Journal of Sound and Vibration (2003) **259**(2), 481–489
doi:10.1006/jsvi.2002.5156, available online at http://www.idealibrary.com on IDEAL®

ON THE USE OF THE DEGENERATE PLATE AND THE ABSOLUTE NODAL CO-ORDINATE FORMULATIONS IN MULTIBODY SYSTEM APPLICATIONS

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(*Received* 11 *February* 2002)

1. INTRODUCTION

The purpose of this paper is to discuss the fundamental differences between the degenerate plate element formulations [1–3] and the plate and shell formulations obtained using the absolute nodal co-ordinate formulation introduced for the large rotation and deformation analysis of flexible bodies [4–8]. It is demonstrated in this investigation that existing large rotation degenerate plate formulations, that employ a displacement field that is linear in the nodal co-ordinates, can only be used in the framework of incremental solution procedures due to the limitations that arise from the kinematic motion description. Such limitations do not apply to the absolute nodal co-ordinate formulation, and as a consequence, this formulation can be used with a non-incremental solution procedure. The assumptions used in the degenerate plate formulations are explained using Rodriguez formula for the finite rotation [4]. It is shown in this investigation that, due to these assumptions, the displacement field in the degenerate plate formulations cannot be defined in the global inertial frame of reference, unless this displacement field is expressed as a non-linear function of the nodal co-ordinates. As a consequence, the use of degenerate plate and shell formulations that provide information about the nodal rotations leads to a non-linear inertia matrix and non-zero centrifugal and Coriolis inertia forces. Furthermore, the existing degenerate plate formulations do not ensure the continuity of all the displacement gradients at the nodal points and they may lead to linearized kinematic equations for large rotation problems. As demonstrated in this paper, the use of linearized equations for the finite rotations can lead to a violation of the principle of work and energy. This is not the case with the finite element absolute nodal co-ordinate formulation that is based on a displacement field defined in the global co-ordinate system. This could be achieved by using global displacements and slopes as nodal co-ordinates, thereby avoiding interpolation of rotations or unit vectors. The absolute nodal co-ordinate formulation leads to a constant mass matrix, and as a consequence, the centrifugal and Coriolis inertia forces are identically equal to zero. Unlike the degenerate plate

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formulations, there is no assumption made on the amount of deformation or rotation within the element. Using this formulation, one can avoid linearization of finite rotations that leads to incorrect integrals of motion and *energy drift* that is typical in many finite element formulations.

2. FINITE ROTATION PROBLEM

In this section, some basic kinematics concepts related to the problem of finite rotations are reviewed. The material presented in this section will be used in the following sections to shed light on some of the assumptions used in the degenerate plate formulations.

The transformation matrix $\mathbf{A}^i$ that defines the orientation of a co-ordinate system $X^iY^iZ^i$ in another co-ordinate system XYZ as the result of a finite rotation θ can be expressed using Rodriguez formula as follows [4]:

$$\mathbf{A}^i = \mathbf{I} + \tilde{\mathbf{v}} \sin\theta + 2\tilde{\mathbf{v}}^2 \sin^2\left(\frac{\theta}{2}\right). \tag{1}$$

In this equation, $\mathbf{I}$ is the identity matrix, $\tilde{\mathbf{v}}$ is the skew symmetric matrix associated with a unit vector $\mathbf{v}$ along the axis of rotation, and θ is the angle of rotation about the axis of rotation. It is important to point out that the vector $\mathbf{v}$ must be defined in the co-ordinate system XYZ. This is crucial in the discussion that will be presented in the following section.

In the case of infinitesimal rotations, the preceding equation reduces to

$$\mathbf{A}^i = \mathbf{I} + \tilde{\mathbf{v}}\theta, \tag{2}$$

where in the preceding equation, the Taylor expansion for $\sin\theta$ is used;

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots . \tag{3}$$

Let $\mathbf{e}_1^i$, $\mathbf{e}_2^i$, and $\mathbf{e}_3^i$ be unit vectors along the axes of the co-ordinate system $X^iY^iZ^i$. If two infinitesimal rotations θ_1 and θ_2 are performed about the axes along the unit vectors $\mathbf{e}_1^i$ and $\mathbf{e}_2^i$, respectively, the final orientation of the unit vector $\mathbf{e}_3^i$ is given by the equation.

