



ON THE SUCCESSIVE APPROXIMATION METHOD FOR THREE-DIMENSIONAL ANALYSIS OF RADIALLY INHOMOGENEOUS TUBES WITH AN ARBITRARY CYLINDRICAL ANISOTROPY

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(Received 5 November 2001, and in final form 22 March 2002)

1. INTRODUCTION

In the last decade, the semi-analytical successive approximation method (SAM), first proposed by Soldatos and Hadjigeorgiou [1], has emerged as a convenient tool for three-dimensional static and dynamic analysis of cylindrically anisotropic laminated hollow cylinders. It has been successfully employed in a series of publications that dealt with dynamic [2–13], static [6, 9, 14], thermoelastic [6, 15] and stability analysis [6, 16–18] of structural components having various geometrical and/or material characteristics and subjected to a variety of lateral or edge support conditions. The method is based on treating a given hollow cylinder as consisting of N thin coaxial cylindrical layers (real or fictitious), so that for each of them the variable coefficients of the governing differential system may be replaced (truncated) by their constant values taken at an interim radial coordinate inside the layer. Furthermore, the obtained exponential solutions for a six-component “displacement-traction” vector are incorporated into the boundary-value problem formulated in terms of the $(6N \times 6N)$ matrix or a product of N (6×6) matrices. Accuracy of the approximation is stipulated by smallness of the constituent layers’ thickness, which may be reduced in successive steps. The efficiency of convergence has been exemplified in references [1, 3, 5, 9, 13, 14] by means of successful numerical comparison with results based on other analytical methods.

The SAM has been initially put forward in reference [1] on somewhat qualitative grounds and was applied to vibrations of homogeneous isotropic cylinders. Subsequent studies have elaborated the analytical part of the method by building it into the different versions of the sextic formalism for the truncated governing system of equations [5, 8–17], which fitted cylindrically anisotropic laminates made of homogeneous orthotropic and monoclinic layers. Those efforts served substantially towards simplifying the method’s numerical implementation. For all that, the considerations have so far been confined with radially homogeneous materials with a symmetry as low as monoclinic, in which case the explicit form of the analytical framework tended to get quite lengthy.

This short note has therefore a dual objective. Firstly, the SAM is combined with the concise sextic formalism making it appropriate for the general case when the layers consist

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of a radially inhomogeneous material with an arbitrary cylindrical anisotropy. Secondly, the mathematical nature of the SAM is elucidated by means of direct comparison of the (exponential) solution of the truncated governing system for a given layer with the formal solution of the governing system in its original form.

2. THE SYSTEM OF GOVERNING EQUATIONS

Consider plane harmonic elastic waves propagating along the axis z of an infinite cylindrically anisotropic elastic medium, whose density ρ and elasticity tensor c_{ijkl} referred to the cylindrical co-ordinate system may depend on the radial co-ordinate r . The displacement \mathbf{u} and the radial-stress vector $\mathbf{t}_r = \mathbf{e}_r \boldsymbol{\sigma}$ are sought in the form

$$\begin{aligned} \mathbf{u} &= \mathbf{U}^{(n)}(r) \exp[i(n\theta + k_z z - \omega t)], \\ \mathbf{t}_r &= \mathbf{Y}^{(n)}(r) \exp[i(n\theta + k_z z - \omega t)], \end{aligned} \quad n = 0, 1, 2, \dots \quad (1)$$

The equation of motion (or equilibrium at $\omega = 0$) in the absence of external forces may be cast into the form

$$\left[\frac{i}{r} \mathbf{G}(r) - \frac{d}{dr} \right] \boldsymbol{\eta}^{(n)}(r) = \mathbf{0}, \quad (2)$$

where

$$\boldsymbol{\eta}^{(n)}(r) = (\mathbf{U}^{(n)}(r), i r \mathbf{Y}^{(n)}(r))^T, \quad i \mathbf{G} = \mathbf{g}_0 + r \mathbf{g}_1 + r^2 \mathbf{g}_2, \quad (3, 4)$$

