{aligned}
&A_j(\tau;\varepsilon) = \varepsilon A_{1j}(\tau_0,\tau_1,\tau_2,\ldots) + \varepsilon^2 A_{2j}(\tau_0,\tau_1,\tau_2,\ldots) + \varepsilon^3 A_{3j}(\tau_0,\tau_1,\tau_2,\ldots) + \cdots,\\
&\tau_n = \varepsilon^n \tau, \quad n = 0,1,2,\ldots,\\
&\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \varepsilon^2\frac{\partial}{\partial\tau_2} + \cdots \equiv D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots,\\
&\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \equiv D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0D_2) + \cdots,
\end{aligned}
\tag{19}
$$

where ε is a measure of the amplitude of the response and is small compared to unity.

For the purpose of studying the parametric resonance of the non-autonomous differential equations, one sets $\hat{M} = \varepsilon\bar{M}$ and $\dot{\xi} = V_\xi$. After manipulating these equations and then equating coefficients of equal power of ε, one obtains order one and two:

$$\varepsilon^1: \quad D_0^2 A_{1j} + \omega_j^2 A_{1j} = 2\bar{M}\hat{g}\hat{S}_j = 2\bar{M}\hat{g}\sin(j\pi V_\xi \tau_0), \tag{20}$$

$$
\begin{aligned}
\varepsilon^2: \quad D_0^2 A_{2j} + \omega_j^2 A_{2j} = &-2D_0D_1A_{1j} + 2\mu\bar{M}\hat{g}\sum_{n=1}^{\infty} R_{jn}A_n\\
&- 2\bar{M}\sum_{n=1}^{\infty}[\hat{S}_{jn}D_0^2A_{1n} + 2V_\xi R_{jn}D_0A_{1n} - V_\xi^2 A_{1n}].
\end{aligned}
\tag{21}
$$

It is shown in equation (20) that unbounded oscillation occurs when the frequency ω_j is near $(j\pi V_\xi)$. Hence, in the following, the conditions considered are related to the cases when the natural frequency ω_j is away from $(j\pi V_\xi)$ and the coefficient of friction μ is set to be zero.

From equation (20), it is seen that the amplitude, A_{1j}, is harmonic in τ_0, and its solution can be represented as

$$
\begin{aligned}
A_{1j} &= a_j\cos(\omega_j\tau_0 + \phi_j) + \frac{2\bar{M}\hat{g}}{(\omega_j^2 - (j\pi V_\xi)^2)}\sin j\pi V_\xi\tau_0\\
&\equiv a_j\cos\beta_j + 2\bar{M}\Lambda_j\sin j\pi V_\xi\tau_0,
\end{aligned}
\tag{22}
$$

where $a_j = a_j(\tau_1,\tau_2,\ldots)$ is the amplitude of response; $\phi_j = \phi_j(\tau_1,\tau_2,\ldots)$ is the phase angle and $\Lambda_j = \hat{g}/(\omega_j^2 - (j\pi V_\xi)^2)$. Here, for convenience, rewriting equation (22) as

$$
\begin{aligned}
A_{1j} = &H_j(\tau_1,\tau_2,\ldots)\exp(\hat{\mathrm{i}}\omega_j\tau_0) + \bar{H}_j(\tau_1,\tau_2,\ldots)\exp(-\hat{\mathrm{i}}\omega_j\tau_0)\\
&- \hat{\mathrm{i}}\bar{M}\Lambda_j(\exp(\hat{\mathrm{i}}j\pi V_\xi\tau_0) - \exp(-\hat{\mathrm{i}}j\pi V_\xi\tau_0)), \quad j = 1,2,3,\ldots,
\end{aligned}
\tag{23}
$$

where $\hat{\mathrm{i}} = \sqrt{-1}$ and $\bar{H}_j$ is the complex conjugate of H_j. $H_j = \frac{1}{2}a_j\exp(\hat{\mathrm{i}}\phi_j)$, $j = 1,2,3,\ldots$, with ϕ_j being the phase of the jth mode.

To seek the solution of A_{2j} defined by equation (21), one substitutes equation (22) into equation (21) and obtains

$$
\begin{aligned}
D_0^2A_{2j} + \omega_j^2A_{2j} = &2\omega_j[(D_1a_j)\sin\beta_j + a_j(D_1\phi_j)\cos\beta_j]\\
&+ \frac{1}{2}\bar{M}\sum_{n=1}^{\infty}\{\omega_n^2 a_n[(\cos\beta_{nj}^{-+} + \cos\beta_{nj}^{--}) - (\cos\beta_{nj}^{++} + \cos\beta_{nj}^{+-})]
\end{aligned}
\tag{24}
$$

$$+\bar{M}V_\xi\sum_{n=1}^{\infty}(n\pi)\omega_n a_n[(\cos\beta_{nj}^{+-}-\cos\beta_{nj}^{++})-(\cos\beta_{nj}^{--}-\cos\beta_{nj}^{-+})]$$

$$+\frac{1}{2}\bar{M}V_\xi^2\sum_{n=1}^{\infty}(n\pi)^2 a_n[(\cos\beta_{nj}^{-+}+\cos\beta_{nj}^{--})-(\cos\beta_{nj}^{++}+\cos\beta_{nj}^{+-})]$$