$$(\mathbf{e}_3^i)_f = \mathbf{A}_1^i \mathbf{A}_2^i \mathbf{e}_3^i. \tag{4}$$

In this case, $\mathbf{A}_1^i$ and $\mathbf{A}_2^i$ are the transformation matrices that result from the two infinitesimal rotations θ_1 and θ_2, respectively; and in this case of infinitesimal rotations, the order of rotation is immaterial. Using equation (2) and the definition of the axes of rotation, it can be shown that the preceding equation leads to

$$(\mathbf{e}_3^i)_f = \mathbf{e}_3^i + \theta_1(\mathbf{e}_1^i \times \mathbf{e}_3^i) + \theta_2(\mathbf{e}_2^i \times \mathbf{e}_3^i). \tag{5}$$

This equation can also be written as

$$(\mathbf{e}_3^i)_f = \mathbf{e}_3^i + \theta_2\mathbf{e}_1^i - \theta_1\mathbf{e}_2^i. \tag{6}$$

The last two terms on the right-hand side of this equation represent the change in the vector $\mathbf{e}_3^i$ as the result of the two infinitesimal rotations θ_1 and θ_2. It is important to note that the vectors $\mathbf{e}_1^i$ and $\mathbf{e}_2^i$ must be, according to Rodriguez formula, defined in the co-ordinate system XYZ. This fact is crucial in understanding the basic assumptions underlying the degenerate plate formulations.

3. DEGENERATE PLATE FORMULATIONS

There are several degenerate plate and shell element formulations in the finite element literature [1–3]. In most of the degenerate plate formulations, the displacement field is described as the sum of two vectors. The first vector is an isoparametric representation that can be conveniently used to describe curved structures such as shells. The nodal co-ordinates used in this vector are three translation nodal displacements. In the second vector, two unit vectors that are tangent to the plate mid-surface are introduced and two rotations are used as nodal co-ordinates for each node. This second part defines the contribution of the displacement field due to the change in the orientation of the normal to the plate mid-surface. Such a two-part plate element displacement field allows for conveniently representing configurations of shell structures and at the same time it provides information about some rotations at the nodes. In this section, we consider the following displacement field as an example of degenerate plate formulations [1–3]:

$$\mathbf{u} = \sum_{k=1}^{n_n} \mathbf{N}_k \mathbf{u}_k + \sum_{k=1}^{n_n} \mathbf{N}_k \varsigma \frac{t_k}{2} \mathbf{V}_k \boldsymbol{\alpha}_k. \tag{7}$$

In this equation, $\mathbf{u}$ is the displacement vector of an arbitrary point on the plate element, $\mathbf{N}_k$ is the element shape function matrices associated with the co-ordinates of node k, $\mathbf{u}_k$ is the three-dimensional vector of translation nodal displacements, ς is the natural co-ordinate, t_k is the plate thickness at node k, $\mathbf{V}_k = [\mathbf{e}_{kx} \;\; \mathbf{e}_{ky}]^{\mathrm{T}}$, $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$ are unit vectors tangent to the mid-surface of the plate at node k, $\boldsymbol{\alpha}_k = [\alpha_{kx} \;\; \alpha_{ky}]^{\mathrm{T}}$, α_{kx} and α_{ky} are the two rotations at node k, and n_n is the number of nodes of the element. For the following discussion, it is more convenient to write the preceding equation in the form

$$\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b, \tag{8}$$

where

$$\mathbf{u}_a = \sum_{k=1}^{n_n} \mathbf{N}_k \mathbf{u}_k, \quad \mathbf{u}_b = \sum_{k=1}^{n_n} \mathbf{N}_k \varsigma \frac{t_k}{2} \mathbf{V}_k \alpha_k. \tag{9}$$

It is important to note that the vector $\mathbf{u}_a$, by selecting a proper number of nodes, is sufficient to describe an arbitrary displacement of the plate including deformations and rigid body motion. However, this part of the displacement field does not provide information about the rotations or slopes, and as a consequence, continuity of some of these kinematics variables is not ensured at the element nodal points. Since rotations are crucial in the classical formulations of plate and shell structures, the second vector $\mathbf{u}_b$ is introduced for the purpose of achieving continuity of some rotation parameters.

3.1. CONSISTENCY

The vector $\mathbf{u}_b$ is systematically derived using Rodriguez formula of the infinitesimal rotation as described in the preceding section. Modification of this vector in order to use higher order terms in the rotations must be made according to Rodriguez formula, otherwise inconsistent formulation is obtained. Nonetheless, introducing the vector $\mathbf{u}_b$ of the displacement field can lead to difficulties if the vector of displacement $\mathbf{u}$ is not defined in the proper co-ordinate system. This fact, which is explained below, is crucial in understanding the fundamental differences between the degenerate plate formulations and the finite element absolute nodal co-ordinate formulation briefly reviewed in the following section.