$$\begin{aligned} \mathbf{g}_0 &= \begin{pmatrix} -\mathbf{Q}^{-1} \mathbf{R} \boldsymbol{\kappa} & -i \mathbf{Q}^{-1} \\ -i \boldsymbol{\kappa} (\hat{\mathbf{T}} - \mathbf{R}^T \mathbf{Q}^{-1} \mathbf{R}) \boldsymbol{\kappa} & -\boldsymbol{\kappa} \mathbf{R}^T \mathbf{Q}^{-1} \end{pmatrix}, \\ \mathbf{g}_1 &= i k_z \begin{pmatrix} -\mathbf{Q}^{-1} \mathbf{P} & \hat{\mathbf{0}} \\ i [\boldsymbol{\kappa} (\mathbf{S} - \mathbf{R}^T \mathbf{Q}^{-1} \mathbf{P}) + (\mathbf{S}^T - \mathbf{P}^T \mathbf{Q}^{-1} \mathbf{R}) \boldsymbol{\kappa}] & -\mathbf{P}^T \mathbf{Q}^{-1} \end{pmatrix}, \\ \mathbf{g}_2 &= \begin{pmatrix} \hat{\mathbf{0}} & \hat{\mathbf{0}} \\ i k_z^2 [(\hat{\mathbf{M}} - \mathbf{P}^T \mathbf{Q}^{-1} \mathbf{P}) - \rho \omega^2 \mathbf{I}] & \hat{\mathbf{0}} \end{pmatrix}, \end{aligned} \quad (5)$$

“T” means transpose, $\hat{\mathbf{0}}$ and \mathbf{I} are the (3×3) zero and unity matrices, and slightly modified Ting’s notations [19] are used for the matrices

$$\begin{aligned} Q_{ik} &= c_{irkr} = Q_{ki}, \quad \hat{T}_{ik} = c_{i\theta k\theta} = \hat{T}_{ki}, \quad \hat{M}_{ik} = c_{izkz} = \hat{M}_{ki}, \\ R_{ik} &= c_{irk\theta}, \quad P_{ik} = c_{irkz}, \quad S = c_{i\theta kz}, \quad i, k = r, \theta, z; \end{aligned} \quad \boldsymbol{\kappa} = \begin{pmatrix} in & -1 & 0 \\ 1 & in & 0 \\ 0 & 0 & in \end{pmatrix}. \quad (6)$$

The matrices \mathbf{g}_0 , \mathbf{g}_1 , \mathbf{g}_2 depend on r if the material is radially inhomogeneous. Due to the particular choice of the variables (3), the system matrix has a particular symmetry,

described by the relation

$$\mathbf{G}(r) = \mathbf{T}\mathbf{G}^+(r)\mathbf{T}, \quad \text{where } \mathbf{T} = \begin{pmatrix} \hat{\mathbf{0}} & \mathbf{I} \\ \mathbf{I} & \hat{\mathbf{0}} \end{pmatrix} \quad (7)$$

(the superscript “+” meaning adjoint of a matrix), which furnishes the solutions with corresponding algebraic properties.

3. RELATION BETWEEN THE EXPONENTIAL AND THE EXACT FORMAL SOLUTION

Denote the inner, outer and middle radii of a given hollow cylinder by R_1 , R_2 and $R = \frac{1}{2}(R_1 + R_2)$ respectively. The idea behind SAM is to assume that the cylinder is thin enough, i.e., its thickness $H = R_2 - R_1$ is small comparatively to R , and to approximate the governing differential system by taking the system matrix $i\mathbf{G}(r)/r$ as a constant one, referred to $r = R$ [1]. Then equation (2) becomes

$$\left[\frac{i}{R}\mathbf{G}(R) - \frac{d}{dr} \right] \boldsymbol{\eta}^{(n)}(r) = \mathbf{0}, \quad (8)$$

yielding exponential partial solutions $\boldsymbol{\eta}_\alpha^{(n)}(r)$. Taking $r = R_2$, these may be written in the form $\boldsymbol{\eta}_\alpha^{(n)}(R_2) = \mathbf{W}(R_2, R_1)\boldsymbol{\eta}_\alpha^{(n)}(R_1)$ ($\alpha = 1, \dots, 6$), where the propagator matrix (the fundamental solution in matrix form [20, 21]) is

$$\mathbf{W}(R_2, R_1) = \exp[i\mathbf{G}(R)h], \quad (9)$$

$h = H/R$ being the dimensionless thickness. The matrix exponential may be evaluated by using either its series expansion or its exact definition through the eigenspectrum of $\mathbf{G}(R)$ [20, 21].