$$+4\bar{M}^2\sum_{n=1}^{\infty}(n\pi V_\xi)^2\Lambda_n[\sin(2n-j)\pi V_\xi\tau_0-\sin(2n+j)\pi V_\xi\tau_0],$$

where

$$\beta_{nj}^{++}=\beta_n+(n+j)\pi V_\xi\tau_0=(\omega_n+(n+j)\pi V_\xi)\tau_0+\phi_n,$$

$$\beta_{nj}^{+-}=\beta_n-(n+j)\pi V_\xi\tau_0=(\omega_n-(n+j)\pi V_\xi)\tau_0+\phi_n,$$

$$\beta_{nj}^{-+}=\beta_n+(n-j)\pi V_\xi\tau_0=(\omega_n+(n-j)\pi V_\xi)\tau_0+\phi_n,$$

$$\beta_{nj}^{--}=\beta_n-(n-j)\pi V_\xi\tau_0=(\omega_n-(n-j)\pi V_\xi)\tau_0+\phi_n.$$

It is known that a multi-degree-of-freedom dynamic system with parametric excitation experiences multi-component parametric resonance when two or more internal frequencies and the excitation frequency are commensurable or nearly commensurable. For a dynamic system with finite degrees of freedom similar to that defined by equation (24), parametric resonance under primary resonance may exist when

(1) $\omega_k\approx\Omega=r\pi V_\xi$, $r=1,2,\ldots$, and $\omega_q-\omega_p\approx(q+p)\pi V_\xi$ with $k=p,q$ and $q>p$,
(2) $\omega_k\approx\Omega=r\pi V_\xi$, $r=1,2,\ldots$, and $\omega_q-\omega_p\approx(q-p)\pi V_\xi$ with $k=p,q$ and $q>p$,
where ω_k is the dimensionless internal frequency of the kth mode of vibration and Ω is the frequency of excitation.

For the purpose of studying the effects of multi-component parametric resonance to the motion of the beam–mass system, the set of $\omega_q\approx\Omega$ and $\omega_q-\omega_p\approx(q+p)\pi V_\xi(q>p)$ is selected so that the two-component parametric resonance defined by equation (24) exists. The resonant phenomena produced by other sets of possibilities can be obtained by similar ways.

In order to express the commensurable relations of $\omega_q\approx\Omega$ and $\omega_q-\omega_p$ to $(q+p)\pi V_\xi$, the detuning parameters σ_{pq} and σ_q are introduced:

$$\Omega=r\pi V_\xi=\omega_q+\varepsilon\sigma_q, \tag{25}$$

$$(q+p)\pi V_\xi=\omega_q-\omega_p+\varepsilon\sigma_{pq}, \tag{26}$$

where $\omega_p=K_{pq}\omega_q$ with $K_{pq}=\sqrt{((p\pi)^4+\hat{k})/(q\pi)^4+\hat{k}}$. The relationship between σ_q and σ_{pq} can be determined from equations (25) and (26) which yields

$$\varepsilon\sigma_{pq}=\left(K_{pq}-1+\frac{q+p}{r}\right)\omega_q+\frac{q+p}{r}\varepsilon\sigma_q\equiv\hat{K}_{pq}\omega_q+\hat{v}_{pq}\varepsilon\sigma_q. \tag{27}$$

Therefore, $\Omega\tau_0=(\omega_q\tau_0+\phi_q)+(\sigma_q\tau_1-\phi_q)\equiv\beta_q+\delta_q$ and for the differences of the arguments of the cosine and sine functions of unequal and equal arguments one has

$$\beta_{pp}^{--}=(\omega_p-(p-p)\pi V_\xi)\tau_0+\phi_p=\beta_p,$$

$$\beta_{pp}^{-+}=(\omega_p+(p-p)\pi V_\xi)\tau_0+\phi_p=\beta_p,$$

$$\beta_{qq}^{--} = (\omega_q - (q-q)\pi V_\xi)\tau_0 + \phi_q = (\omega_q\tau_0 + \phi_q) \equiv \beta_q,$$

$$\beta_{qq}^{-+} = (\omega_q + (q-q)\pi V_\xi)\tau_0 + \phi_q = \beta_q,$$

$$\beta_{qp}^{+-} = (\omega_q - (q+p)\pi V_\xi)\tau_0 + \phi_q = (\omega_p\tau_0 + \phi_p) - (\sigma_{pq}\tau_1 + \phi_p - \phi_q) \equiv \beta_p - \delta_{pq},$$

$$\beta_{pq}^{++} = (\omega_p + (q+p)\pi V_\xi)\tau_0 + \phi_p = \beta_q + \delta_{pq},$$

where $\delta_q = \sigma_q\tau_1 - \phi_q$ and $\delta_{pq} = \sigma_{pq}\tau_1 + \phi_p - \phi_q$. Therefore, $\delta_q = \delta_q(\tau_1, \tau_2, \ldots)$ and $\delta_{pq} = \delta_{pq}(\tau_1, \tau_2, \ldots)$ are two new phase angles. From the definition of σ_q and σ_{pq} one has

$$D_1\phi_q = \sigma_q - D_1\delta_q \quad \text{and} \quad D_1\delta_{pq} = \sigma_{pq} + D_1\phi_p - D_1\phi_q. \tag{28}$$