If the vector $\mathbf{u}$ is selected to define the *location*, not the displacement, of an arbitrary point on the plate element, the tangent vectors at an arbitrary point on the plate mid-surface are defined as

$$\mathbf{t}_x = \frac{\partial \mathbf{u}}{\partial x}, \quad \mathbf{t}_y = \frac{\partial \mathbf{u}}{\partial y}. \tag{10}$$

In this equation, x and y are the local spatial co-ordinates of the element. For a general plate or shell configuration, these two vectors are not necessarily orthogonal. They are orthogonal for flat plates that undergo rigid-body displacements. The cross product of these two vectors defines the normal to the plate mid-surface. In general, the tangent and normal vectors can be normalized and then used with simple cross product operations to define the two orthogonal unit tangent vectors $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$ that are used in the displacement field [1]. In view of the discussion presented in the preceding section, the vectors $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$ must be defined in the co-ordinate system in which the element configuration is to be defined. Therefore, if the global system is used and accurate expressions are used for the two vectors, $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$, these two vectors become highly non-linear functions of the nodal co-ordinates. As a consequence, the assumed displacement field $\mathbf{u}$ becomes a highly non-linear function of the nodal co-ordinates. That is to say, in principle, the degenerate plate formulation presented in this section, if implemented correctly, cannot lead to an expression for the locations of the points on the elements in the global system that is linear in the nodal co-ordinates. For this reason, most finite element implementations of the degenerate plate formulations are incremental and require the use of a co-rotational procedure. An updating scheme is used to define the vectors $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$ as explained by Bathe [1].

3.2. SUMMARY AND REMARKS

Based on the discussion presented in this section, some conclusions that are important to demonstrate the basic differences between the degenerate plate formulations and the absolute nodal co-ordinate formulation are summarized as follows:

1. In the case of large rotation problems, the displacement field of the degenerate plate formulations that is linear in the nodal co-ordinates cannot be used to define the global locations of arbitrary points on the finite element in an inertial frame of reference. This is due to the fact that the unit tangents depend on the plate configuration.
2. As a consequence of equation (1), degenerate plate formulations are often implemented using incremental solution procedures that require elaborate updating schemes and the use of local element co-ordinate systems.
3. In large rotation dynamic problems, existing degenerate plate formulations do not lead to a constant mass matrix for the finite element due to the nature of the co-ordinates and solution procedure used. As a result, the centrifugal and Coriolis inertia forces are not equal to zero and are highly non-linear functions of the nodal co-ordinates and velocities.
4. Degenerate plate formulations ensure continuity of some rotations. However, a rotation is defined by an angle and an axis of rotation. This fact sheds light on the approximations made in some existing finite element formulations when the unit vectors $\mathbf{e}_{kx}$ and $\mathbf{e}_{ky}$ are determined.
5. Degenerate plate formulations do not ensure the continuity of all the displacement gradients. They ensure continuity of some rotations at the nodal points, provided that

the unit tangents are correctly determined. Furthermore, representation of the drilling degree of freedom adds more complexity to the degenerate plate formulations.

4. ABSOLUTE NODAL CO-ORDINATE FORMULATION

The absolute nodal co-ordinate formulation was introduced for the large deformation analysis of flexible bodies in multibody system applications that are characterized by large rigid-body rotations. The motion of the bodies exhibits a strong coupling between the reference displacement and the body deformations. While most solution procedures used in general purpose finite element codes are based on incremental methods, most multibody dynamics solution procedures are non-incremental. This is due to the fact that multibody simulations are carried out for a long period of time, and the use of the incremental solution procedures can lead to significant error accumulation. In order to have non-incremental large deformation procedure that can be easily implemented in multibody solution algorithms, one must use a fully non-linear formulation in which no linearization is made with regard to the large rotations. The absolute nodal co-ordinate formulation has been found to be an effective method that satisfies this requirement, and does not require special measures in the numerical integration in order to satisfy the principle of work and energy. In this section, we briefly review the basic kinematics equations in order to demonstrate the fundamental differences between the absolute nodal co-ordinate formulation and the degenerate plate formulations.

Unlike the degenerate plate formulations, in the absolute nodal co-ordinate formulation, the displacement field is used to define the location of the arbitrary point on the finite element in the global co-ordinate system. In this formulation, the global position vector of an arbitrary point on the element can be written as [4–8]

$$\mathbf{r} = \mathbf{S}(x, y, z)\mathbf{e}. \tag{11}$$

In this equation, $\mathbf{S}$ is the element shape function matrix, x, y, and z are the local spatial co-ordinates of the element, and $\mathbf{e}$ is the vector of nodal co-ordinates. The shape function matrix $\mathbf{S}$ and the vector of nodal co-ordinates $\mathbf{e}$ must be selected such that the element can describe an arbitrary rigid-body displacement and large deformation. Examples of two- and three-dimensional finite elements developed using the general description of equation (11) can be found in the literature [4–8].

