Journal of Sound and Vibration (2003) **259**(2), 457–460
doi:10.1006/jsvi.2002.5170, available online at http://www.idealibrary.com on IDEAL®

FRACTIONAL VAN DER POL EQUATIONS

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(*Received* 21 *March* 2002)

The standard van der Pol oscillator is modelled by a differential equation for which the elastic restoring force is "harmonic," i.e., it is a linear function of the dependence variable [1],

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \tag{1}$$

where ε is a positive parameter. Mathematical analysis shows that equation 1 has an essentially unique, stable limit-cycle toward which all other solutions approach as $t \to \infty$. However, it is of interest to consider modifications to equation (1) in which the dependent variable x and/or its first derivative occur to some fractional power. Such non-linear differential equations will be called fractional van der Pol oscillator equations. Particular examples include the following two equations:

$$\ddot{x} + x^{1/3} = \varepsilon(1 - x^2)\dot{x}, \tag{2}$$

$$\ddot{x} + x = \varepsilon(1 - x^2)(\dot{x})^{1/3}. \tag{3}$$

Equation (2) has been studied by Mickens [2] using phase-space techniques and the application of the Liénard–Levinson–Smith theorem [3]; the method of harmonic balance allowed the calculation of an analytic approximation to the limit-cycle solution. The main purpose of this Letter-to-the-Editor is to investigate equation (3).

To proceed, write equation (3) as a system of two first order ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \varepsilon(1 - x^2)y^{1/3}. \tag{4}$$

It follows that $(\bar{x}, \bar{y}) = (0, 0)$ is the only fixed-point. Using the same energy argument, as presented in reference [2], it also follows that this fixed-point is unstable.

A comparison of equation (3) with the structural form of the differential equation occurring in the Liénard–Levinson–Smith theorem [3] clearly shows that this theorem cannot be applied to it. However, the calculation of the nullclines [1] for equation (3), in the (x, y) phase space, gives

(1) $\mathrm{d}y/\mathrm{d}x = 0$, along the curve

$$y^{1/3} = \frac{x}{\varepsilon(1 - x^2)}; \tag{5}$$

(2) $\mathrm{d}y/\mathrm{d}x = \infty$, along $y = 0$ or the x-axis. With this information, it is clearly seen that the trajectories in phase space for equation (3) are topologically the same as for the standard van der Pol equation. Hence, a unique and stable limit-cycle is expected to exist for equation (3).

0022-460X/02/$35.00

The method of slowly varying amplitude and phase [3], sometimes known as the first approximation of Krylov and Bogoliubov [4], can be used to determine an analytical approximation to the limit-cycle solution of equation (3). For the non-linear differential equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \quad 0 < \varepsilon \ll 1, \tag{6}$$

a first approximation (in ε) to the periodic solutions is given by

$$x(t, \varepsilon) \simeq a(t, \varepsilon)\sin[t + \phi(t, \varepsilon)], \tag{7}$$

where $a(t, \varepsilon)$ and $\phi(t, \varepsilon)$ are calculated from

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} f(a\sin\psi,\ a\cos\psi)\cos\psi\ \mathrm{d}\psi, \tag{8a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} f(a\sin\psi,\ a\cos\psi)\sin\psi\ \mathrm{d}\psi. \tag{8b}$$

For the particular case of equation (3), the function $f(x, \dot{x})$ becomes, in equations (8), the expression

$$f(x, \dot{x}) \to f(a\sin\psi,\ a\cos\psi) = (1 - a^2\sin^2\psi)(a\cos\psi)^{1/3}. \tag{9}$$

The way to get around the difficulty posed by the one-third power cosine term is to expand it into a Fourier series. *A priori*, this expansion is expected to take the form

$$(\cos\psi)^{1/3} = b_1\cos\psi + b_2\cos 3\psi + b_3\cos 5\psi + \cdots. \tag{10}$$

These coefficients can be easily determined by numerically integrating the standard Fourier integral expressions. Carrying out this procedure [5], the following values are obtained: $b_1 = 1{\cdot}15946$, $b_2 = -0{\cdot}231888$, $b_3 = 0{\cdot}11594$, $b_4 = -0{\cdot}07378$, etc. Substituting equation (10) into equations (8), the integrations can be done to give

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{\varepsilon}{2}\right)a^{1/3}\left[b_1 - (b_1 - b_2)\left(\frac{a^2}{4}\right)\right], \tag{11}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0. \tag{12}$$

The second equation can be easily solved to give $\phi(t, \varepsilon) = \phi_0$, where ϕ_0 is a constant. Consequently, to this level of calculation, the angular frequency of the period solution is 2π, which is the same as that for the free oscillator, i.e., the case where $\varepsilon = 0$. Note that this result also holds true for the standard van der Pol equation [3, 4] given by equation (1).