Returning to equation (24) the solvability conditions are the vanishing of the secular terms. These are respectively:

$$4\omega_p\hat{\mathrm{i}}(D_1H_p + \hat{\mathrm{i}}\tfrac{1}{2}\bar{M}\alpha_p^*H_p) + \bar{M}\Lambda_q^*H_q\exp(-\hat{\mathrm{i}}\sigma_{pq}\tau_1) = 0, \tag{29}$$

$$4\omega_q\hat{\mathrm{i}}(D_1H_q + \hat{\mathrm{i}}\tfrac{1}{2}\bar{M}\alpha_q^*H_q) + \bar{M}\Lambda_p^*H_p\exp(\hat{\mathrm{i}}\sigma_{pq}\tau_1) = 4\hat{\mathrm{i}}\bar{M}^2F_{km}\mathrm{e}^{\hat{\mathrm{i}}\sigma_q\tau_1}, \tag{30}$$

where $\alpha_p^* = [\omega_p^2 + (p\pi V_\xi)^2]$, $\alpha_q^* = [\omega_q^2 + (q\pi V_\xi)^2]$, $\Lambda_p^* = (\omega_p + p\pi V_\xi)^2$, $\Lambda_q^* = (\omega_q - q\pi V_\xi)^2$ and $F_{km} = [(k\pi V_\xi)^2\Lambda_k\delta_{(2k+p)r} - (m\pi V_\xi)^2\Lambda_m\delta_{(2m-p)r}]$ with k, $m = 1, 2, \ldots$, and δ_{mn} being the Dirac delta symbol. The main purpose of equations (29) and (30) is to study the response of the motion.

To determine the solutions and correspondingly the local stability of the two-component parametric resonance, we follow the procedures outlined in reference [14] and let

$$H_k = \tfrac{1}{2}(x_k - \hat{\mathrm{i}}z_k)\exp(\hat{\mathrm{i}}\theta_k\tau_1), \quad k = p, q. \tag{31}$$

Here x_k and z_k are real and $\theta_k = \mathrm{d}\phi_k/\mathrm{d}\tau_1$.

For the resonant case, one substitutes equation (31) into the resonant equations defined by equations (29) and (30) and separates the real and imaginary parts. The result yields

$$x_p' + \left(\theta_p + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right)z_p - \frac{\bar{M}\Lambda_q^*}{4\omega_p}z_q = 0, \tag{32}$$

$$z_p' - \left(\theta_p + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right)x_p + \frac{\bar{M}\Lambda_q^*}{4\omega_p}x_q = 0, \tag{33}$$

$$x_q' + \left(\theta_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right)z_q - \frac{\bar{M}\Lambda_p^*}{4\omega_q}z_p = 2\frac{\bar{M}^2}{\omega_q}F_{km}, \tag{34}$$

$$z_q' - \left(\theta_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right)x_q + \frac{\bar{M}\Lambda_p^*}{4\omega_q}x_p = 0, \tag{35}$$

where $(\ )' = \mathrm{d}/\mathrm{d}\tau_1$. The solutions of the two-component resonance with the excitation frequency being near the lower resonant frequency ω_p then can be obtained by setting $x_k' = z_k' = 0$ and $x_k^2 + z_k^2 = a_k^2$, $k = p$, q. This gives

$$a_p = \frac{\bar{M}\Lambda_q^*/4\omega_p}{[(\sigma_q - \sigma_{pq}) + (\bar{M}/2\omega_p)\alpha_p^*]}a_q = \frac{\varepsilon\bar{M}\Lambda_q^*/4\omega_p}{[-\hat{K}_{pq}\omega_q + (1-\hat{v}_{pq})\varepsilon\sigma_q + (\varepsilon\bar{M}/2\omega_p)\alpha_p^*]}a_q, \tag{36}$$

$$\begin{aligned} a_q &= \frac{(2\bar{M}^2/\omega_q)F_{km}((\sigma_q - \sigma_{pq}) + (\bar{M}/2\omega_p)\alpha_p^*)}{(\sigma_q + (\bar{M}/2\omega_q)\alpha_q^*)((\sigma_q - \sigma_{pq}) + (\bar{M}/2\omega_p)\alpha_p^*) - \bar{M}^2 \Lambda_p^* \Lambda_q^*/16\omega_p\omega_q} \\ &\equiv \frac{(2\varepsilon\bar{M}^2/\omega_q)F_{km}[-\hat{K}_{pq}\omega_q + (1-\hat{v}_{pq})\varepsilon\sigma_q + (\varepsilon\bar{M}/2\omega_p)\alpha_p^*]}{\Delta}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Delta &= \varepsilon^2\left[\left(\sigma_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right)\left((\sigma_q - \sigma_{pq}) + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right) - \frac{\bar{M}^2\Lambda_p^*\Lambda_q^*}{16\omega_p\omega_q}\right] \\ &= \left(\varepsilon\sigma_q + \frac{\varepsilon\bar{M}}{2\omega_q}\alpha_q^*\right)\left(-\hat{K}_{pq}\omega_q + (1-\hat{v}_{pq})\varepsilon\sigma_q + \frac{\varepsilon\bar{M}}{2\omega_p}\alpha_p^*\right) - \frac{(\varepsilon\bar{M})^2\Lambda_p^*\Lambda_q^*}{16\omega_p\omega_q}. \end{aligned}$$