4.1. PLATE ELEMENT

Unlike the degenerate plate formulations and large rotation vector formulations [9], the vector of nodal co-ordinates $\mathbf{e}$ in equation (11) does not include any finite or infinitesimal rotations. It consists of global displacements and slope nodal co-ordinates; each node in the four-node plate elements proposed in reference [8] has 12 degrees of freedom that include three translation co-ordinates and nine position vector gradients. These proposed plate elements that can describe an arbitrary large rotation and deformation have 48 nodal co-ordinates. The 12 nodal co-ordinates for node k are defined as

$$\mathbf{e}_k = \begin{bmatrix} \mathbf{r}_k^{\mathrm{T}} & \dfrac{\partial \mathbf{r}_k^{\mathrm{T}}}{\partial x} & \dfrac{\partial \mathbf{r}_k^{\mathrm{T}}}{\partial y} & \dfrac{\partial \mathbf{r}_k^{\mathrm{T}}}{\partial z} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{r}_k^{\mathrm{T}} & \mathbf{r}_{ks}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{12}$$

In this equation, $\mathbf{r}_k$ is the three-dimensional vector of nodal translations, and $\mathbf{r}_{ks}$ is the nine-dimensional vector of the position vector gradients at node k. Therefore, equation

(11) can be written as

$$\mathbf{r} = \sum_{k=1}^{4} \mathbf{S}_{kt}\mathbf{r}_k + \sum_{k=1}^{4} \mathbf{S}_{kr}\mathbf{r}_{ks}. \tag{13}$$

In this equation, $\mathbf{S}_{kt}$ and $\mathbf{S}_{kr}$ are shape function matrices [8]. Note that the assumed displacement field of equation (13), unlike the degenerate plate element formulation, is linear in the nodal co-ordinates regardless of the amount of rotation or deformation within the element. No unit vectors or rotations are used in this equation, and no interpolation of rotations or slopes is required, yet all information about the tangents and normal to the plate mid-surface can be easily obtained from the preceding equation. This assumed displacement field defines the *location* of the arbitrary points on the plate or shell element in an inertial global system and not in an element co-ordinate system. Details of plate and shell element shape functions based on the absolute nodal co-ordinate formulation as well as highly non-linear large deformation and large rotation problems solved using equation (13) are presented in the literature [8].

4.2. FUNDAMENTAL DIFFERENCES

The kinematics description of equation (13) is fundamentally different from the description used in the degenerate plate formulation presented in the preceding section for the following reasons:

1. In the case of large rotation problems, the displacement field of the plate element based on the absolute nodal co-ordinate formulation is linear in the nodal co-ordinates. This displacement field can be used, unlike the degenerate plate formulation, to define the global locations of arbitrary points on the finite element in an inertial frame of reference.
2. As a consequence of equation (1) and since no linearization is made, the absolute nodal co-ordinate formulation can be implemented using non-incremental solution procedures that do not require elaborate updating schemes nor the definition or interpolation of unit vectors or finite rotations.
3. In large rotation dynamic problems, the absolute nodal co-ordinate formulation leads to a constant mass matrix for the finite element due to the nature of the co-ordinates used. As a result, the centrifugal and Coriolis inertia forces are identically equal to zero.
4. Because the position vector gradients are used as nodal co-ordinates, the absolute nodal co-ordinate formulation provides information about all the rotations at the nodal point. To the authors' knowledge, this is the only known finite element dynamic formulation that provides this information and at the same time leads to a constant mass matrix.
5. The absolute nodal co-ordinate formulation ensures the continuity of all rotations and displacement gradients and can be applied to beam, plate and shell problems as demonstrated in several publications [5–8].

5. CONSERVATION OF ENERGY

The dynamic equations of motion lead to constants or integrals of motion. Formulated correctly, the dynamic equations lead to an integral of motion that defines the principle of work and energy. Solved accurately, the equations of motion should lead to a solution that satisfies the principle of work and energy. Solutions obtained using most multibody formulations satisfy the principle of work and energy since these formulations employ fully non-linear equations. The absolute nodal co-ordinate formulation also leads to a non-linear formulation that ensures slope continuity and leads to exact modelling of the rigid-body dynamics. This is crucial in multibody applications that are inherently non-linear due to the large reference displacements and the constraints that restrict the motion of the system components. Linearization and slope discontinuity in addition to using incremental solution procedures, if not properly handled, can lead to serious problems when multibody system applications are considered.

