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EXACT POWER SERIES SOLUTIONS FOR AXISYMMETRIC VIBRATIONS OF CIRCULAR AND ANNULAR MEMBRANES WITH CONTINUOUSLY VARYING DENSITY IN THE GENERAL CASE

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1. INTRODUCTION

In a recent paper by Bala Subrahmanyam and Sujith [1], a theoretical study of axisymmetric vibrations of solid circular and annular membranes with continuously varying density has been presented continuing the works of previous investigations [2–7]. Exact solutions are found for two families of functional dependencies of the membrane density with respect to the cylindrical radial co-ordinate [1]. One family is for the case where the density ρ_1 varies as

$$\rho_1(r) = \rho_0 f_1(r), \qquad f_1(r) = \frac{A}{r^2} + Br^t + Cr^m, \tag{1}$$

while the other family corresponds to the case where the density ρ_2 varies as

$$\rho_2(r) = \rho_0 f_2(r), \qquad f_2(r) = \frac{[1 + \alpha \log(r)]^\sigma}{r^2}, \quad \alpha \neq 0. \tag{2}$$

In the present work, a quasi-analytical approach is presented so as to find eigenfrequencies and eigensolutions (mode displacements) in the general case where the density $\rho(r)$ can be written as an infinite power series expansion in the radial co-ordinate, i.e.,

$$\rho(r) = \rho_0 f(r), \qquad f(r) = \sum_{n=0}^{\infty} f_n r^n. \tag{3}$$

Since any C^∞ function has a Taylor series expansion, membrane densities which are C^∞ functions of the radial co-ordinate are covered by the present formulation. In addition, the most relevant density variations, if not all, can be well *approximated* by a polynomial/power series expansion in the radial co-ordinate. In the latter case, the present formulation can also be used to find membrane solutions.

2. THEORY

In the following, an exact method is described so as to obtain the displacement $W(r)$ for the axisymmetric vibrational modes in the case of a solid circular membrane or an annular membrane of outer radius R and inner radius R_0. The exact power series solution method, proposed in this letter, does not impose any requirements on the functional form of the

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membrane density except that it can be written as an infinite power series in the radial coordinate r. The governing differential equation for the displacement $W(r)$ is [1]

$$\frac{\mathrm{d}^2 W}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}+\Omega^2 f(r)W(r)=0, \quad 0\leqslant r_0\leqslant r\leqslant 1, \tag{4}$$

where

$$\rho(r)=\rho_0 f(r), \qquad f(r)=\sum_{n=0}^{\infty} f_{n1} r^n, \tag{5, 6}$$

$$r_0=\frac{R_0}{R}, \qquad \Omega=\omega R\sqrt{\rho_0/S} \tag{7, 8}$$

and S is the tension per unit length. The coefficients f_{n1} are not restricted but $f(r=1)=\sum_{n=0}^{\infty} f_{n1}$ must be a convergent series ($\sum_{n=0}^{\infty} f_{n1}<\infty$). In order to handle the problem of annular and solid circular membranes it is convenient to introduce the variable

$$r^*=1-r \tag{9}$$

and equation (4) becomes

$$\frac{\mathrm{d}^2 W}{\mathrm{d}r^{*2}}+\frac{1}{(r^*-1)}\frac{\mathrm{d}W}{\mathrm{d}r^*}+\Omega^2 f(r^*)W(r^*)=0, \quad 0\leqslant r^*\leqslant 1-r_0, \tag{10}$$

where

$$f(r^*)=\sum_{n=0}^{\infty} f_{n2} r^{*n}\equiv\sum_{n=0}^{\infty} f_{n1} r^n=f(r). \tag{11}$$

The boundary conditions for the solid circular membrane is

$$\frac{\mathrm{d}W}{\mathrm{d}r}(r=0)=-\frac{\mathrm{d}W}{\mathrm{d}r^*}(r^*=1)=0, \tag{12}$$

$$W(r=1)=W(r^*=0)=0, \tag{13}$$

while for the annular membrane, the boundary conditions become

$$W(r=r_0)=W(r^*=1-r_0)=0, \tag{14}$$

$$W(r=1)=W(r^*=0)=0. \tag{15}$$

Next, the Frobenius method [8] is employed so as to solve equation (10) and the associated boundary conditions. The Frobenius method is based on the assumption that W can be written as a series expansion in r^*:

$$W(r^*)=\sum_{n=0}^{\infty} a_n r^{*n+k}, \tag{16}$$

where k is unspecified (in general, at this point, k can be any real constant). Insertion of equations (11) and (16) into equation (10) [after multiplying the latter equation by (r^*-1)] and demanding that $W(r^*=0)=0$ as well as $a_0\neq 0$ gives

