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MINIMUM STIFFNESS OF AN INTERNAL ELASTIC SUPPORT TO MAXIMIZE THE FUNDAMENTAL FREQUENCY OF A VIBRATING BEAM

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1. INTRODUCTION

The fundamental frequency (below which no vibration could occur) can be increased if a beam has additional internal point supports. If the supports are rigid, Courant and Hilbert [1] showed that the optimum locations of the supports should be at the nodal points of a higher vibration mode, and the fundamental frequency is correspondingly raised. For elastic supports, Akesson and Olhoff [2] demonstrated that the optimum locations are still the same as the case of the rigid supports, with no decrease in fundamental frequency, provided the support stiffness exceeds a certain minimum value. Such minimum support stiffness phenomenon also occurs in the buckling of beams [3]. The minimum stiffness prediction is very important in the design of beams, since the bracing or support material can be reduced without any loss of performance.

There exist other literature on the vibration of beams with internal elastic supports (e.g. references [4–8]), but reference [2] is the only source which discussed the minimum stiffness. They used finite elements to find the stiffness criterion for the cantilever beam. The present note presents the optimum location and the minimum stiffness of internal support for beams with other end conditions. For accuracy, we shall use the exact characteristic equation to compute the eigenfrequencies.

2. FORMULATION

Let the beam be of length L and xL be the distance from the left end. If the transverse displacement is $w(x)\cos(\omega t)$, the governing equation for vibration of a slender beam is [4]

$$w''''(x) - \lambda^4 w = 0, \tag{1}$$

where $\lambda^4 =$ (mass per length ρ) L^4 (frequency ω^2)/ρ(flexural rigidity D) is the square of the normalized frequency. The general solution to equation (1) is a linear combination of $\sinh(\lambda x)$, $\cosh(\lambda x)$, $\sin(\lambda x)$, $\cos(\lambda x)$. The elastic support is at $x = b$. Let the subscript I denote the segment $0 \leqslant x \leqslant b$ and the subscript II denote the segment $b \leqslant x \leqslant 1$. Thus, the solution for segment I is

$$w_{\mathrm{I}}(x) = C_1[\sinh(\lambda x) - \sin(\lambda x)] + C_2[\cosh(\lambda x) - \cos(\lambda x)], \tag{2a}$$

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$$w_{\mathrm{I}}(x) = C_1 \sinh(\lambda x) + C_2 \sin(\lambda x), \quad w_{\mathrm{I}}(x) = C_1 \cosh(\lambda x) + C_2 \cos(\lambda x). \tag{2b,c}$$

$$w_{\mathrm{I}}(x) = C_1 [\sinh(\lambda x) + \sin(\lambda x)] + C_2 [\cosh(\lambda x) + \cos(\lambda x)] \tag{2d}$$

for clamped, simply supported, sliding, and free left end conditions respectively. Similarly, the solution for segment II is

$$w_{\mathrm{II}}(x) = C_3 \{\sinh[\lambda(x-1)] - \sin[\lambda(x-1)]\} + C_4 \{\cosh[\lambda(x-1)] - \cos[\lambda(x-1)]\}, \tag{3a}$$

$$w_{\mathrm{II}}(x) = C_3 \sinh[\lambda(x-1)] + C_4 \sin[\lambda(x-1)],$$

$$w_{\mathrm{II}}(x) = C_3 \cosh[\lambda(x-1)] + C_4 \cos[\lambda(x-1)]. \tag{3b, c}$$

$$w_{\mathrm{II}}(x) = C_3 \{\sinh[\lambda(x-1)] + \sin[\lambda(x-1)]\} + C_4 [\cosh[\lambda(x-1)] + \cos[\lambda(x-1)]\} \tag{3d}$$

for the above-mentioned four kinds of right end conditions. At the support, the two segments are matched for displacement, slope, moment and shear:

$$w_{\mathrm{I}}(b) = w_{\mathrm{II}}(b), \quad w'_{\mathrm{I}}(b) = w'_{\mathrm{II}}(b), \quad w''_{\mathrm{I}}(b) = w''_{\mathrm{II}}(b), \tag{4–6}$$

$$w'''_{\mathrm{I}}(b) - \gamma w_{\mathrm{I}}(b) = w'''_{\mathrm{II}}(b), \tag{7}$$