5.1. SIMPLE EXAMPLE

In order to further elaborate on some of the difficulties that can be encountered when the equations of motion of a simple system are linearized, we consider the simple pendulum example shown in Figure 1. The non-linear equation of motion of this single-degree-of-freedom rigid-body system is given by

$$I_O\ddot{\theta} + mg\frac{l}{2}\sin\theta = 0, \tag{14}$$

where θ is the angle of rotation of the pendulum, I_O is the mass moment of inertia of the pendulum about the fixed point O, m and l are respectively the pendulum mass and length, and g is the gravity constant. Recall that $\ddot{\theta} = \dot{\theta}\,(\mathrm{d}\dot{\theta}/\mathrm{d}\theta)$ and, substituting into the preceding equation, one obtains the following constant *energy integral of motion* for this conservative system:

$$\tfrac{1}{2}I_O\,\dot{\theta}^2 - mg\frac{l}{2}\cos\theta = C_1, \tag{15}$$

where C_1 is a constant. This equation, that states that the change in the system kinetic energy is equal to the change in the system potential energy, shows that the kinetic energy has an upper limit. Therefore, an accurate numerical solution of the non-linear relationship of equation (14) must satisfy the principle of work and energy as defined for the pendulum system by equation (15). Most multibody algorithms do satisfy the principle of work and energy since the equations of motion are not linearized, and as a consequence, there is no need for using incremental solution procedures.

On the other hand, a linearization of equation (14) produces the equation.

$$I_O\ddot{\theta} + mg\frac{l}{2}\theta = 0. \tag{16}$$

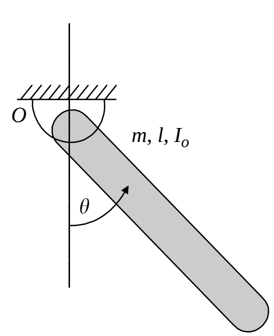

Figure 1. Simple pendulum.

Using again the fact that $\ddot{\theta} = \dot{\theta}(\mathrm{d}\dot{\theta}/\mathrm{d}\theta)$ into the preceding equation and integrating leads to the following constant of motion:

$$\tfrac{1}{2} I_O \dot{\theta}^2 + mg\frac{l}{4}\theta^2 = C_2, \tag{17}$$

where C_2 is a constant. The solution of equation (16) satisfies equation (17), but it does not satisfy the energy constant of equation (15). Equation 17 is an approximation for the principle of work and energy only when the reference rotation as defined in the global system is small. If the reference rotation is finite, the integral of motion of equation (16) is no longer an accurate representation of the principle of work and energy. As the result, any solution procedure, incremental or non-incremental, of equation (16) can lead to *energy drift* as the angle θ increases since such a solution is not required to satisfy equation (15).

6. SUMMARY AND CONCLUSIONS

In this paper, the basic differences between the degenerate plate formulations and the plate formulations obtained using the absolute nodal co-ordinate formulation are discussed. It was demonstrated that the kinematics description of the degenerate plate formulations, that is linear in the nodal co-ordinates, cannot be used to define the global locations of arbitrary points in an inertial co-ordinate system. Such a kinematics description has to be a non-linear function of the nodal co-ordinates in order to achieve this global definition in the degenerate plate formulations [10]. As a result, degenerate plate formulations lead to non-linear mass matrix. The absolute nodal co-ordinate formulation, on the other hand, can be used to define the global configurations of the finite elements by using a displacement field that is linear in the nodal co-ordinates. This can be achieved by using global displacement and slope co-ordinates. This displacement description leads to a constant mass matrix, and as a consequence, the centrifugal and Coriolis inertia forces are identically equal to zero. Since the absolute nodal co-ordinate formulation does not lead to any linearization, this formulation does not require the use of special measures in the numerical integrator to satisfy the principle of work and energy.

The simple pendulum example, discussed in this paper, demonstrates that violation of the principle of work and energy can be an indication that the differential equations of motion are not correctly defined. System of dynamic equations satisfies certain integrals. If the dynamic equations are correctly defined, the numerical solution of these equations must satisfy the principle of work and energy. An energy violation can be a clear indication of a problem associated with the description of the large reference rotation. The simple pendulum example also explains the reason for introducing, in many non-linear multibody formulations, the concept of the *floating frame of reference* that leads to accurate modelling of the reference rotations [4, 11]. The floating frame of reference formulation does not lead to an energy drift since the reference rotations are not linearized while the small elastic deformations are defined in the body co-ordinate system.

ACKNOWLEDGEMENTS

This research was supported, in part, by the National Science Foundation.

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