$$k=1, \tag{17}$$

if terms proportional to r^{*k-2} are equated (corresponding to $n=0$). Again, employing the identity principle for infinite power series to terms proportional to r^{*n} leads to the following recursion formula:

$$a_0 = 1, \qquad a_1 = \tfrac{1}{2} a_0,$$
$$a_2 = \tfrac{2}{3} a_1 - \tfrac{1}{6} \Omega^2 a_0 f_{02}, \qquad a_3 = \tfrac{3}{4} a_2 + \tfrac{1}{12} \Omega^2 a_0 f_{02} - \tfrac{1}{12} \Omega^2 (a_1 f_{02} + a_0 f_{12}),$$

$$a_{n+1} = \frac{n+1}{n+2} a_n + \Omega^2 \frac{1}{(n+1)(n+2)} \sum_{m=0}^{n-2} a_{n-2-m} f_{m2} - \Omega^2 \frac{1}{(n+1)(n+2)} \sum_{m=0}^{n-1} a_{n-1-m} f_{m2}, \quad n \geqslant 3. \tag{18}$$

Next, the possible Ω values must be determined by use of the second boundary condition. In the solid membrane case, the discrete set of possible Ω values are those for which

$$\frac{\mathrm{d}W}{\mathrm{d}r}(r = 0) = -\frac{\mathrm{d}W}{\mathrm{d}r^*}(r^* = 1) = 0, \tag{19}$$

i.e.,

$$\sum_{n=0}^{\infty} a_n (n+1) = 0, \tag{20}$$

where the last equation follows from equations (16), (17), and (19).

Similarly, in the annular membrane case, the discrete set of possible Ω values are those for which

$$W(r = r_0) = W(r^* = 1 - r_0) = \sum_{n=0}^{\infty} a_n (1 - r_0)^{n+1} = 0, \tag{21}$$

employing equations (14), (16), and (17). By solving equations (18) and (20) [or equation (21)] numerically, a set of discrete Ω values ($\equiv \Omega_n$) are found and so a discrete set of W_n solutions have been determined. Each of the W_n solutions represents a vibrational mode of the axisymmetric membrane.

3. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the Frobenius infinite power series expansion method is applied to three different mass density profiles of the membrane.

3.1. EXAMPLE 1

Consider first the case where

$$f(r) = 1 + \alpha r^2, \tag{22}$$

which has been studied in reference [1] analytically and numerically in reference [5]. In Table 1, values of Ω corresponding to the fundamental and second frequency coefficients are given for the case $\alpha = \frac{1}{2}$. The agreement with references [1, 5] is excellent. The same conclusion is reached for other α values.

3.2. EXAMPLE 2

In Table 2, data for the case

$$f(r) = -r^2 \log(r), \tag{23}$$

TABLE 1

Calculated eigenfrequencies corresponding to the membrane density variation $f(r) = 1 + \alpha r^2$ considered in example 1. The first column is the value r_0 of the inner radius for annular membranes [$r_0 = 0$ for solid circular membranes]. The second and third columns are the eigenfrequencies Ω for the fundamental and second-vibrational mode respectively

r_0/mode	1	2
0	2·2819	5·1412
0·1	3·0735	6·3195
0·2	3·4969	7·1061
0·3	3·9943	8·0659
0·4	4·6321	9·3186
0·5	5·5071	11·0525
0·6	6·8050	13·6366
0·7	8·9547	17·9269
0·8	13·2394	26·4893
0·9	26·0725	52·1498

TABLE 2

Calculated eigenfrequencies corresponding to the membrane density variation $f(r) = -r^2 \log(r)$ considered in example 2. The first column is the value r_0 of the inner radius for annular membranes [$r_0 = 0$ for solid circular membranes]. The second and third columns are the eigenfrequencies Ω for the fundamental and second-vibrational mode respectively

r_0/mode	1	2
0	6·5809	16·6039
0·1	8·3526	18·8578
0·2	9·4279	20·6835
0·3	10·8386	23·3231
0·4	12·8513	27·2916
0·5	15·9230	33·5169
0·6	21·0259	44·0042
0·7	30·6690	63·9557
0·8	53·5231	111·3861
0·9	144·1913	299·7721

are shown. This particular example is not covered by the method proposed in reference [1]. The function f given by equation (23) can be written as an infinite power series expansion in r^* as follows:

$$\begin{aligned} -r^2\log(r) &= -(1-r^*)^2\log(1-r^*) \\ &= (2r^* - 1 - r^{*2})\left(-r^* - \frac{r^{*2}}{2} - \frac{r^{*3}}{3} - \cdots\right) \\ &= \sum_{n=0}^{\infty}\frac{r^{*n+1}}{n+1} + \sum_{n=0}^{\infty}\frac{r^{*n+3}}{n+1} - 2\sum_{n=0}^{\infty}\frac{r^{*n+2}}{n+1}. \end{aligned} \tag{24}$$