As for the amplitude, examination of equation (11) shows that it has two fixed-points, the first at $\bar{a}^{(1)} = 0$, the second at

$$\bar{a}^{(2)} = 2\left(\frac{b_1}{b_1 - b_2}\right)^{1/2}. \tag{13}$$

Since $b_1 > 0$ and $b_2 < 0$, it follows that

$$\bar{a}^{(2)} < 2 \tag{14}$$

and noted that for the standard van der Pol equation the value for the amplitude of the non-trivial limit-cycle, to this level of calculation, is $\bar{a} = 2$. Further inspection of the right-hand side of equation (11) shows that the fixed-point $\bar{a}^{(1)}$ is unstable, while that at $\bar{a}^{(2)}$ is stable. Putting all of these results together gives the following approximation to the

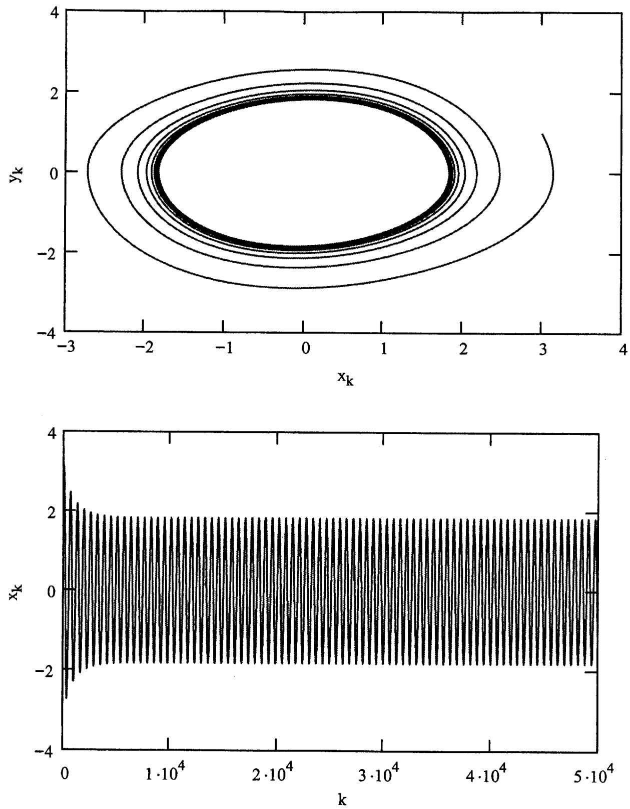

Figure 1. The parameter values are $h = \Delta t = 0{\cdot}01$ and $\varepsilon = 0{\cdot}10$, with initial conditions, $x_0 = 3$, $y_0 = 1$. (a) Phase-plane plot of x versus y, (b) plot of $x_k = x(t_k)$ versus k, where $t_k = hk$.

periodic solution of equation (3):

$$x(t, \varepsilon) \simeq 2\left(\frac{b_1}{b_1 - b_2}\right)^{1/2} \sin(t + \phi_0). \tag{15}$$

Using the numerically derived values for b_1 and b_2, the amplitude is

$$a = 2\left(\frac{b_1}{b_1 - b_2}\right)^{1/2} = 1{\cdot}82574. \tag{16}$$

A confirmation of the above analysis, regarding the behavior of the solutions to equation (3), was done by numerical integration of the differential equation for many values of the initial conditions, $x_0 = x(0)$ and $y_0 = \mathrm{d}x(0)/\mathrm{d}t$; the parameter ε; and the step-size, $h = \Delta t$. Figure 1 shows a typical result. In summary, if $0 < \varepsilon \lesssim 1$, then all initial conditions (x_0, y_0) lead to an asymptotic approach in time to a limit-cycle. The numerical derived value for the amplitude was determined to be $a = 1{\cdot}826$ which is in excellent agreement with that given in equation (16).

It should be clear, from the work presented here, how to proceed to calculate analytical approximations to periodic solutions of other "fractional" non-linear oscillator equations. However, for these cases, in general, the method of harmonic balance must be used [1]. An example of such an equation is

$$\ddot{x} + x^{1/3} = \varepsilon(1 - x^2)(\dot{x})^{1/3}. \tag{17}$$

ACKNOWLEDGMENTS

The work reported here was supported in part by research grants from DOE and the MBRS-SCORE Program at Clark Atlanta University.

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