As shown in equation (37), unbounded solutions exist if $\Delta = 0$. In other words, the growth of small amplitude of vibrations into large amplitude regime occurs if the denominator Δ is close to zero and unbounded solutions exist if $\Delta = 0$. Note that for the case of single resonant mode, one-component resonance, the solution reduces to

$$a_q = \frac{(2\varepsilon\bar{M}^2/\omega_q)\,F_{km}}{(\varepsilon\sigma_q + (\varepsilon\bar{M}/2\omega_q)\alpha_q^*)}. \tag{38}$$

Therefore, in the case of one-component resonance, the occurrences of unbounded solutions and the growth of response do not exist.

The local stability of a fixed point with respect to a small perturbation for each resonant case, hence, can be determined by the eigenvalues λ which are given by the zero of the determinant of the perturbation equations. For this, a small perturbation is superimposed on x_k and z_k $(k = p, q)$ and we have

$$x_k = x_k^0 + \hat{x}_k \quad \text{and} \quad z_k = z_k^0 + \hat{z}_k. \tag{39}$$

Here $x_k^0, z_k^0, \hat{x}_k$ and $\hat{z}_k$ are the fixed points and the disturbances respectively. The determinant then can be obtained by substituting equation (39) into equations (32)–(35). The result yields

$$\begin{vmatrix} \lambda & \theta_p + \frac{\bar{M}}{2\omega_p}\alpha_p^* & 0 & -\frac{\bar{M}\Lambda_q^*}{4\omega_p} \\ -\left(\theta_p + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right) & \lambda & \frac{\bar{M}\Lambda_q^*}{4\omega_p} & 0 \\ 0 & -\frac{\bar{M}\Lambda_p^*}{4\omega_q} & \lambda & \theta_q + \frac{\bar{M}}{2\omega_q}\alpha_q^* \\ \frac{\bar{M}\Lambda_p^*}{4\omega_q} & 0 & -\left(\theta_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right) & \lambda \end{vmatrix} = 0, \tag{40}$$

where $\theta_p = D_1\phi_p = \sigma_p$ and $\theta_q = D_1\phi_q = \sigma_{pq} + \sigma_p$. Thus, the characteristic equation of equation (40) has the form

$$\lambda^4 + r_1\lambda^2 + r_2^2 = 0, \tag{41}$$

where

$$r_1 = \left[\left(\sigma_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right)^2 + \left((\sigma_q - \sigma_{pq}) + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right)^2 + \frac{\bar{M}^2}{2\omega_p\omega_q}\Lambda_p^*\Lambda_q^*\right],$$

$$r_2^2 = \left[\left(\sigma_q + \frac{\bar{M}}{2\omega_q}\alpha_q^*\right)\left((\sigma_q - \sigma_{pq}) + \frac{\bar{M}}{2\omega_p}\alpha_p^*\right) - \frac{\bar{M}^2}{16\omega_p\omega_q}\Lambda_p^*\Lambda_q^*\right]^2.$$

Note that from equations (37) and (41), it is found that r_2 has the same form as Δ defined by equation (37) and since r_1 and r_2^2 are always greater than zero, the solutions are stable (bounded solutions) except $r_2 = 0$ (unbounded solutions).

3. NUMERICAL RESULTS AND DISCUSSIONS

Numerical results refer to an assumed model wherein a finite inextensible beam rests on a uniform elastic foundation and carries a mass rolling on it with constant velocity. The accuracy of the model is verified by numerically integrating equation (18), by the Runge–Kutta method with sixth order accuracy. The existence and validity of perturbation solutions are also proved by numerically integrating the modulation equations, equations (29)–(30), by the Runge–Kutta method.

Figure 3 shows the trajectories of the moving mass with two different constant velocities versus the position of the mass along the beam. The parameters used in this figure are exactly the same as those used in reference [3]. The parameters used are $\hat{v}_{st} = \hat{M}\hat{g}/48$ and $\hat{M} = 0{\cdot}5$ with two different constant velocities $\dot{\xi} = 0{\cdot}75\pi$ (top plot) and $\dot{\xi} = 0{\cdot}5\pi$ (bottom plot). The accuracy of the model then was tested by comparison of its results with the