Now return to the original form of the governing system, having variable coefficients. To start with, it appears instructive to assume for the moment that the matrices \mathbf{g}_0 , \mathbf{g}_1 , \mathbf{g}_2 , comprising the system matrix, pairwise commute with each other and so the matrices $\mathbf{G}(r_1)$ and $\mathbf{G}(r_2)$ commute at any r_1, r_2 . Then equation (2) admits a simple “scalar-like” solution which, being taken for $r = R_2$, may be written in matrix form as follows:

$$\begin{aligned} \mathbf{M}^{(com)}(R_2, R_1) &= \exp \left[\int_{R_1}^{R_2} ds \frac{i\mathbf{G}(s)}{s} \right] \\ &= \exp[i\mathbf{G}(R)h] + \frac{1}{24} \left(\frac{i\mathbf{G}(r)}{r} R^3 \right)''_{r=R} h^3 + O(h^4), \end{aligned} \quad (10)$$

where the superscript reminds one of the commutativity model assumption, the prime hereafter denotes differentiation, and O means “of the order of”. It is noteworthy that, due to a standard expedient of taking the reference point at the centre of the integration interval (R_2, R_1) rather than at the edge point, the error of replacing of the integrand by its constant value begins with the term of the cubic order in h (i.e., there is no quadratic-order term).

In general, the involved matrices do not commute indeed and so the solution equation (10) is invalid [20, 21]. In fact, the differential system (2) does not admit a closed-form explicit solution in a generic case. Instead, its formal matrix solution is a limit of the product of an infinite number of exponentials (the multiplicative integral), which may be

expressed in terms of the Peano expansion

$$\mathbf{M}(r, R_1) = \mathbf{I}_{(6)} + \int_{R_1}^r ds \frac{i\mathbf{G}(s)}{s} + \int_{R_1}^r ds_1 \frac{i\mathbf{G}(s_1)}{s_1} \int_{R_1}^{s_1} ds_2 \frac{i\mathbf{G}(s_2)}{s_2} + \dots \quad (11)$$

($\mathbf{I}_{(6)}$ is the (6×6) identity matrix), which is convergent for any continuous $i\mathbf{G}(r)/r$, but its convergence is known to be slow [20, 21]. At the same time, a simple speculation shows that the power-series expansion in h of the difference between equation (10) and the exact solution $\mathbf{M}(R_2, R_1)$ defined by equation (11) appears only in the terms proportional to h^m , $m \geq 3$. Indeed, this difference is entirely due to the matrix non-commutativity, so it is not involved in the single-integral term of equation (11) whereas its contribution resulting from each of the n -tuple integrals starts from the term proportional to h^{n+1} . Accordingly, calling on equations (11) and (4), it may be shown that

$$\mathbf{M}(R_2, R_1) = \mathbf{W}(R_2, R_1)$$

$$+ \frac{1}{12} \left\{ \frac{1}{2} \left(\frac{i\mathbf{G}(r)}{r} R^3 \right)''_{r=R} + [i\mathbf{G}(R), \mathbf{g}_0(R) - \mathbf{g}_2(R)R^2 - \Delta^{(1)}R] \right\} h^3 + O(h^4), \quad (12)$$

where $\mathbf{W}(R_2, R_1)$ is the exponential given by equation (9); the symbol $[,]$ stands for the commutator of enclosed matrices: $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$; and

$$\begin{aligned} \left(\frac{i\mathbf{G}(r)}{r} R^3 \right)''_{r=R} &= 2\mathbf{g}_0(R) - 2(\mathbf{g}'_0 - \mathbf{g}'_2 R^2)_{r=R} R + \Delta^{(2)} R^2, \\ \Delta^{(1)} &= (\mathbf{g}'_0 + \mathbf{g}'_1 R + \mathbf{g}'_2 R^2)_{r=R}, \quad \Delta^{(2)} = (\mathbf{g}''_0 + \mathbf{g}''_1 R + \mathbf{g}''_2 R^2)_{r=R}. \end{aligned} \quad (13)$$

The inhomogeneity setting implies that the radial variation of the elasticity tensor $\mathbf{c}(r)$ is much greater than small h , that is, $rc'/c \gg h$, where c means a typical value of the stiffness. In the case of a radially homogeneous medium (or, more precisely, if $rc'/c \sim h \ll 1$), equation (12) simplifies to the form

$$\mathbf{M}(R_2, R_1) = \mathbf{W}(R_2, R_1) + \frac{1}{12} \{ \mathbf{g}_0(R) + [i\mathbf{G}(R), \mathbf{g}_0(R) - \mathbf{g}_2(R)R^2] \} h^3 + O(h^4), \quad (14)$$

Thus, equation (12) (or equation (14) in case of homogeneity) specifies, to the leading order in small h , the discrepancy admitted on replacing the exact solution $\mathbf{M}(R_2, R_1)$ of the original governing system (2) by the exponential solution $\mathbf{W}(R_2, R_1)$ [equation (9)] of the system with constant (truncated) coefficients. The discrepancy appears in the term of the order $O(h^3)$ and consists of two terms, the first one following merely from the approximate integration (see equation (10)), while the second one, in the form of a commutator, is due to the matrix non-commutativity.