where $\gamma =$ (spring constant c) L^3/D is the normalized stiffness. Equations (2) and (3) are then substituted into equations (4)–(7). For non-trivial solutions, exact characteristic equation is obtained. The frequency parameter λ is then solved by a bisection algorithm to any desired accuracy. In order to find the minimum stiffness, the following scheme is used. First consider the beam without the support. Form equation (2) and the appropriate boundary conditions on the right end, obtain the *second* eigenfrequency, say λ^*. Using the corresponding eigenfunction, the single interior nodal location is determined, say at b^*. According to references [1, 2], these are the maximum fundamental frequency and the optimum location of the interior beam support. The next step is to set $b = b^*$ and use the characteristic equation obtained from equations (2)–(7) to find the minimum stiffness such that λ^* becomes the fundamental frequency.

3. RESULTS AND DISCUSSION

We illustrate in more detail the clamped–clamped beam. Using equation (2a) and the boundary conditions $w_{\mathrm{I}}(1) = w'_{\mathrm{I}}(1) = 0$, we find that the characteristic equation without an internal support is

$$\cosh(\lambda)\cos(\lambda) - 1 = 0. \tag{8}$$

The first two roots are 4·73004 and 7·85321 and we set $\lambda^* = 7{\cdot}8532$. The eigenfunction is

$$w_{\mathrm{I}} = \sinh(\lambda^* x) - \sin(\lambda^* x) - \frac{\sinh(\lambda^*) - \sin(\lambda^*)}{\cosh(\lambda^*) - \cos(\lambda^*)}[\cosh(\lambda^* x) - \cos(\lambda^* x)]. \tag{9}$$

By setting equation (9) to zero, a root search gives $x = b^* = 0{\cdot}5$. (In this symmetric clamped–clamped case, one could have inferred the node is at the midpoint.) Now setting

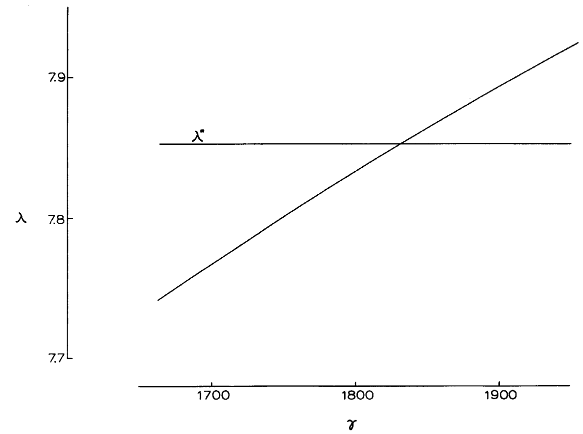

Figure 1. Variation of the two lowest frequencies with respect to stiffness for the clamped–clamped case at the fixed location $b^* = 0{\cdot}5$. The intersection is at γ^*, λ^*.

TABLE 1

Optimum location b^*, minimum stiffness γ^* and the maximum frequency attained λ^* with various end conditions. C = clamped, S = simply supported, Sl = sliding, F = free

Ends	C–C	C–S	C–Sl	C–F	S–S	S–Sl	S–F	Sl–Sl	Sl–F
λ^*	7·8532	7·0686	5·4978	4·6941	2π	$3\pi/2$	3·9266	π	2·3650
b^*	$\frac{1}{2}$	0·5575	0·7169	0·7834	$\frac{1}{2}$	$\frac{2}{3}$	0·7358	$\frac{1}{2}$	0·5517
γ^*	1834	1377	619·4	266·9	995·9	402·0	163·6	113·7	33·50

$b = b^*$ and using the characteristic equation obtained from equations (2)–(7), we vary γ to obtain to lowest two frequencies. Figure 1 shows the results. The horizontal line λ^* represents a mode independent of the stiffness. The slanted curve is another mode whose frequency increases with stiffness. (In the clamped–clamped case, the two modes can be identified as symmetric and antisymmetric modes respectively.) The lowest (fundamental) frequency becomes constant at the intersection of the two curves, at $\gamma^* = 1834$ which is the minimum stiffness for a fundamental frequency of λ^*.

Using similar methods, we found the optimum location and the minimum stiffness for a variety of end conditions, given in Table 1.

For the clamped–free case, our minimum stiffness of 266·87 compares well with 226·9 obtained by Akesson and Olhoff [2]. Note that there is no free–free case since an interior support would not improve the frequency of zero due to rigid rotation.

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