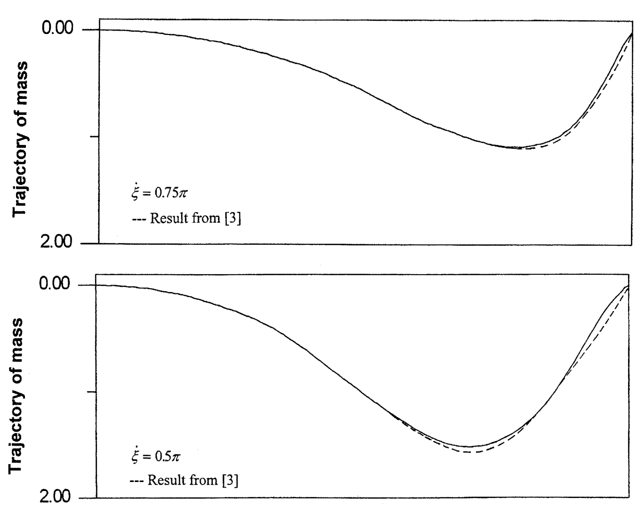

Figure 3. The trajectory of mass versus the position of the mass along the beam with the same parameters as used in reference [3] for $\dot{\xi} = 0{\cdot}75\pi$ and $\dot{\xi} = 0{\cdot}5\pi$.

results reported by Ting *et al.* [3]. It is known that the last is in agreement with experimental observations. It is mentioned here that in order to retain for sufficient accuracy, the dimension n used in equation (18) was 30.

Without loss of generality and considering the commensurable relations among frequencies and the probability of occurrence in nature, the following set of commensurable relations of vibrating modes to determine the basic characteristics of the occurrence of two-component parametric resonance was selected. For the case of $\Omega = r\pi V_\xi \approx \omega_q$ and $(q+p)\pi V_\xi \approx \omega_q - \omega_p$ one chose $p = 1$ (fundamental mode), $q = 2$ (second mode) and $r = 4$ to present the occurrence of two-component parametric resonance. It is recalled that V_ξ is the dimensionless velocity of the attached mass moving along the beam; ω_j is the dimensionless natural frequency and is defined by $\omega_j = \sqrt{(j\pi)^4 + \hat{k}}$ where $\hat{k}$ is the foundation stiffness.

In addition to the stability analysis, as mentioned previously, the existence and validity of perturbation solutions is verified by numerically integrating the modulation equations.

From equation (37), it is found that unbounded solutions exist if $\Delta = 0$ and the growth of small amplitude of vibrations into large motion regime occurs if Δ is near zero. Hence, boundaries of the unbounded solutions, as functions of the velocity of the mass V_ξ, must

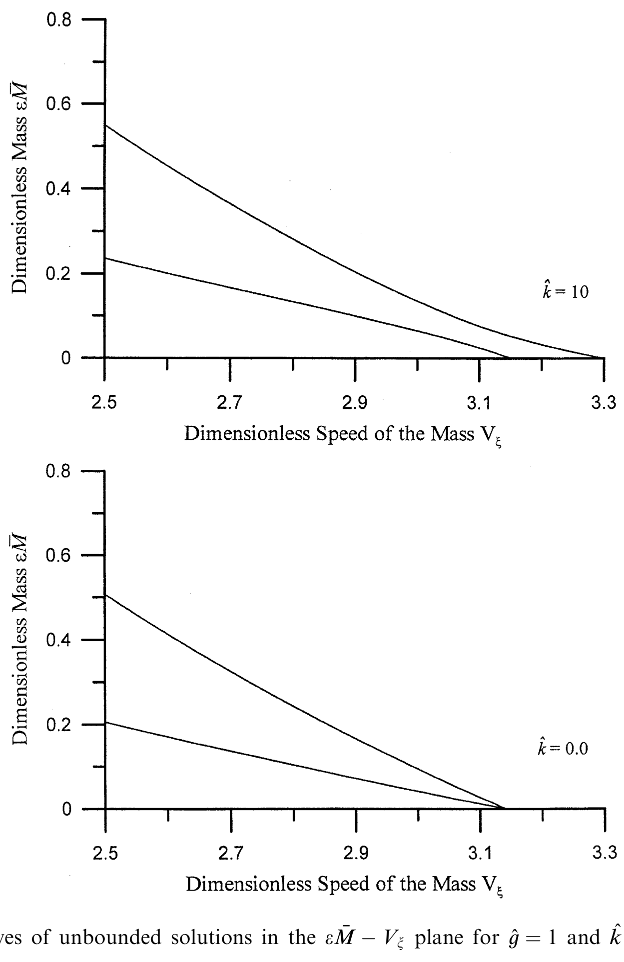

Figure 4. The curves of unbounded solutions in the $\varepsilon\bar{M} - V_\xi$ plane for $\hat{g} = 1$ and $\hat{k} = 0$ (lower plot) and $\hat{k} = 10$ (upper plot).