The values given in Table 2 correspond to the first two Ω solutions fulfilling the relevant boundary conditions [either equations (12) and (13) or equations (14) and (15)].

3.3. EXAMPLE 3

Finally, consider the case

$$f(r) = 1 + \alpha r + \beta r^2 + \gamma r^3, \tag{25}$$

where α, β, and γ are constants. The expression for f in terms of r^* co-ordinates becomes

$$f(r) = 1 + \alpha + \beta + \gamma + (-\alpha - 2\beta - 3\gamma)r^* + (\beta + 3\gamma)r^{*2} - \gamma r^{*3}. \tag{26}$$

TABLE 3

Calculated eigenfrequencies corresponding to the membrane density variation $f(r) = 1 + \alpha r + \beta r^2 + \gamma r^3$ considered in example 3. Four cases (I)–(IV) are considered as described in the main text. The first column is the value r_0 of the inner radius for annular membranes [$r_0 = 0$ for solid circular membranes]. The second, third, fourth, and fifth columns are the eigenfrequencies Ω for the fundamental vibrational mode in cases (I), (II), (III)*, and* (IV) *respectively*

r_0/case	I	II	III	IV
0	1·9211	1·8655	2·0607	1·7911
0·1	2·4955	2·4155	2·6643	2·2851
0·2	2·7950	2·7045	2·9703	2·5394
0·3	3·1459	3·0457	3·3244	2·8374
0·4	3·5954	3·4860	3·7747	3·2204
0·5	4·2123	4·0942	4·3902	3·7480
0·6	5·1277	5·0016	5·3027	4·5338
0·7	6·6451	6·5115	6·8156	5·8408
0·8	9·6717	9·5314	9·8366	8·4549
0·9	18·7424	18·5963	18·9008	16·3035

TABLE 4

Calculated eigenfrequencies corresponding to the membrane density variation $f(r) = 1 + \alpha r + \beta r^2 + \gamma r^3$ considered in example 3. Four cases (I)–(IV) *are considered as described in the main text. The first column is the value r_0 of the inner radius for annular membranes [$r_0 = 0$ for solid circular membranes]. The second, third, fourth, and fifth columns are the eigenfrequencies Ω for the second vibrational mode in cases* (I), (II), (III), *and* (IV) *respectively*

r_0/case	I	II	III	IV
0	4·2669	4·1704	4·5027	3·9558
0·1	5·1273	4·9943	5·4372	4·7065
0·2	5·6810	5·5249	6·0163	5·1764
0·3	6·3556	6·1766	6·7064	5·7478
0·4	7·2365	7·0352	7·5950	6·4959
0·5	8·4574	8·2346	8·8168	7·5371
0·6	10·2787	10·0360	10·6336	9·0976
0·7	13·3058	13·0450	13·6517	11·7022
0·8	19·3529	19·0761	19·6865	16·9225
0·9	37·4893	37·1986	37·8080	32·6129

In Table 3, the fundamental frequency coefficient values are given for the four cases (I) $\alpha = 1$, $\beta = 0$, $\gamma = 1$; (II) $\alpha = 1$, $\beta = 1$, $\gamma = 0$; (III) $\alpha = 0$, $\beta = 1$, $\gamma = 1$; (IV) $\alpha = 1$, $\beta = 1$, $\gamma = 1$.

In Table 4, the second frequency coefficient values of Ω are given for the same four cases (I)–(IV).

As expected, the natural frequency Ω increases with increasing r_0 value in all examples considered.

4. CONCLUSIONS

A general quasi-analytical model based on the Frobenius power series expansion method is described so as to handle vibrations of solid circular and annular membranes with continuously varying density. The method given in this work serves as an extension to previous analytical works [1–7] as it can be used to handle *any* density variation which can be represented as an infinite power series expansion in the radial co-ordinate. Natural frequency results are finally computed for three examples of varying membrane density. One of the three examples has been considered in references [1, 5] as well and excellent agreement between the present results and references [1, 5] is obtained for this particular example.

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