The next step of the SAM is to utilize the approximate solution, obtained for a thin cylindrical layer, in solving the boundary problem for the aggregate thick cylinder consisting of N coaxial layers, either made of different materials in case of a composite cylinder or fictitious ones for a uniform (yet probably radially inhomogeneous) one. With regard for the continuity condition at the interfaces, the "displacement-stress" vectors at the inner ($r = R_1$) and outer ($r = R_{N+1}$) surfaces of the aggregate cylinder satisfy the relation $\boldsymbol{\eta}_\alpha(R_{N+1}) = \mathbf{M}_\Sigma \boldsymbol{\eta}_\alpha(R_1)$, where $\mathbf{M}_\Sigma = \mathbf{M}_N \cdots \mathbf{M}_1$. Suppose that the constituent layers $j = 1, \dots, N$ (say, equidistant ones) are thin, that is, $h_j = 2H_\Sigma/N(R_{j+1} + R_j) \ll 1$, where R_j is the inner radius of the j th layer, H_Σ is the thickness of the aggregate cylinder. As a result, the exact matrix solutions $\mathbf{M}_j(R_j, R_{j+1})$ for each layer may be replaced by the corresponding exponential $\mathbf{W}_j(R_j, R_{j+1}) = \exp[i\mathbf{G}_j((R_j + R_{j+1})/2)h_j]$ with an error $O(h_j^3)$, so that the total error of replacing \mathbf{M}_Σ by the product of N exponentials is of the order of

$\sum_{j=1}^N h_j^3 \lesssim (H_\Sigma/R_1 N)^2$. It may indeed be made, however, small by taking sufficiently large N , therefore providing the approximate relation

$$\mathbf{\eta}_z(R_{N+1}) = \mathbf{W}_\Sigma \mathbf{\eta}_z(R_1), \quad \mathbf{W}_\Sigma = \mathbf{W}_N \cdots \mathbf{W}_1. \quad (15)$$

This relation is then subject to the boundary conditions at the lateral surfaces. For instance, in the case of wave propagation in the given cylinder either with one surface free of traction and the other clamped, or having both surfaces clamped or traction-free, the corresponding dispersion equations are in the form of vanishing determinant of the appropriate (3×3) blocks of \mathbf{W}_Σ . The edge boundary conditions for a simply-supported finite cylinder may be accommodated in the standard fashion, discretizing appropriately the axial wavenumber k_z .

4. CLOSURE

The successive approximation method (SAM) of three-dimensional dynamic and static analysis for thick uniform or composite hollow cylinders relies on the approximation (truncation) of the governing system of equations for a constituent thin cylindrical layer. This procedure has been outlined for the general case when a thin cylinder consists of a radially inhomogeneous material with an arbitrary cylindrical anisotropy. The relation between the formal solution, in the form of the Peano expansion, for the original governing system and the exponential solution of the approximate system has been examined. It has been demonstrated that within the SAM approach, as far as the dimensionless thickness h of a fictitious layer may be set anyhow small, the exact formal solution may be approximated with any desired accuracy by the exponential expanded up to $O(h^2)$ inclusive. Further to this conclusion, the actual computational performance of the method has shown [1, 3, 13, 14] that very accurate numerical results can be obtained even with the use of h values which are not necessarily very small and/or for rather large k_z and ω values, such that altogether may demand extending of the series expansion in h to the higher order terms. The broader area of the SAM applicability is apparently related to the fact that any coefficient of the exponential expansion in h contains a higher power of the system matrix and hence a higher maximum power of k_z and ω , than the coefficient of the same order in the expansion of the approximation discrepancy.

Practically, exact through-thickness distributions of displacements and stresses, thus obtained by the SAM for curved structural components, can be built and appropriately stored in a computer's memory, in the convenient form of "subroutines", in a manner similar to that met in structural analysis problems dealing with corresponding flat plate components [22–24]. It then follows, as an additional benefit of the SAM application, that such highly accurate through thickness displacement distributions can be merged with a generalised, two-dimensional, "smeared" deep shell model (see, for instance, reference [25]) in the sense already described in references [26–31] for flat plates and shallow shell panels. Apart from that such a merging essentially endows the two-dimensional shell theory the ability to yield exact three-dimensional elasticity results for simply supported edges, it further extends the new stress analysis method [26–31] to shell structures that are subjected to different sets of edge boundary conditions.

ACKNOWLEDGMENTS

This work has been supported by the EPSRC (Grant No GR/R28584/01).

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