be determined. One recalls

$$\Delta = \left(\varepsilon\sigma_q + \frac{\varepsilon\bar{M}}{2\omega_q}\alpha_q^*\right)\left(-\hat{K}_{pq}\omega_q + (1-\hat{v}_{pq})\varepsilon\sigma_q + \frac{\varepsilon\bar{M}}{2\omega_p}\alpha_p^*\right) - \frac{(\varepsilon\bar{M})^2\Lambda_p^*\Lambda_q^*}{16\omega_p\omega_q} \tag{42}$$

$$= \left[\frac{1}{16\omega_p\omega_q}(4\alpha_p^*\alpha_q^* - \Lambda_p^*\Lambda_q^*)\right](\varepsilon\bar{M})^2 + \left[-\frac{1}{2}\alpha_q^*\hat{K}_{pq} + \varepsilon\sigma_q\left((1-\hat{v}_{pq})\frac{\alpha_q^*}{2\omega_q} + \frac{\alpha_p^*}{2\omega_p}\right)\right](\varepsilon\bar{M}) + [-\hat{K}_{pq}\omega_q(\varepsilon\sigma_q) + (1-\hat{v}_{pq})(\varepsilon\sigma_q)^2].$$

In the above equation, $\varepsilon\bar{M}$ is the mass and has to be no less than zero. Therefore, the positive roots of equation (42) of $\varepsilon\bar{M}$ imply the existence of unbounded solutions.

Figure 4 shows the variations of the mass $\varepsilon\bar{M}$ with the dimensionless speed of the mass, V_ξ, for $\hat{g}=1$ and $\hat{k}=0$ (lower plot) and $\hat{k}=10$ (upper plot). In this figure, solid lines denote the values of the corresponding parameters such that the unstable motion (unbounded solutions) occurs. This figure indicates that there exist two possibilities of occurrence of unbounded solutions and correspondingly the growth of response for small values of $\varepsilon\bar{M}$. However, only one possibility exists if $\varepsilon\bar{M}$ is not small. As an example, if $\hat{k}$

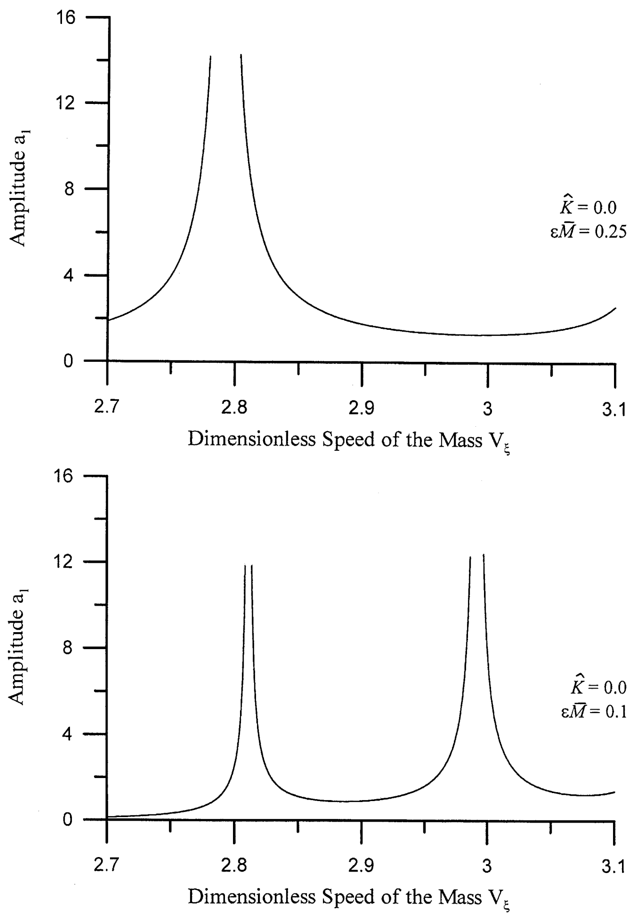

Figure 5. The amplitude a_1 versus the speed of the sass V_ξ for $\hat{g}=1$, $\hat{k}=0$ and $\varepsilon\bar{M}=0{\cdot}1$ (bottom plot) and $\varepsilon\bar{M}=0{\cdot}25$ (top plot).

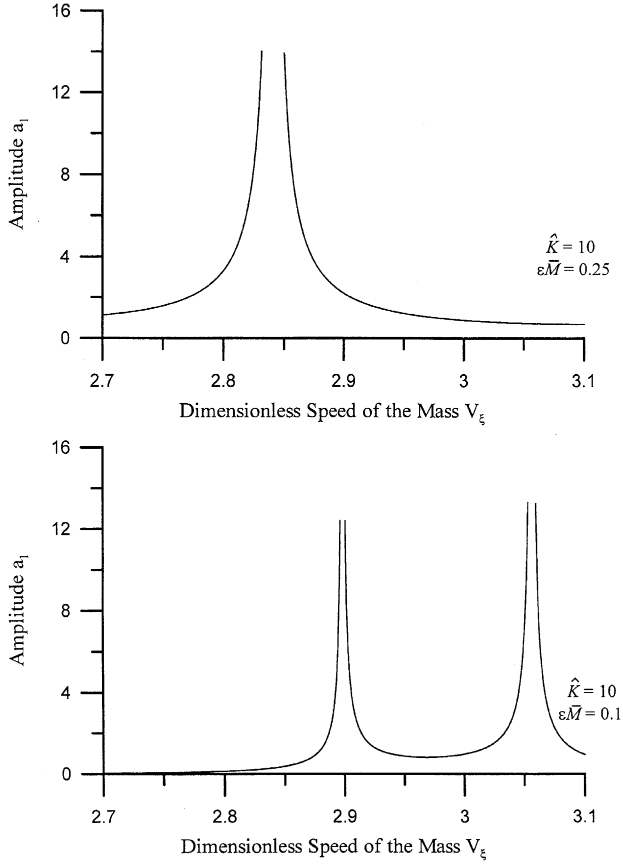

Figure 6. The amplitude a_1 versus the speed of the mass V_ξ for $\hat{g} = 1$, $\hat{k} = 10$ and $\varepsilon\bar{M} = 0{\cdot}1$ (lower plot) and $\varepsilon\bar{M} = 0{\cdot}25$ (upper plot).

and $\varepsilon\bar{M}$ are chosen to be 0·0 and 0·1, respectively, then unbounded solutions of the system exist when V_ξ is near either 2·811 or 2·992. If $\varepsilon\bar{M}$ is set to be 0·25 unbounded solution occurs only when V_ξ is close to 2·792. This is verified in Figure 5. In this figure, the lower plot is related to the case for $\varepsilon\bar{M} = 0{\cdot}1$ and the upper plot to that for $\varepsilon\bar{M} = 0{\cdot}25$.

Figure 6 illustrates the same manner as does Figure 5, except $\hat{k} = 10$. From Figures 5 and 6, one found that the width of the regions of the occurrence of large motions increases as the mass $\varepsilon\bar{M}$ increases. However, it decreases with the increase of the foundation stiffness $\hat{k}$. In addition, the occurrence of large amplitude of vibrations shifts to larger values of speed of mass as foundation stiffness shifts to higher values.

Figure 7 presents the long-time behavior of the amplitude a_1 for $\varepsilon\bar{M} = 0{\cdot}25$, $\hat{g} = 1$, and $\hat{k} = 0$ with two different values of V_ξ, $V_\xi = 2{\cdot}95$ (bottom plot) and 2·82 (top plot). Figure 8 illustrates similar information to that shown in Figure 6 except that in this figure, $\varepsilon\bar{M} = 0{\cdot}1$ and the speed of the mass V_ξ is set to be 2·82 (lower one) and 2·975 (upper one). The results of these figures clearly show that in certain conditions, the growth of small amplitude of vibrations into large motion regime does occur.

4. CONCLUSIONS

In this study, the weak form of the occurrence of two-component parametric resonance is obtained under a primary resonance. The mechanics of a beam–mass system and the

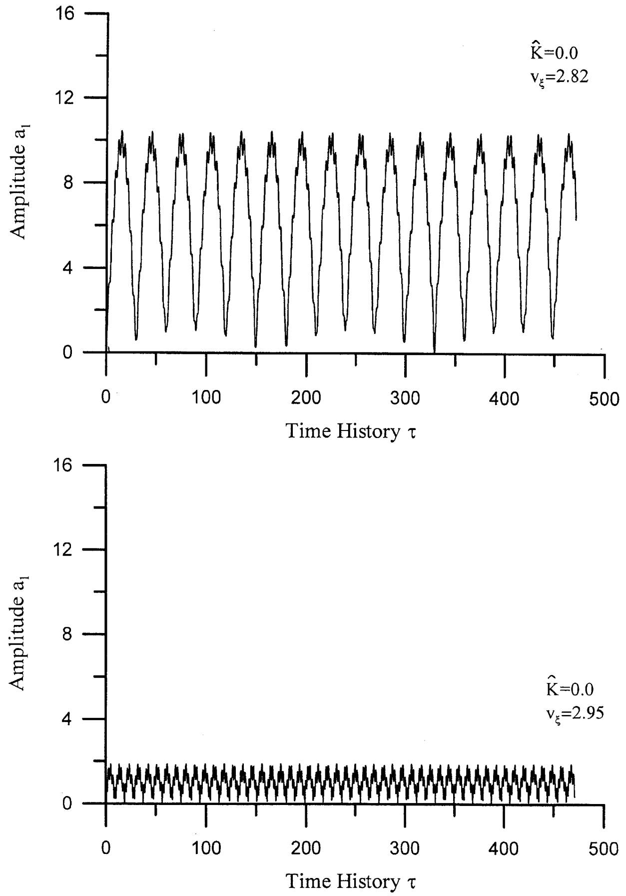

Figure 7. Time history of the amplitude a_1 for $\varepsilon\bar{M} = 0{\cdot}25$, $\hat{g} = 1$, and $\hat{k} = 0$ with two different values of V_ξ, $V_\xi = 2{\cdot}95$ (bottom plot) and $2{\cdot}82$ (top plot).

phenomena produced by the existence of the two-component parametric resonance are studied.

Results of the study show that for the case of two-component resonance, new regions of the growth of small amplitude of vibrations into large motion regime for the first mode are found even if the excitation is not close to the fundamental mode. This is due to the occurrence of modal interactions caused by the existence of a two-component parametric resonance. However, this phenomenon, the growth of response for the fundamental mode of the beam, could not be found in the case of single resonant mode if the excitation due to the motion of attached mass is not near the first mode.

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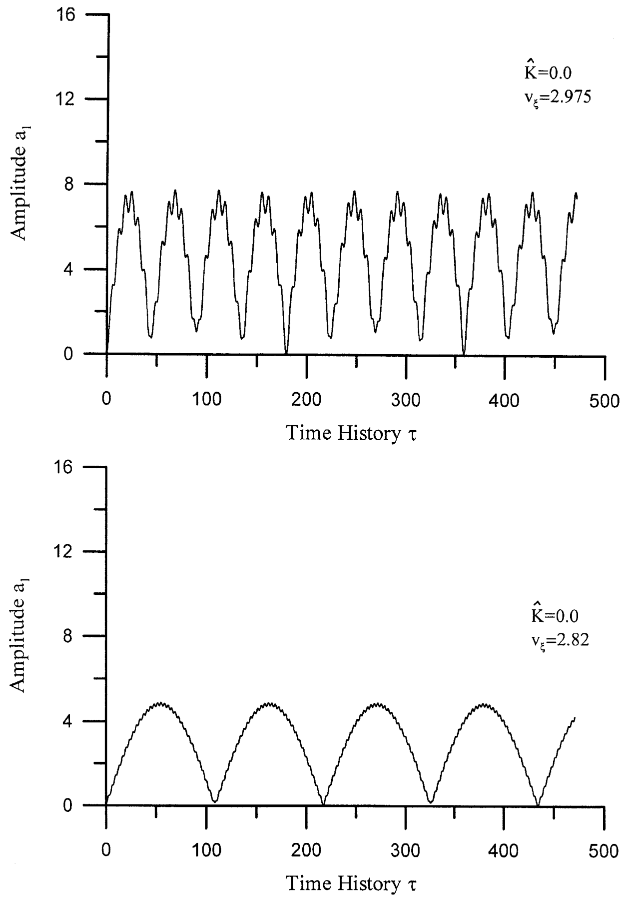

Figure 8. Time history of the amplitude a_1 for $\varepsilon\bar{M}=0{\cdot}1$, $\hat{g}=1$, and $\hat{k}=0$ with two different values of V_ξ, $V_\xi=2{\cdot}82$ (lower one) and 2·975 (upper one).

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APPENDIX A: NOMENCLATURE

E	Young's modulus of the beam
H	the horizontal component of the tension in the beam
I	the area moment of inertia of the beam
$M, \hat{M}$	the mass and the dimensionless mass of the moving mass
$\hat{M}$	$\left(=\dfrac{M}{ml}=\varepsilon\bar{M}\right)$
N	the reaction of beam on the mass
R_{jn}	$(=(n\pi)\cos n\pi\xi \sin j\pi\xi)$
S_{jn}	$(=(n\pi)^2 \sin n\pi\xi \sin j\pi\xi = (n\pi)^2\hat{S}_{jn})$
$\hat{S}_n$	$(=\sin n\pi\xi)$
$T, \hat{T}$	the axial force and dimensionless axial force in the beam respectively
V	the transverse force in the beam
V_ξ	dimensionless velocity of the moving mass
a_j	the amplitude of response of the jth mode
$\mathbf{a}_M$	acceleration of the moving mass
$\mathbf{f}$	external force
$\mathbf{f}_m$	tangential propelling force with f being a prescribed function of time $(=Mf\hat{\boldsymbol{\tau}})$
$\mathbf{g}$	acceleration due to gravity $(=g\mathbf{j})$
$\hat{g}$	$\left(=\dfrac{ml^3}{EI}g\right)$
H_j	$\frac{1}{2}a_j\exp(\hat{\mathrm{i}}\phi_j)$
$\bar{H}_j$	complex conjugate of H_j
$\hat{\mathrm{i}}$	$(=\sqrt{-1})$
$\mathbf{i}$	unit vector of the horizontal co-ordinate (x-axis) for the beam
$\mathbf{j}$	unit vector of the transverse co-ordinate (y-axis) for the beam, positive downward
$k, \hat{k}$	foundation stiffness per unit length and dimensionless foundation stiffness respectively
l	length of span
m	the mass per unit length of the beam
$\mathbf{n}$	unit normal vector to the beam configuration
$\mathbf{r}$	$((x+u), v)^{\mathrm{T}}$ position vector and components at time t
$s, \bar{s}(t)$	the arc length and the position of the moving mass along the beam respectively
$\hat{v}$	dimensionless transverse displacement
$\hat{\boldsymbol{\tau}}$	unit tangent vector to the beam configuration
θ	the angle between the neutral axis of the beam and the x-axis
τ	dimensionless time
μ	coefficient of friction
η	dimensionless arc
ξ	dimensionless position of the moving mass along the beam
ω_j	the jth dimensionless natural frequency of the beam

ϕ_j	the phase angle of the jth mode of vibration
δ_i, δ_{ij}	phase angles
$\delta(s-\bar{s})$	Dirac delta function
$\bar{\delta}_{ij}$	Dirac delta symbol
σ_j	the detuning parameter between the excitation frequency and the frequency of the jth mode of vibration
σ_{ij}	the detuning parameter between the frequencies of the ith and the jth modes.